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Statistical stability for diffeomorphisms with dominated splitting

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Abstract. Alves, Bonatti and Viana have recently shown that physical (or Sinai–Ruelle–Bowen) invariant measures exist for partially hyperbolic diffeomorphisms with mostly expanding center-unstable direction. In this paper we prove that such systems are statistically stable, that is, nearby diffeomorphisms have nearby physical measures.

1. *Introduction*

Much of the remarkable recent progress in dynamics arose from a probabilistic approach to the understanding of complicated dynamical systems, where one focuses on the statistical properties of typical orbits and typical is meant in the sense of large volume in the ambient space. Here we deal with discrete time systems, namely diffeomorphisms $f : M \to M$ on compact manifolds. The most basic statistical data are the time averages

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(z)},$$

where δ_w represents the Dirac measure at a point w. Given any invariant probability, the ergodic theorem asserts that time averages do exist at almost every point. In addition, if the measure is ergodic, then the time average coincides with the space average, that is, the invariant probability itself. In general, invariant measures are singular with respect to volume, and so these observations say nothing about the behavior of positive volume (Lebesgue measure) sets of orbits.

A *physical* measure is an invariant probability measure for which the time average exists and coincides with the space average for a set of initial conditions with positive Lebesgue measure. This set is called the *basin* of the measure. Physical measures are often also called SRB measures, after Sinai, Ruelle and Bowen, who introduced this notion a little over 30 years ago, and proved that for uniformly hyperbolic (Axiom A) diffeomorphisms

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and flows time averages exist for Lebesgue almost every point and coincide with one of finitely many SRB measures. See [**6, 7, 23, 25**].

The problem of the existence and finiteness of SRB measures and their stability under perturbations of the system, beyond the Axiom A setting, has remained at the center of dynamics ever since. We refer the reader to [**4, 27, 29**] for surveys of much of the progress obtained.

Some ten years ago, Palis [**16**] conjectured that every dynamical system can be approximated by another having only finitely many SRB measures, whose basins cover Lebesgue almost every point of the ambient manifold. Moreover, these SRB measures should be stable under perturbations of the system, in a stochastic sense. Also of special interest to us here are the papers of Alves, Bonatti and Viana [**3, 5**], where they prove the existence and finiteness of SRB measures for partially hyperbolic diffeomorphisms, roughly, under the assumption that the central direction is either 'mostly contracting' [**5**] or 'mostly expanding' [**3**]. Indeed, we are going to prove that a large class of systems as in the latter case are statistically stable.

Statistical stability means that time averages do not change much under small deterministic perturbations: nearby systems should have physical measures that are nearby; the precise definition will be given in the next section. This terminology was introduced in [**1, 2**], where it was proved that a large class of systems with non-uniformly expanding behavior are statistically stable. Statistical stability is germane, although not formally related to the notion of stochastic stability introduced by Kolmogorov and Sinai and much developed by the pioneering work of Kifer; see [**12, 13**] and references therein. Before that, statistical stability was well known for uniformly expanding smooth maps and for uniformly hyperbolic diffeomorphisms [**24**] restricted to the basin of their attractors. More recently, in [**9, 10**] it was proved that the same is true for an open class of partially hyperbolic attractors whose central direction is mostly contracting.

In this paper we prove statistical stability for a large class of diffeomorphisms admitting an invariant dominated splitting with mostly expanding center-unstable direction. Precise definitions and statements of our results follow.

1.1. *Statement of results.* Let us consider a compact Riemannian boundaryless manifold M. We denote by $\|\cdot\|$ the norm on the tangent bundle TM of M and by m the normalized Riemannian volume form on M, which we call the *Lebesgue measure* on M.

Given an integer $k \geq 1$ and $\nu \in (0, 1)$, we say that a function is of class $C^{k+\nu}$ if it has k continuous derivatives and if its kth derivative is Hölder continuous with exponent ν. We denote by $\mathrm{Diff}^r(M)$ the set of diffeomorphisms of M of class C^r, endowed with the C^r topology.

Let $f \in \mathrm{Diff}^r(M)$ with $r > 1$. The *time average* of a continuous function $\varphi : M \to \mathbb{R}$ along the orbit of a point $x \in M$ is defined by

$$\tilde{\varphi}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)).$$

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If μ is an invariant probability measure, then the *basin* $B(\mu)$ of μ is the set

$$B(\mu) = \left\{ x \in M : \tilde{\varphi}(x) = \int_M \varphi \, d\mu, \text{ for all } \varphi \in C(M; \mathbb{R}) \right\}.$$

An invariant measure μ is an *SRB measure* or *physical measure* if $B(\mu)$ has positive Lebesgue measure.

Let $f \in \mathrm{Diff}^r(M)$ with $r > 1$, and let $U \subseteq M$ be a open set such that $\overline{f(U)} \subseteq U$. Then, $\Lambda = \bigcap_{n\geq 0} f^n(U)$ is called an *attracting set*, and any neighborhood V of Λ such that $\overline{f(V)} \subseteq V$ is called a *trapping region* of Λ. It is easy to see that $\Lambda = \bigcap_{n\geq 0} f^n(V)$, for any trapping region V of Λ. Moreover, the open set $\bigcup_{n\geq 0} f^{-n}(V)$ is independent of V; it is called the *basin* of Λ.

1.1.1. *Partially hyperbolic diffeomorphisms.* The attracting set Λ has a *dominated splitting* if there is a continuous Df-invariant decomposition $T_\Lambda M = E^{cu} \oplus E^{cs}$ of the tangent bundle $T_\Lambda M$ of M over Λ and constants $C \geq 0$ and $0 < \lambda < 1$ satisfying

$$\|Df^n|E^{cs}_x\| \cdot \|Df^{-n}|E^{cu}_{f^n(x)}\| \leq C\lambda^n, \tag{1}$$

for every $x \in \Lambda$, and for every $n \geq 1$. We say that the attracting set Λ is *partially hyperbolic* if its has a dominated splitting such that the subbundle E^{cs} is *uniformly contracting* (or else E^{cu} is *uniformly expanding*), that is, $\|Df^n|E^{cs}_x\| \leq C\lambda^n$ for every $x \in \Lambda$ and $n \geq 1$; then, we denote E^{cs} by E^s (respectively, we denote E^{cu} by E^u).

The diffeomorphism f has *non-uniform expansion along the center-unstable direction* if there exists a constant $c_0 > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| \leq -c_0 < 0, \tag{2}$$

for all x in a full Lebesgue measure subset of U.

If there exists a neighborhood $\mathcal{N} \subseteq \mathrm{Diff}^r(M)$ of f, $r > 1$, such that every diffeomorphism $g \in \mathcal{N}$ has non-uniform expansion along the center-unstable direction, with the same constant c_0, then we say that f is *robustly non-uniformly expanding along the center-unstable direction*.

The following theorem was proved by Alves *et al*:

THEOREM 1.1. [**3**, Theorem A] *Consider $f \in \mathrm{Diff}^r(M)$, $r > 1$. Assume that $\Lambda \subseteq M$ is a partially hyperbolic attracting set with non-uniform expansion along the center-unstable direction. Then f admits finitely many ergodic SRB measures, and the union of their basins covers a full Lebesgue measure subset of the basin of Λ.*

Definition 1. We say that $f_0 \in \mathrm{Diff}^r(M)$, $r > 1$, is *C^r-statistically stable* if for every sequence $f_n \in \mathrm{Diff}^r(M)$ converging to f_0 in the C^r-topology, and for every sequence $(\mu_n)_{n\geq 1}$ of SRB measures for f_n, the weak* accumulation measures of $(\mu_n)_{n\geq 1}$ are in the convex hull of finitely many SRB measures for f_0.

The main result in the present work is the following.

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THEOREM A. *Consider $f \in \mathrm{Diff}^r(M)$, $r > 1$. Let Λ be a partially hyperbolic attracting set with trapping region U for f. Suppose that f is robustly non-uniformly expanding along the center-unstable direction. Then f is C^r-statistically stable.*

In fact, the conclusion of Theorem A holds in much more generality, as we explain in the sequel.

1.1.2. *Diffeomorphisms with a dominated splitting.* In fact, partial hyperbolicity is not strictly necessary for Theorem 1.1: Alves *et al* obtained the same conclusion under weaker hypotheses [**3**, Proposition 6.3], namely, that the attracting set Λ admits a dominated splitting such that the diffeomorphism is non-uniformly expanding along the center-unstable direction on the trapping region, and satisfies a technical condition that they call the existence of *simultaneous hyperbolic times* for the two subbundles in the splitting. Our proof of Theorem A also shows that diffeomorphisms satisfying these hypotheses are statistically stable.

Of particular interest is an open class $\mathcal{N}$ of C^r-diffeomorphisms, $r > 1$, constructed by Bonatti and Viana [**5**] on the torus $M = \mathbb{T}^n$, $n \geq 4$, such that every $f \in \mathcal{N}$ admits a dominated splitting on M but is not uniformly hyperbolic and, yet, is robustly transitive (all small perturbations have dense orbits). A precise description will be recalled later (§4.1.2). These diffeomorphisms are non-uniformly expanding along the center-unstable direction, and they satisfy the simultaneous hyperbolic times condition. It follows from the mentioned result in [**3**] that every $f \in \mathcal{N}$ admits a finite number of SRB measures. In fact, it was later proved by Tahzibi [**26**] that the SRB measure is unique. Most interesting, he also showed that if f is conservative, then the SRB measure coincides with Lebesgue measure and the diffeomorphism is stably ergodic. Here we obtain the following.

THEOREM B. *If $f \in \mathcal{N}$, then is C^r-statistically stable.*

1.1.3. *Mostly contracting central direction.* We begin by recalling the following notion, introduced in [**5**].

Definition 2. Let $f \in \mathrm{Diff}^r(M)$, $r \geq 1$, have an attracting set Λ with a partially hyperbolic splitting $TM = E^{cu} \oplus E^{cs}$ such that the center-unstable subbundle $E^u = E^{cu}$ is uniformly expanding. We say that f is *mostly contracting* along the center-stable subbundle if

$$\limsup_{n \to +\infty} \frac{1}{n} \log \|Df^n | E^{cs}_x\| < 0 \tag{3}$$

for a positive Lebesgue measure set of points x in every disk D contained in an unstable local manifold.

Bonatti and Viana [**5**] showed, among other results, that the same conclusion as in Theorem 1.1 holds for diffeomorphisms f which are mostly contracting along the center-stable direction. See also [**9**] for further important results on SRB measures and decay of correlations for this class of systems.

We shall extend the definition to the case where there is only a dominated splitting, and prove the following.

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THEOREM C. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, have an attracting set Λ exhibiting a dominated splitting with non-uniformly expanding along the center-unstable direction and mostly contracting along the center-stable direction. Then f has finitely many SRB measures and the union of their basins covers a full Lebesgue measure subset of the basin of Λ. In addition, if f is robustly non-uniformly expanding along the central-unstable direction and every diffeomorphism in a C^r-neighborhood of f is mostly contracting along the center-stable direction, then f is C^r-statistically stable.*

1.1.4. *Sketch of the proof and organization of the paper.* An important ingredient in the proof of Theorem 1.1 in [**3**] is the construction of Gibbs *cu*-states (see Definition 3 below). Gibbs *cu*-states are a non-uniform version of the Gibbs *u*-states introduced by Pesin and Sinai [**18**]. See [**4**, Ch. 11] for a detailed account of Gibbs *u*-states and their relations with SRB measures.

In §2 we extend several aspects of this theory to Gibbs *cu*-states, obtaining a general description of such measures in the setting of diffeomorphisms with a dominated splitting (Theorem 2.1). In §3 we outline the constructive proof in [**3**] that Gibbs *cu*-states exist. Next, using certain uniform bounds obtained from the construction together with general properties of Gibbs *cu*-states established previously in §2, we prove that the set of Gibbs *cu*-states is closed (Theorem 3.4). An important point here is to show that the construction given by Alves *et al* actually yields *all* of the Gibbs *cu*-states of the system (Theorem 3.3). Finally, §4 is dedicated to studying the relationship between Gibbs *cu*-states and SRB measures. In particular, we prove that in our setting every SRB measure must be a Gibbs *cu*-state (Corollary 4.1). In that section we also prove the main Theorems A, B and C.

2. *Gibbs cu-states*

In this section we assume that $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting $T_\Lambda M = E^{cu} \oplus E^{cs}$.

Definition 3. An invariant measure μ supported in the attracting set Λ is a *Gibbs cu-state* if the dim E^{cu} largest Lyapunov exponents are positive μ-almost everywhere and the conditional measures of μ along the corresponding local strong-unstable manifolds are almost everywhere absolutely continuous with respect to Lebesgue measure on these manifolds.

Alves *et al* give a constructive proof of the existence of Gibbs *cu*-states under the hypothesis that the f is non-uniformly expanding along the center unstable direction (see [**3**] or §3.1 for more details).

The main goal of this section is study the Gibbs *cu*-state in the setting of diffeomorphisms with an attracting set with dominated splitting and obtain a general description of such measures.

2.1. *Preliminary results.* We first recall some well-known properties for dominated splitting, Pesin theory and conditional measures to be used in this work.

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2.1.1. *Attracting set with dominated splitting.* Let $f \in \mathrm{Diff}^r(M)$, $r > 1$. Let $U \subseteq M$ be an open set such that $\overline{f(U)} \subseteq U$ and let $\Lambda = \bigcap_{n\geq 1} f^n(U)$ be an attracting set. Then, the set Λ is a compact set, maximal invariant for f in U.

We will assume that the attracting set Λ has *dominated splitting*. It follows from the definition (see (1) in §1.1.1) that the angles between the subbundles E^{cu} and E^{cs} are uniformly bounded away from zero.

Given $0 < a < 1$, we define the *center-unstable cone field* $C_a^{cu} = (C_a^{cu}(x))_{x\in\Lambda}$ of width a by

$$C_a^{cu}(x) = \{v_1 + v_2 \in E_x^{cs} \oplus E_x^{cu} : \|v_1\| \leq a\|v_2\|\}. \tag{4}$$

This cone field is positively invariant under Df. We define the center-stable cone field $C_a^{cs} = (C_a^{cs}(x))_{x\in\Lambda}$ of width a in a similar way, just reversing the roles of the subbundles in (4).

Of course, we may extend continuously the two subbundles to U (it is not necessary for the extension to preserve the invariance of the splitting under Df) and so extend the cone fields to the neighborhood U also (now the positive invariance of C^{cs} is preserved). So the domination property is open in $\mathrm{Diff}^r(M)$ as a consequence of the existence of cone fields. Both the splitting and the cone fields vary continuously with the diffeomorphism in $\mathrm{Diff}^r(M)$.

Given a disk D we denote by dist_D the Riemannian metric induced on D by the Riemannian metric on the manifold M. This is the distance 'inside' the disk D. In the same way, we denote by m_D the Lebesgue measure induced by m on the disk D.

An essential ingredient here is the Hölder control of the Jacobian given by the domination. We say that a function is (C, ν)-Hölder if it is Hölder continuous with constant $C \geq 0$ and exponent $\nu > 0$.

LEMMA 2.1. *There exists $\xi > 0$ such that, given $L > 0$ and any C^2 disk $D \subseteq U$ transverse to the center stable direction E^{cs}, then there exists $C_1 > 0$ such that*

$$x \mapsto \log|\det(Df|T_x f^n(D))|$$

is (C_1, ξ)-Hölder on every domain of diameter L inside any $f^n(D)$, $n \geq 1$.

We refer the reader to the proof in [**3**, §2]. We observe that this constant C_1 depends only on the diffeomorphism f.

2.1.2. *Pesin theory.* Now we assume that $f \in \mathrm{Diff}^r(M)$, $r > 1$, has an attracting set with dominated splitting. Let μ be an f-invariant measure supported on Λ.

Oseledets theorem [**15**] (see also [**14**]) ensures that for μ-almost every point $x \in \Lambda$ there exist unique Lyapunov exponents and Lyapunov subspaces forming a splitting of $T_x M$, depending measurably of x.

Assume that μ has $u = \dim E^{cu}$ positive Lyapunov exponents and denote by $\lambda(x)$ the smallest of them. Then, the direct sum of Lyapunov space associated to these Lyapunov exponents is equal to E_x^{cu}. This follows from the domination condition.

In general, the angles between Lyapunov subspaces are not bounded away from zero, but the angles may decrease at most subexponentially fast. However, by the

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domination condition the angle between E^{cu}_x and the direct sum of complementary Lyapunov subspaces is bounded away from zero.

Pesin theory [**11, 17, 20, 21**] ensures that x has a local unstable manifold $W^u_{\rm loc}(x)$ which is a C^r-embedded disk through x and there exists a constant C_x, depending measurably on x, such that:

(PT1) $W^u_{\rm loc}(x)$ is tangent to E^{cu}_x at x;

(PT2) $\mathrm{dist}_{W^u_{\rm loc}(f^{-n}(x))}(f^{-n}(x_1), f^{-n}(x_2)) \le C_x e^{-n\lambda(x)} \mathrm{dist}_{W^u_{\rm loc}(x)}(x_1, x_2)$, for all $x_1, x_2 \in W^u_{\rm loc}(x)$ and for all $n \ge 1$;

(PT3) $f^{-1}(W^u_{\rm loc}(x)) \subseteq W^u_{\rm loc}(f^{-1}(x))$.

The size of $W^u_{\rm loc}(x)$ also depends measurably on x. From this measurable dependence follows the existence of *hyperbolic blocks* $(\Lambda_n)_{n\in\mathbb{N}}$ (see [**11, 17, 20, 21**]). An hyperbolic block $\Lambda_n \subseteq \Lambda$ is a compact subset of Λ such that:

(HB1) $C_x < n$ and $\lambda_x > 1/n$ for every $x \in \Lambda_n$;

(HB2) $\Lambda_n \subseteq \Lambda_{n+1}$ and $f(\Lambda_n) \subseteq \Lambda_{n+1}$ for all $n \in \mathbb{N}$ and $\mu(\Lambda_n) \to 1$ as $n \to \infty$;

(HB3) the C^r disks $W^u_{\rm loc}(x)$ vary continuously with $x \in \Lambda_n$, in particular the size of $W^u_{\rm loc}(x)$ is bounded away from zero for $x \in \Lambda_n$.

Using the properties above, we can describe the support of every measure with $\dim E^{cu}$ Lyapunov exponents.

Let $u = \dim E^{cu}$ and $s = \dim E^{cs}$. A *cylinder* $\mathcal{C}$ is a diffeomorphic image of $B^u \times B^s$ where B^u and B^s are balls in $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively, and a 'disk' D is a diffeomorphic image of B^u . We say that a C^1-disk D *crosses* $\mathcal{C}$ if it is contained in $\mathcal{C}$ and is a graph over B^u. We also denote by $B(x, \delta)$ the ball around $x \in M$ with radius $\delta > 0$.

LEMMA 2.2. *Let $f \in \mathrm{Diff}^r(M)$ exhibiting an attracting set Λ with dominated splitting. Let μ be an f-invariant measure supported in Λ such that μ-almost every $x \in \Lambda$ has $u = \dim E^{cu}$ positive Lyapunov exponents in the E^{cu}_x direction. For μ-almost every point x and every $\delta > 0$ small enough, there exists a cylinder containing x, $\mathcal{C}(x, \delta) \subseteq B(x, \delta)$ and a family $\mathcal{K}(x, \delta)$ of disjoint unstable disks crossing the cylinder $\mathcal{C}(x, \delta)$ such that their union $K(x, \delta)$ has positive μ-measure.*

Proof. Let μ be a f-invariant measure such that μ-almost every $x \in \Lambda$ has u positive Lyapunov exponents in the E^{cu}_x direction, for instance a Gibbs *cu*-state. For such generic x, there exist an hyperbolic block Λ_n, such that $x \in \Lambda_n$, and a unique C^1-embedded disk $W^u_{\rm loc}(x)$ tangent to E^{cu}_x at x, such that the diameter of $f^n(W^u_{\rm loc}(x))$ converges exponentially fast to zero as $n \to \infty$. The C^1-disk $W^u_{\rm loc}(x)$ depends in a continuous way on the point x in Λ_n. In particular, there exists a uniform lower bound on the size of $W^u_{\rm loc}(x)$ in Λ_n: there is $\delta_n > 0$ such that the pre-image of $W^u_{\rm loc}(x)$ under the exponential map of M at x contains the graph of a C^1 map defined from $B(0, \delta_n) \cap E^{cu}_x$ to E^{cs}_x.

Given any $0 < \delta < \delta_n/4$ and $x \in \Lambda_n$, we can define the tubular neighborhood $\mathcal{C}(x, \delta)$ of $W^u_{\rm loc}(x)$ as the image under the exponential map of M at y of all of the vectors of norm less than $\delta > 0$ in the orthogonal complement of E^{cu}_y in T_yM, for all $y \in W^u_{\rm loc}(x)$. If $\delta > 0$ is small enough then this neighborhood $\mathcal{C}(x, \delta)$ is a cylinder and it comes equipped with the canonical projection π onto $W^u_{\rm loc}(x)$ which is a C^1 map. We denote by $\mathcal{K}(x, \delta)$ the family of local strong-unstable manifolds at points of Λ_n that cross $\mathcal{C}(x, \delta)$ and by $K(x, \delta)$ the union of the disks in the family $\mathcal{K}(x, \delta)$.

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There exist $y_1, \ldots, y_k \in \Lambda_n$ such that $\Lambda_n \subseteq \bigcup_{j=1}^{k} \mathcal{C}(y_j, \delta)$, because Λ_n is compact. We may suppose that each of these cylinders has positive μ-measure, and we obtain a covering ($\mu \mod 0$) of Λ_n. As a consequence, for all $j = 1, \ldots, k$ we have $\mu(\Lambda_n \cap \mathcal{C}(y_j, \delta)) > 0$. On the other hand, for each $z \in \Lambda_n \cap \mathcal{C}(y_j, \delta)$, we have that $W^u_{\delta_n}(z)$ crosses $\mathcal{C}(y_j, \delta)$, because $\delta < \delta_n/4$. Then, for all $j = 1, \ldots, k$,

$$\mu(K(y_j, \delta)) > \mu(\Lambda_n \cap \mathcal{C}(y_j, \delta)) > 0.$$

We consider the set of $x \in \operatorname{supp}\mu$ such that $x \in \Lambda_n$ for some $n \geq 1$. This set has full μ-measure. For $\delta > 0$, there exists $y \in \Lambda_n$ such that $x \in \mathcal{C}(y, \delta)$. Since x is in $\operatorname{supp}\mu$, $\mathcal{C}(y, \delta)$ must have positive μ measure. It is clear that δ can be chosen arbitrary small. To obtain the statement, we write $\mathcal{C}(x, \delta) = \mathcal{C}(y, \delta)$ and $\mathcal{K}(x, \delta) = \mathcal{K}(y, \delta)$. □

Let $\mathcal{W}^u = \{W^u_{\text{loc}}(x)\}$ be the Pesin unstable lamination. Then the associated holonomy maps preserve zero Lebesgue measure sets [**17, 21**]. More precisely, let x be in some hyperbolic block Λ_n, and x_1, x_2 be points of $W^u_{\text{loc}}(x)$. Let Σ_1 and Σ_2 be small smooth disks transverse to $W^u_{\text{loc}}(x)$ at x_1 and x_2. Given any point $y \in \Lambda_n$ close to x the local unstable manifold $W^u_{\text{loc}}(y)$ intersects each cross section Σ_i at exactly one point y_i. This defines the *holonomy map* $H^u : y_1 \to y_2$ which is an homeomorphism between subsets of Σ_1 and Σ_2, respectively. Moreover, H^u is absolutely continuous, that is, it maps zero Lebesgue measure sets to zero Lebesgue measure sets.

Remark 1. Analogous properties hold in the case of negative Lyapunov exponents.

2.1.3. *Conditional measures along the unstable foliation.* Let $x \in \operatorname{supp}\mu$ be fixed and let $C(x, \delta)$, $\mathcal{K}(x, \delta)$ and $K(x, \delta)$ be the sets defined in Lemma 2.2 above. Because $\mu(K(x, \delta)) > 0$, we can define $\mu|K(x, \delta)$, the restriction measure of μ on $K(x, \delta)$, by

$$\mu|K(x, \delta)(B) = \frac{\mu(K(x, \delta) \cap B)}{\mu(K(x, \delta))}$$

for all measurable subsets $B \subseteq K(x, \delta)$.

Let $\mathcal{P}$ be the coarsest partition of $K(x, \delta)$ such that every local unstable manifold $W^u_{\text{loc}}(x)$ is entirely contained in some atom of $\mathcal{P}$. In such case the atoms are embedded manifolds corresponding to local unstable manifolds. Then $\mathcal{P}$ is a measurable partition in the sense of Rokhlin [**22**]: it may be written as the product

$$\mathcal{P} = \bigvee_{n=1}^{\infty} \mathcal{P}_n$$

of an increasing sequence of finite partitions $\mathcal{P}_n$ of $K(x, \delta)$. Moreover, to each atom of $\mathcal{P}$, there corresponds a unique disk D which is an unstable local manifold, so we may identify $\mathcal{P} = \mathcal{K}(x, \delta)$.

Let $\pi_s : K(x, \delta) \to B^s$ be the projection on B^s along the center-unstable leaves. We can induce a measure $\hat{\mu}$ on B^s given by

$$\hat{\mu}(A) = \mu(\pi_s^{-1}(A) \cap K(x, \delta))$$

for measurable $A \subseteq B^s$.

By Rokhlin's disintegration theorem [**22**], there exists a family $\{\mu_D : D \in \mathcal{K}(x, \delta)\}$ of conditional probability measures satisfying:

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(CP1) $\mu_D(D) = 1$ for $\hat{\mu}$-almost every $D \in \mathcal{K}(x, \delta)$;
(CP2) $\mu(E) = \int \mu_D(E)\, d\hat{\mu}(D)$ for every measurable subset E of $K(x, \delta)$.

The family is essentially unique: given another choice $\{\mu'_D : D \in \mathcal{K}(x, \delta)\}$ of measures satisfying (CP1) and (CP2) then $\mu_D = \mu'_D$ for $\hat{\mu}$-almost every $D \in \mathcal{K}(x, \delta)$.

Given $D \in \mathcal{K}(x, \delta)$, let $P_n \in \mathcal{P}$ be the element of the partition $\mathcal{P}_n$ that contains D, then

$$\lim_{n\to\infty} \frac{1}{\nu(P_n)} \int_{P_n} \varphi\, d\mu = \int \varphi\, d\mu_D,$$

for every continuous function $\varphi : M \to \mathbb{R}$.

By definition, conditional measures of a Gibbs *cu*-state are absolutely continuous along the unstable manifolds, that is, for μ-almost every x and every $\delta > 0$, the conditional measures μ_D defined above are absolutely continuous with respect to Lebesgue measure m_D in D, for $\hat{\mu}$-almost every $D \in \mathcal{K}(x, \delta)$.

2.2. *Gibbs cu-states and dominated splitting.* We begin by studying the relations between a Gibbs *cu*-state and its ergodic components.

Let ξ be an invariant measure for f. Let $R(f)$ be the set of *regular points* of f, that is, the set of points in M such that the Birkhoff averages exist and

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x)) = \lim_{n\to-\infty} \frac{1}{n} \sum_{j=0}^{n+1} \varphi(f^j(x))$$

for all $\varphi \in C^0(M; \mathbb{R})$. It is well known that this set has full measure with respect to any f-invariant measure ξ.

Given a point x let us denote by ξ_x the probability measure given by the time average along the orbit of x

$$\int \varphi\, d\xi_x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f^j(x)$$

for every continuous $\varphi : M \to \mathbb{R}$. According to the ergodic decomposition theorem (cf. [**14**]) ξ_x is well defined and ergodic for every x in a set $\Sigma(f) \subseteq M$ that has full measure with respect to any invariant measure. Moreover, for every bounded measurable function $\varphi : M \to \mathbb{R}$ we can write

$$\int \varphi\, d\xi = \iint \varphi\, d\xi_x\, d\xi(x),$$

for every such φ the integral $\int \varphi\, d\xi_x$ coincides with the time average ξ-almost everywhere, and $x \in \operatorname{supp} \xi_x$ for ξ-almost all x.

Let μ be a Gibbs *cu*-state. Given a point in the support of μ and $\delta > 0$, we fix a cylinder $\mathcal{C}$, a family of disks $\mathcal{K}$ crossing $\mathcal{C}$ and the union of such disks K as in Lemma 2.2.

Fix B a measurable subset of M such that

$$m_D(B \cap D) = 0 \quad \text{for every } D \in \mathcal{K},$$

and B is maximal among all measurable sets with this property. Observe that $\mu(B) = 0$, because μ is absolutely continuous along the leaves on $\mathcal{K}$. Let $\tilde{Z} = K \cap \Sigma(f) \cap R(f) \setminus B$. Then $\mu(\tilde{Z}) > 0$ and let $(\mu|\tilde{Z})$ be the restriction of μ to $\tilde{Z}$.

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Let A be any measurable subset of $\tilde{Z}$ such that $m_D(A \cap D) = 0$ for every $D \in \mathcal{K}$. Then $\mu(A)$ must be zero, since we took $\mu(B)$ maximal. This means that $(\mu \mid \tilde{Z})$ is absolutely continuous with respect to the product $m_D \times \hat{\mu}$, where $\hat{\mu}$ stands for the quotient measure induced by $(\mu \mid \tilde{Z})$ on $\mathcal{K}$. As a consequence, the conditional measures $(\mu \mid \tilde{Z})_D$ of $(\mu \mid \tilde{Z})$ on the disks $D \in \mathcal{K}$ are absolutely continuous with respect to Lebesgue measure m_D for $\hat{\mu}$-almost all $D \in \mathcal{K}$.

On the other hand, for any measurable set $A \subseteq \tilde{Z}$,

$$\mu(A) = \int \mu_x(A)\, d\mu(x),$$

where the integral is taken over $\Sigma(f) \subseteq M$.

Let us denote by $\mathbf{1}_A$ the characteristic function of the measurable subset A. Then we have (as already mentioned)

$$\mu_x(A) = \int \mathbf{1}_A\, d\mu_x = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_A(f^j(x))$$

μ-almost everywhere. So $\mu_x(A)$ can be non-zero only if x has some iterate in $A \subseteq \tilde{Z}$, for μ-almost every point x. Let $k(z)$ denote the first backward return time to $\tilde{Z}$ of a point $z \in \tilde{Z}$, this means $k(z)$ is the smallest positive integer such that $f^{-k(z)}(z) \in \tilde{Z}$. This is defined μ-almost everywhere, by Poincare's recurrence theorem. Observing also that $\mu_z = \mu_{f^j(z)}$ for every z and every integer $j \in \mathbb{Z}$, we have

$$\mu(A) = \int_{\Sigma(f)} \mu_x(A)\, d\mu(x) = \int_{\tilde{Z}} k(z)\mu_z(A)\, d\mu(z)$$

for any measurable subset A of $\tilde{Z}$.

LEMMA 2.3. *Let λ be a finite measure on a measure space Z, with $\lambda(Z) > 0$. Let $\mathcal{K}$ be a measurable partition of Z, and $(\lambda_z)_{z\in Z}$ be a family of finite measures on Z such that:*

(1) *the function $z \to \lambda_z(A)$ is measurable, and constant on each element of $\mathcal{K}$, for any measurable set $A \subset Z$;*

(2) *$\{w : \lambda_z = \lambda_w\}$ is a measurable set with full λ_z-measure, for every $z \in Z$.*

Assume that $\lambda(A) = \int l(z)\lambda_z(A)\, d\lambda$ for some measurable function $l : Z \to \mathbb{R}_+$ and any measurable subset A of Z. Let $\{\tilde{\lambda}_\gamma : \gamma \in \mathcal{K}\}$ and $\{\tilde{\lambda}_{z,\gamma} : \gamma \in \mathcal{K}\}$ be the disintegrations of λ and λ_z, respectively, into conditional probability measures along the elements of the partition $\mathcal{K}$. Then $\tilde{\lambda}_{z,\gamma} = \tilde{\lambda}_\gamma$ for λ-almost every $z \in Z$ and $\hat{\lambda}_z$-almost every γ, where $\hat{\lambda}_z$ is the quotient measure induced by λ_z on $\mathcal{K}$.

The reader can see the proof in [**3**, Lemma 6.2].

LEMMA 2.4. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting. If μ is a Gibbs cu-state of f supported in Λ, then every ergodic component of μ is a Gibbs cu-state.*

Proof. If we take $Z = \tilde{Z}$, $\lambda = (\mu|\tilde{Z})$, $\mathcal{K} = \mathcal{K}$, $\lambda_z = (\mu_z|\tilde{Z})$ and $l(z) = k(z)$ for each $z \in \tilde{Z}$ in Lemma 2.3, then the conditional probability measures $(\mu_z|\tilde{Z})_D$ of $(\mu_z|\tilde{Z})$ along the disks $D \in \mathcal{K}$ coincide almost everywhere with the corresponding conditional

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measures μ_D of $(\mu|\tilde{Z})$. Recall that we had already shown that the latter are almost everywhere absolutely continuous with respect to Lebesgue measure on the corresponding disks $D \in \mathcal{K}$. Moreover, the u-largest Lyapunov exponents are positive μ_D almost everywhere. □

Remark 2. If μ is an invariant measure with ergodic decomposition $(\mu_x)_{x\in\Sigma(f)}$, and μ_x is a Gibbs *cu*-state for μ-almost every point x, then μ is also a Gibbs *cu*-state. In fact, μ-almost every point x is in a disk D contained in a local unstable manifold, and so its largest Lyapunov exponents are positive. Since D is completely contained in an unique ergodic component, $\mu_D = \mu_{x,D}$. However, $\mu_{x,D}$ is absolutely continuous respect to Lebesgue measure, so μ_D is also absolutely continuous.

In what follows we assume that μ is an ergodic Gibbs *cu*-state. Given $z \in \operatorname{supp}\mu$ we denote by μ_D the conditional measure of μ in $D = W^u_{\mathrm{loc}}(z)$ and we denote by ρ_z the density of the conditional measure μ_D of μ along the unstable disk D through z. Our next goal is to characterize this density.

LEMMA 2.5. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and let μ be an ergodic Gibbs cu-state of f supported in Λ. For every x and y in the same local unstable manifold, the product*

$$\prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}$$

converges and is bounded away from zero and infinity.

Proof. Let $x, y \in W^u_\varepsilon(z)$ and set $J^u_k(x) = |\det Df^{-1}|E^{cu}_{f^k(x)}|$, $k \geq 0$. Lemma 2.1 implies that the map $x \to \log(J^u_k(x)^{-1})$ is (C_1, ξ)-Hölder. Let $\lambda > 0$ be the smallest Lyapunov exponent for z in the E^{cu}-direction. Then, for $N \geq 1$,

$$\begin{aligned}\left|\log \prod_{k=0}^{N} \frac{J^u_k(x)}{J^u_k(y)}\right| &\leq \sum_{k=0}^{N} |\log J^u_k(x) - \log J^u_k(y)| \\ &\leq \sum_{k=0}^{N} C_1 \operatorname{dist}_{f^k(W^u_\varepsilon(z))}(f^k(x), f^k(y))^\xi \\ &\leq \sum_{k=0}^{N} C_1 C^\xi e^{-k\lambda\xi} \operatorname{dist}_{W^u_\varepsilon(z)}(x, y)^\xi.\end{aligned}$$

The symmetry of the product (we may exchange x and y) and the convergence of the series imply that the product converges for all $x, y \in W^u_\varepsilon(z)$ and is non-zero. Moreover, the convergence is absolute and Hölder with respect to x and y, so the product is bounded away from zero and infinity. □

Remark 3. The convergence of the product depends only on f and on the smallest Lyapunov exponent along the center-unstable disks.

PROPOSITION 2.1. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and let μ be an ergodic Gibbs cu-state of f supported in Λ.*

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For μ-almost every $z \in \operatorname{supp}\mu$ and for every $x,\ y \in W^u_{\mathrm{loc}}(z)$,

$$\frac{\rho_z(x)}{\rho_z(y)} = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}.$$

The densities are Hölder continuous and bounded away from zero and infinity.

Proof. We fix a generic z and $W^u_{\mathrm{loc}}(z) = D$. Since μ_z is absolutely continuous with respect to Lebesgue measure in D, there exists some $\rho : D \to \mathbb{R}$ which is measurable and positive μ_z-almost everywhere such that

$$\mu_z(B) = \int_B \rho\, dm_D$$

for all Borelean subsets $B \subseteq D$. Let ρ_n be the density of $\mu_{f^{-n}(z)}$ on $f^{-n}(D)$. By change of variables we have for $x \in W^u_{\mathrm{loc}}(z)$ that

$$\rho(x) = C\rho_n(f^{-n}(x)) \prod_{k=0}^{n-1} J^u_k(x) \tag{5}$$

for any $n \geq 0$ where $C > 0$ is a constant of normalization depending of z and n. Then, for every $x,\ y \in W^u_{\mathrm{loc}}(z)$,

$$\frac{\rho(x)}{\rho(y)} = \frac{\rho_n(f^{-n}(x))}{\rho_n(f^{-n}(y))} \prod_{k=0}^{n-1} \frac{J^u_k(x)}{J^u_k(y)}.$$

By Lemma 2.5 the right-hand product converges to a non-zero value, so the quotient $\rho_n(f^{-n}(x))/\rho_n(f^{-n}(y))$ also converges. We claim that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ such that

$$\frac{\rho_{n_k}(f^{-n_k}(x))}{\rho_{n_k}(f^{-n_k}(y))} \to 1$$

when $k \to \infty$. This claim completes the proof.

For $\delta > 0$, let $\mathcal{C}(z,\delta)$, $\mathcal{K}(z,\delta)$ and $K(z,\delta)$ be the sets defined in Lemma 2.2. Let $\varepsilon > 0$ be fixed and Λ_ε be a compact subset of $K(z,\delta)$ such that $\mu(K(z,\delta)\setminus\Lambda_\varepsilon) < \varepsilon$ and the map

$$\begin{aligned} \Lambda_\varepsilon \subseteq B^u \times B^s &\to \mathbb{R} \\ (x, w) &\mapsto \rho_w(x) \end{aligned}$$

is continuous. In particular, we may assume the following: given $k \in \mathbb{N}$, there exists $\delta_k > 0$, not depending on $w \in \Lambda_\varepsilon$, such that

$$\left|\frac{\rho_w(x)}{\rho_w(y)} - 1\right| < \frac{1}{k}$$

for all $x, y \in W^u_{\mathrm{loc}}(z) \cap \Lambda_\varepsilon$ satisfying $\operatorname{dist}_{W^u_{\mathrm{loc}}(w)}(x, y) < \delta_k$. As $\mu(\Lambda_\varepsilon) > 0$, Poincare's recurrence theorem together with the properties (PT1) and (PT2) of the unstable Pesin lamination (see §2.1.2) imply that for μ-almost every $w \in \Lambda_\varepsilon$ there exists a n_k large enough such that:

(1) there exists $\tilde{w} \in \Lambda_\varepsilon$ such that $f^{-n_k}(W^u_{\mathrm{loc}}(w)) \subseteq W^u_{\mathrm{loc}}(\tilde{w})$;
(2) $\operatorname{diam} f^{-n_k}(W^u_{\mathrm{loc}}(w)) \leq \delta_k$.

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Since μ is f-invariant, from item (1) it follows that $\rho_{n_k,w} = \rho_{\tilde{w}}$ for $\mu_{f^{-n_k}(w)}$-almost every point in $f^{-n_k}(W^u_{\mathrm{loc}}(w))$. Now item (2) implies that for every $x, y \in W^u_{\mathrm{loc}}(w) \cap \Lambda_\varepsilon$

$$\mathrm{dist}_{W^u_{\mathrm{loc}}(\tilde{w})}(f^{-n_k}(x), f^{-n_k}(y)) = \mathrm{dist}_{f^{-n_k}(W^u_{\mathrm{loc}}(w))}(f^{-n_k}(x), f^{-n_k}(y)) \le \delta_k$$

and so

$$\left|\frac{\rho_{n_k,w}(f^{-n_k}(x))}{\rho_{n_k,w}(f^{-n_k}(y))} - 1\right| = \left|\frac{\rho_{\tilde{w}}(f^{-n_k}(x))}{\rho_{\tilde{w}}(f^{-n_k}(y))} - 1\right| < \frac{1}{k}$$

and this finishes the proof. □

If x has a local strong unstable manifold $W^u_{\mathrm{loc}}(x)$, we define the *global unstable manifold* of x as the set

$$W^u(x) = \bigcup_{n\ge 0} f^n(W^u_{\mathrm{loc}}(f^{-n}(x))).$$

The next result follows from the fact that the densities ρ are bounded away from zero and infinity.

COROLLARY 2.1. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and let μ be a Gibbs cu-state of f supported in Λ. Then the support of μ contains global unstable manifolds whose union has full μ-measure.*

Proof. Let Λ_n be a hyperbolic block. Then for all $x \in \Lambda_n$, there exists $W^u_\delta(x)$ with $\delta > 0$ uniform on x. Moreover $\mu(\Lambda_n) > 1 - \varepsilon_n$, with $\varepsilon_n \to 0$ as $n \to \infty$. It is sufficient to prove that for μ-almost every $x \in \Lambda$, one has $W^u_\delta(x) \subseteq \mathrm{supp}\,\mu$.

For each $x \in \mathrm{supp}\,\mu$, we can construct a cylinder $\mathcal{C}$ that contains $W^u_\delta(x)$ and such that if $z \in \Lambda_n \cap \mathcal{C}$, then $W^u_\delta(z)$ crosses $\mathcal{C}$. Suppose there is $y \in W^u_\delta(x)$ such that $y \notin \mathrm{supp}\,\mu$. Then there exists a small neighborhood $y \in V \subseteq \mathcal{C}$ such that $\mu(V) = 0$. By the disintegration of μ we have

$$\mu(V) = \int \mu_z(V \cap W^u_\delta(z))\, d\hat{\mu}(z),$$

but each μ_z has strictly positive density. Then there exists a neighborhood of x having zero $\hat{\mu}$-measure, which contradicts the fact that x is in the support of μ. □

The next lemma plays an important role in the following sections.

LEMMA 2.6. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and let μ be a Gibbs cu-state of f supported in Λ. For μ-almost every $x \in \Lambda$ and every $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}(x,\delta)$ such that $\hat{\mu}$-almost every disk $D \in \mathcal{K}(x,\delta)$ satisfies $B(\mu) \cap R(f) \cap D$ has full Lebesgue measure in D.*

Proof. We observe that $\mu(B(\mu) \cap R(f)) = 1$. For almost every $x \in B(\mu) \cap R(f) \cap \mathrm{supp}\,\mu$ and every $\delta > 0$ small enough, there exists a cylinder $\mathcal{C}$ and a family $\mathcal{K}$ of disk crossing $\mathcal{C}$ such that the union of those disks has positive μ-measure. Let us consider the probability measure $(\mu|K)$. Then

$$(\mu|K)(B(\mu) \cap R(f) \cap K) = 1.$$

By disintegration, $\hat{\mu}|K$-almost every disk $D \in \mathcal{K}$ satisfies

$$\mu_D(D \cap B(\mu) \cap R(f)) = \mu_D(D).$$

Because μ_D is absolutely continuous with respect to Lebesgue measure and the density ρ_D is bounded from zero and infinity, it follows that $m_D(B(\mu) \cap R(f) \cap D) = m_D(D)$. □

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We can summarize the results of this section in the following theorem.

THEOREM 2.1. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and let μ be a Gibbs cu-state for f. Then:*

(1) *for μ-almost every point x and every $\delta > 0$ small enough, there exists a cylinder containing x, $\mathcal{C}(x,\delta) \subseteq B(x,\delta)$ and a family $\mathcal{K}(x,\delta)$ of disjoint unstable disks crossing the cylinder $\mathcal{C}(x,\delta)$ such that their union $K(x,\delta)$ has positive μ-measure;*

(2) *denoting by ρ_z the density of the conditional measure μ_D along the disk $D = W^u_{\mathrm{loc}}(z)$ through z, then, for μ-almost every $z \in \operatorname{supp}\mu$ and for every $x,\ y \in D$, we have*

$$\frac{\rho_z(x)}{\rho_z(y)} = \prod_{k=0}^{\infty} \frac{\det(Df^{-1}|E^{cu}_{f^{-k}(x)})}{\det(Df^{-1}|E^{cu}_{f^{-k}(y)})}, \tag{6}$$

the densities are Hölder continuous functions bounded away from zero and infinity;

(3) *the support of μ contains global unstable manifolds whose union has full μ-measure;*

(4) *every ergodic component of μ is a Gibbs cu-state.*

3. *Gibbs cu-states and non-uniform expansion*

3.1. *Building Gibbs cu-states.* Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and non-uniform expansion along the E^{cu} direction. Alves *et al* [**3**] proved that under this hypothesis there exist Gibbs *cu*-states

THEOREM 3.1. [**3**, Theorem 6.3] *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and non-uniform expansion along the center-unstable direction, then there exist ergodic Gibbs cu-states supported on Λ.*

The goal of this section is to briefly review the construction of Gibbs *cu*-states.

A disk $D \subset U$ is *tangent to the center-unstable cone field* C^{cu} if the tangent subspace to D at each point $x \in D$ is contained in the corresponding cone $C^{cu}(x)$. We fix a C^2 disk D tangent to the center-unstable cone field such that the following hold.

(1) The set of points in D having non-hyperbolic behavior has full Lebesgue measure in the disk. This is possible because we assume that almost every point in U satisfies (1).

(2) There are fixed $\xi > 0$ and $C_1 > 0$ as in Lemma 2.1 such that the functions J_k defined on $f^k(D) \subset U$ by $J_k(x) = \log|\det Df|T_x f^k(D)|$, for $k = 1, \ldots, n$, are (C_1, ξ)-Hölder. These constants depend only on f.

Definition 4. Given $0 < \sigma < 1$, we say that n is a *σ-hyperbolic time* for a point $x \in U$ if

$$\prod_{j=n-k+1}^{n} \|Df^{-1}|E^{cu}_{f^j(x)}\| \le \sigma^k$$

for all $1 \le k \le n$.

Conditions (1) and (2) above imply that there exist many (positive density at infinity) σ-hyperbolic times for points $x \in D$ satisfying (2) with $\sigma < e^{-c_0/3}$. The rate depends on c_0 and f. This follows from an adapted version of Pliss' lemma [**19**], also proved in [**14**] and [**3**].

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PROPOSITION 3.1. *Given any $x \in \tilde{D}$ and any sufficiently large $N \geq 1$, there exist σ-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq N$ for x with*

$$l \geq \frac{-|\log \sigma|}{\sup |\log \|Df^{-1}|E^{cu}\|| - 2|\log \sigma|} N.$$

Remark 4. Hyperbolic times cannot depend continuously on the diffeomorphism f, but the rate

$$\theta = \frac{-|\log \sigma|}{\sup |\log \|Df^{-1}|E^{cu}\|| - 2|\log \sigma|}$$

depends continuously on f (in the C^1-topology) and c_0.

As a consequence of the existence of σ-hyperbolic times, we obtain backward uniform contraction and bounded distortion properties. More precisely (see [**3**]), we have the following.

PROPOSITION 3.2. *There exist $C_2 > 0$ and $\delta_1 > 0$ such that for all $x \in D$, for all σ-hyperbolic times n and for every $y \in D$ such that $\operatorname{dist}_{f^n(D)}(f^n(x), f^n(y)) \leq \delta_1$, we have*

$$\operatorname{dist}_{f^{n-k}(D)}(f^{n-k}(x), f^{n-k}(y)) \leq \sigma^{k/2} \operatorname{dist}_{f^n(D)}(f^n(x), f^n(y)), \tag{7}$$

$$\frac{1}{C_2} \leq \frac{|\det Df^n|T_y D|}{|\det Df^n|T_x D|} \leq C_2. \tag{8}$$

The constant C_2 in (8) above depends on σ, δ_1 and on the Hölder constant C_1. We remark that (8) is similar to the quotient factor in Proposition 2.1.

For each $j \geq 1$, let $\hat{H}_j$ be a finite set of $x \in D$ such that j is a σ-hyperbolic time for x. For $\delta = \delta_1/4$, we denote by $\Delta_j(x, \delta)$ the δ-neighborhood of $f^j(x)$ inside $f^j(D)$. We choose $\hat{H}_j$ such that the balls $\Delta_j(x, \delta)$ are pairwise disjoint. We denote by Δ_j the union of such balls.

We can choose $\hat{H}_j$ satisfying the following (see [**3**, Proposition 3.3 and Lemma 3.4]): there exists a constant $\tau > 0$, depending only on f, such that for any j

$$f_*^j m_D(\Delta_j \cap f^j(U)) \geq f_*^j m_D(\Delta_j \cap f^j(\hat{H}_j)) \geq \tau m_D(\hat{H}_j).$$

Consider the set of accumulation points of $(\Delta_j)_j$:

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j \geq n} \Delta_j}.$$

Observe that $\Delta_j \subseteq f^j(D) \subseteq f^j(U)$. Then, since U is positively invariant,

$$\overline{\bigcup_{j \geq n} \Delta_j} \subseteq \overline{f^n(U)} \subseteq f^{n-1}(U)$$

and so $\Delta_\infty \subseteq \Lambda$.

Given $y \in \Delta_\infty$ there exist a sequence $(j_i)_i \to \infty$, disks $D_i = \Delta(x_i, \delta) \subseteq \Delta_{j_i}$ and points $y_i \in D_i$, $y_i \to y$ as $i \to \infty$. By passing to a subsequence if necessary, we may suppose that the centers x_i converge to some point x and, by the Arzela–Ascoli theorem,

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that the D_i converge to a disk $D(x)$ of radius δ around x. Then y is in the closure $\overline{D(x)}$ of $D(x)$, and $\overline{D(x)} \subseteq \Delta_\infty$ and the points x are in the set

$$\hat{H}_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j \geq n} f^j(\hat{H}_j)}.$$

Observe that D_i is contained in the j_i-iterate of D, which was taken tangent to the center-unstable cone field. So the domination property implies that the angle between D_i and E^{cu} goes to zero as $i \to \infty$. By Proposition 3.2, given $k \geq 1$, f^{-k} is a $\sigma^{k/2}$-contraction on D_i for every large i. Passing to the limit when $i \to \infty$, we get that every f^{-k} is a $\sigma^{k/2}$-contraction on $D(x)$, and that $D(x)$ is tangent to the center-unstable subbundle at every point of $D(x) \subseteq \Lambda$, including x.

In particular we have shown that the subspace E^{cu}_x is indeed uniformly expanding for Df. The domination property means that any expansion that Df exhibits along the complementary direction is weaker than this. Then (see [**17**]), there exists a unique strong-unstable manifold $W^u_{\text{loc}}(x)$ tangent to E^{cu} which is contracted by negative iterates of f at a rate of at least $\sigma^{k/2}$, when k becomes large. Moreover, $D(x)$ is contained in $W^u(x)$ because it is contracted by every f^{-k}, $k \geq 1$, and all its negative iterates are tangent to the center-unstable cone field. Summing up, we have the following.

PROPOSITION 3.3. *The family of disks $D(x)$, with $x \in \hat{H}_\infty$, constructed as above satisfies:*

(1) *the radius of $D(x)$ is $\delta_1/4$ uniformly in $x \in \hat{H}_\infty$;*
(2) *for every $y \in \Delta_\infty$ there exists $x \in \hat{H}_\infty$ such that $y \in \overline{D(x)}$;*
(3) *for all $x \in \hat{H}_\infty$, the subspace E^{cu}_x satisfies*

$$\|Df^{-k}|E^{cu}_x\| \leq \sigma^{k/2} \quad \text{for all } k \geq 0;$$

(4) *$D(x)$ is contained in the corresponding strong-unstable manifold $W^u_{\text{loc}}(x)$;*
(5) *$D(x)$ is tangent to the center-unstable subbundle at every point of $\Lambda \cap D(x)$.*

We now consider the sequence of averages of push-forwards of Lebesgue measure restricted to such a disk D:

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* \, m_D.$$

Remark 5. The argument that follows does not change if we consider φm_D instead of m_D, where φ is a measurable function bounded away from zero and infinity, m_D-almost everywhere.

We decompose μ_n as a sum of two measures ν_n and η_n, where

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f^j_* \, m_D|\Delta_j$$

and $\eta_n = \mu_n - \nu_n$. Observe that the support of ν_n is $\bigcup_{j=0}^{n-1} \Delta_j$.

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Now, we consider any subsequence $(n_k)_k$ such that μ_{n_k} and ν_{n_k} converge to μ and ν respectively. Then the support of ν is contained in the set

$$\Delta_\infty = \bigcap_{n=1}^{\infty} \overline{\bigcup_{j\geq n} \Delta_j} \subseteq \Lambda$$

of accumulation points of $(\Delta_j)_j$. Proposition 3.3 gives a characterization of the support of ν. Moreover [**3**, Proposition 3.5 and Remark 3.6], there is $\alpha_1 = \alpha_1(c_0, f) > 0$ such that, for all $n \geq 1$ and $k \geq n$ large enough,

$$\nu_k(f^n(U)) \geq \nu_k(f^k(U)) \geq \alpha_1.$$

This is because $\overline{f^k(U)} \subseteq f^n(U)$. Then, $\nu(f^n(U)) \geq \alpha_1$ and so

$$\nu(\Lambda) = \nu\left(\bigcap_{n\geq 1} f^n(U)\right) = \liminf_{n\to\infty} \nu(f^n(U)) \geq \alpha_1.$$

Recall from Proposition 3.3 that, given any $y \in \Delta_\infty$, there exist a point $x \in \hat{H}_\infty$ and a disk $D(x)$ of size $\delta_1/4$ around x such that $y \in \overline{D(x)} \subseteq \Delta_\infty$. For any such x and $r > 0$ small, let $C_r(x)$ be the tubular neighborhood of $\overline{D(x)}$, defined as the union of the images under the exponential map at each point $z \in \overline{D(x)}$ of all vectors orthogonal to $\overline{D(x)}$ at z with norm less than or equal to r. We take r to be sufficiently small, so that $C_r(x)$ is a cylinder endowed with the canonical projection $\pi : C_r(x) \to \overline{D(x)}$. We may suppose that the boundary of $C_r(x)$ has zero ν-measure (observe that r depends on the size of the domain of the exponential map, and so depends continuously on f).

For any $\varepsilon > 0$, we can fix a cover of $\overline{D(x)}$ by finitely many domains $D_{x,l} \subseteq \overline{D(x)}$, $l = 1, \ldots, N(\varepsilon)$, small enough so that the intersection of each $C_{x,l} = \pi^{-1}(D_{x,l})$ with any smooth disk γ tangent to the center-unstable cone field has diameter less than ε inside γ. We choose the cover with the least possible $N(\varepsilon)$ and take the $D_{x,l}$ diffeomorphic to the compact ball B^u, so that every $C_{x,l}$ is a cylinder.

We say that a disk γ crosses $C_{x,l}$ if π maps $\gamma \cap C_{x,l}$ diffeomorphically onto $D_{x,l}$. For each $j \geq 0$, let $K_j(x, l)$ be the union of the intersections of $C_{x,l}$ with all the disks in Δ_j that cross $C_{x,l}$ and let $K_\infty(x, l)$ be the union of the intersections of $C_{x,l}$ with all the disks in Δ_∞ that cross $C_{x,l}$. Fixing a small enough ε for at least one of the cylinders $C_{x,l}$ the part of the measure ν that is carried by the disks in $K_\infty(x, l)$ has positive mass $\alpha > 0$, depending on the rate of hyperbolic times, and so depending on c_0 and f (see [**3**, Lemmas 4.2 and 4.3]).

In the following we write $\mathcal{C} = C_{x,l}$, $\tilde{D} = D_{x,l}$ and $\mathcal{K}_j$, $0 \leq j \leq \infty$, is the family of disks whose union is $K_j = K_j(x, l)$.

Observe that from the construction of μ_n the measure $f_*^j m_D|\Delta_j$ is absolutely continuous with respect to Lebesgue measure along $f^j(D)$. Moreover, from Proposition 3.2(8) the density of the normalization of this measure is uniformly bounded from below and from above. The construction preserves this property for ν_n and ν.

Let us introduce $\hat{K} = \bigcup_{0\leq j\leq\infty} K_j \times \{j\}$. In this space, we consider the sequence of finite measures $\hat{\nu}_n$ defined by

$$\hat{\nu}_n(B_0 \times \{0\} \cap \cdots \cap B_{n-1} \times \{n-1\}) = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j m_D(B_j),$$

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and $\hat{\nu}_n(B_n) = 0$ whenever B is in $\bigcup_{n\le j\le\infty} K_j \times \{j\}$. We also consider a sequence of partitions $\mathcal{P}_k$ in $\hat{K}$ constructed as follows. Fix an arbitrary point $z \in \tilde{D}$ and let V be the inverse image $\pi^{-1}(z)$ under the canonical projection. Fix a sequence $\mathcal{V}_k$, $k \ge 1$, of increasing partitions of V with diameter going to zero. Then, by definition, two points $(x, m), (y, n) \in \hat{K}$ are in the same atom of the partition $\mathcal{P}_k$ if:

- the disk in Δ_m containing x and the disk in Δ_n containing y intersect some common element of $\mathcal{V}_k$;
- either $m \ge k$ and $n \ge k$, or $m = n < k$.

It is clear from the construction that for any point $\xi \in K_j$ and every $0 \le j \le \infty$, one has

$$\mathcal{P}_1(\xi) \supset \cdots \supset \mathcal{P}_k(\xi) \supset \cdots,$$

and $\bigcap_{k=1}^{\infty} \mathcal{P}_k(\xi)$ coincides with the intersection of the cylinder $\mathcal{C}$ with the disk in Δ_j that contains ξ. We define $\hat{\pi} : \hat{K} \to \tilde{D}$ by $\hat{\pi}(x, j) = \pi(x)$.

Clearly, any weak* accumulation measure of the sequence $\hat{\nu}_n$ must be supported in $K_\infty \times \{\infty\}$. We have chosen a sequence $(n_k)_k$ such that ν_{n_k} converges to the measure ν. It is easy to see that this is just the same as saying that $\hat{\nu}_k$ converges to the measure $\hat{\nu}$ defined by $\hat{\nu}(B \times \{\infty\}) = \nu(B)$ for any Borel set $B \subseteq \mathcal{C}$, so ν and $\hat{\nu}$ are naturally identified.

PROPOSITION 3.4. *There exist $C_3 > 1$, depending on f only, and a family of conditional measures $(\nu_\gamma)_\gamma$ of $\nu|\mathcal{K}_\infty$ along the disks $\gamma \in \mathcal{K}_\infty$ such that ν_γ is absolutely continuous with respect to the Lebesgue measure m_γ on γ, with*

$$\frac{1}{C_3} m_\gamma(B) \le \nu_\gamma(B) \le C_3 m_\gamma(B) \tag{9}$$

for any Borel set $B \subseteq \gamma$.

The reader can see the proof in [**3**, §4]. The constant C_3 depends on the Lebesgue measure along the disks in the cylinder (so depends on f) and depends on the distortion bound C_2 obtained in Proposition 3.2.

The construction of Gibbs *cu*-states concludes as follows: there exists an ergodic component μ_z of μ having positive measure on $\mathcal{K}_\infty$ which is absolutely continuous along the disks.

Each disk $D \in \mathcal{K}_\infty$ is completely contained in some ergodic component because it is contained in some local-unstable manifold. In particular, all Lyapunov exponents of μ_z in the center unstable direction are larger than $-\log\sigma > 0$. The domination condition implies that all of the other exponents are less than $-\log\sigma + \log\lambda < -\log\sigma$. Again, by Pesin theory, μ_z-almost every point has a local strong-unstable manifold which is an embedded disk whose backward orbits contract at the exponential rate $\log\sigma$. Moreover the disks $D \in \mathcal{K}_\infty$ contain the local strong-unstable manifolds of points in its interior.

Summing up this section, we have the following.

THEOREM 3.2. (Alves *et al* [**3**]) *Any diffeomorphism f with a dominated splitting which is non-uniformly expanding along the center unstable direction has an ergodic Gibbs cu-state. More precisely, there exist a cylinder $\mathcal{C} \subseteq M$ and a family $\mathcal{K}_\infty$ of disjoint disks contained in $\mathcal{C}$ which are graphs over B^u, and a ergodic invariant probability measure μ supported on Λ such that:*

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(1) *the cylinder contains a ball whose radius is uniformly bounded away from zero, depending continuously on the diffeomorphism f;*
(2) *there exists $\alpha > 0$ such that the union of all disks in K_∞ has μ-measure larger than α, depending on f and c_0;*
(3) *μ has absolutely continuous conditional measures along the disk in $\mathcal{K}_\infty$. The densities of the conditional measures are bounded away from zero and infinity by a constant depending on f and c_0;*
(4) *the $u = \dim E^{cu}$ largest Lyapunov exponents are larger than $-\log\sigma > 0$.*

3.2. *Continuity of Gibbs cu-states.* Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and non-uniform expansion along the center-unstable direction. By Theorem 3.1 we know that there exists a Gibbs *cu*-states. We saw in §3.1 that the proof consist of constructing a Gibbs *cu*-state. The next theorem shows that such a construction in fact provides us with *all* Gibbs *cu*-states existent for f.

We denote by $\mathcal{G}(f)$ the class of Gibbs *cu*-states for f constructed in Theorem 3.1.

THEOREM 3.3. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and non-uniformly expanding along the center-unstable direction, then every ergodic Gibbs cu-state supported in Λ is in $\mathcal{G}(f)$.*

Proof. Let μ be an ergodic Gibbs *cu*-state for f supported on Λ. Let also $D \subseteq W^u_{\mathrm{loc}}(x)$ be in the support of μ, such that $D \cap B(\mu)$ has full μ_D-measure in D (cf. Lemma 2.6). We may assume that D satisfies condition (2) in §3.1 taking an iterate of D if necessary [**3**, Corollary 2.4, Proposition 2.9]. Condition (1) is satisfied by ergodicity: consider the function $\varphi(x) = \log \|Df^{-1}|E^{cu}_x\|$. Birkhoff's ergodic theorem implies that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df^{-1}|E^{cu}_{f^j(x)}\| = \int \log \|Df^{-1}|E^{cu}_y\|\, d\mu(y) \le -c_0 < 0$$

for μ_D-almost every point in D, where c_0 depends on the Lyapunov exponents of D in the E^{cu} direction. However, $\mu_D = \rho_D m_D$ where ρ_D is a measurable function bounded away from zero and infinity, so the claim above holds Lebesgue-almost everywhere in D.

Let $\tilde{\mu}$ be a ergodic Gibbs *cu*-state obtained as a weak* accumulation measure of

$$\frac{1}{n}\sum_{j=0}^{n-1} f^j_*\left(\frac{\mu_D}{\mu_D(D)}\right).$$

Of course, $\tilde{\mu} \in \mathcal{G}(f)$ because μ_D is absolutely continuous with respect to Lebesgue measure on D.

Observe that for every continuous $\varphi : M \to \mathbb{R}$ we have

$$\frac{1}{n}\sum_{j=0}^{n-1} f^j_*\left(\frac{\mu_D(\varphi)}{\mu_D(D)}\right) = \frac{1}{\mu_D(D)}\int_D \frac{1}{n}\sum_{j=0}^{n-1} \varphi\circ f^j(x)\, d\mu_D. \tag{10}$$

Denote by F_n the average $(1/n)\sum_{j=0}^{n-1}\varphi\circ f^j$. Each F_n is μ_D-integrable and bounded by $\|\varphi\|$. Also F_n converges pointwisely to $\int \varphi\, d\mu$ because $B(\mu)$ has full μ_D-measure on

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the disk D. The dominated convergence theorem implies that the right-hand side of (10) converges to $\int \varphi \, d\mu$.

On the other hand, the left-hand side of (10) was assumed to have an accumulation measure $\tilde{\mu}$, so it converges to $\int \varphi \, d\tilde{\mu}$. As a consequence $\mu = \tilde{\mu}$. □

Let $f \in \mathrm{Diff}^r(M)$, $r > 1$. Let $U \subseteq M$ be a trapping region for f and let Λ be an attracting set with dominated splitting and robustly non-uniformly expanding along the center-unstable direction. If $g \in \mathrm{Diff}^r(M)$ is close to f, then U is a trapping region for g as well and we denote by $\Lambda(g)$ the attracting set for g in U. The dominated splitting and non-uniformly expanding along the center-unstable direction are properties that hold in a neighborhood of f, so the attracting set $\Lambda(g)$ has dominated splitting and it is also uniformly expanding along the center-unstable direction. Then, by Theorem 3.1, there exists a Gibbs *cu*-state for g, μ_g supported in $\Lambda(g)$.

The next theorem is the key to the proof of the main results in this work.

THEOREM 3.4. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, exhibiting an attracting set Λ with dominated splitting and robustly non-uniformly expanding along the center-unstable direction. Let $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathrm{Diff}^r(M)$ be a sequence converging in the C^r-topology to f. For $n \in \mathbb{N}$, denote by $\Lambda_n = \Lambda(f_n)$ and let μ_n be a Gibbs cu-state for f_n supported on Λ_n. If $(\mu_n)_{n\in\mathbb{N}}$ converge to μ^* in the weak* topology, then μ^* is a Gibbs cu-state for f.*

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of diffeomorphisms converging to f in the C^r-topology, $r > 1$. For $n \in \mathbb{N}$ large enough, we can assume that each f_n, Λ_n exhibits a dominated splitting with non-uniform expansion along the $E^{cu}(f_n)$ direction with constants C, α and c_0 not depending on $n \in \mathbb{N}$. We are assuming that μ_n, $n \in \mathbb{N}$, is a Gibbs *cu*-state supported in Λ_n and the sequence $(\mu_n)_{n\in\mathbb{N}}$ tends to a probability measure μ^* in the weak* topology.

It is clear that μ^* is f-invariant. By Theorem 3.3, each μ_n is a Gibbs *cu*-state in $\mathcal{G}(f)$. Then, for each $n \geq 1$, there exist $(\mathcal{C}_n)_n$ and $(\mathcal{K}^n_\infty)_n$ cylinders and families of disks associated to (f_n, μ_n). From §3.1 we may assume that:

(a) the size of the disks is uniformly bounded from below;

(b) there exists $\alpha > 0$ such that, for all $n \geq 0$, we have

$$\mu_n(K^n_\infty) \geq \alpha > 0, \tag{11}$$

where K^n_∞ is the union of the disks in $\mathcal{K}^n_\infty$, since we are assuming that c_0 is uniform (cf. Proposition 3.1);

(c) there exists $C_3 > 1$ such that for all $n \geq 1$ the family of conditional measures $(\mu_{n,D})_{D\in\mathcal{K}^n_\infty}$ of $\mu_n | K^n_\infty$ along the disks $D \in \mathcal{K}^n_\infty$ satisfies

$$\frac{1}{C_3} m_D(B) \leq \mu_{n,D}(B) \leq C_3 m_D(B) \tag{12}$$

for any Borel set $B \subseteq D$.

We prove that μ^* is a Gibbs *cu*-state by completing the following steps.

(1) We construct a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u such that all of the disks in $\mathcal{K}^*_\infty$ are local uniformly expanding manifolds under f.

(2) The union K^*_∞ of all disks in $\mathcal{K}^*_\infty$ has positive μ^*-measure.

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(3) The restriction of μ^* to that union has absolutely continuous conditional measures along the disks in $\mathcal{K}^*_\infty$.
(4) Almost every ergodic component of μ^* is a Gibbs *cu*-state.

Of course, by Remark 2, μ^* must be a Gibbs *cu*-state, because almost all of its ergodic components are Gibbs *cu*-states. □

We prove these steps in the following lemmas.

LEMMA 3.1. *There exist a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u such that all disks in $\mathcal{K}^*_\infty$ are local unstable manifolds.*

Proof. Let $(\mathcal{C}_n)_n$ and $(\mathcal{K}^n_\infty)_n$ be sequences of cylinders and families of disks associated to μ_n, respectively. By the compactness of M and considering a subsequence if necessary, we may suppose that $\mathcal{C}_n$ converges to $\mathcal{C}^*$.

We claim that $\mathcal{C}^*$ is a cylinder. Indeed, the $\mathcal{C}_n$ are diffeomorphic images of $B^u \times B^s$ where B^u and B^s are compact balls in $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively, corresponding to $\mathcal{C}_n$, $n \geq 1$. Let $(B^u_n)_n$ and $(B^s_n)_n$ be the diffeomorphic images of B^u and B^s in M, respectively. By the Arzela–Ascoli theorem $(B^u_n)_n$ converges to a disk B^u_* and $(B^s_n)_n$ converges to a disk B^s_*. So, for $\mathcal{C}^*$ to be a cylinder, it must satisfy:

(i) the diameters of B^u_n and B^s_n do not go to zero, when n tends to infinity;
(ii) the angle between B^u_n and B^s_n does not go to zero, when n tends to infinity.

On the one hand, by construction each $\mathcal{C}_n$ contains balls with radius uniformly bounded away from zero from Theorem 3.2, so (i) is fulfilled. On the other hand, by the domination property of the family (f_n), (ii) must hold.

Now, we consider the family $\mathcal{K}^*_\infty$ of disks D^u contained in $\mathcal{C}^*$ which are accumulated by sequences $(D^u_n)_n$ of disks, $D^u_n \in \mathcal{K}^n_\infty$, $n \geq 1$. Observe that every disk $D^u_n \in \mathcal{K}^n_\infty$ is tangent to the center-unstable cone field of f_n; by continuity of the splitting with respect to the diffeomorphism, $D^u \in \mathcal{K}^*_\infty$ must be tangent to the center-unstable cone field of f. For any $x, y \in D^u$ let $(x_n)_n$ and $(y_n)_n$ be two sequences of points in D^u_n converging to x and y, respectively. By Proposition 3.3, for all $k \geq 0$ fixed we have

$$\operatorname{dist}(f_n^{-k}(x_n), f_n^{-k}(y_n)) \leq \sigma^{-k/2} \operatorname{dist}(x_n, y_n).$$

Passing to the limit when $n \to \infty$, we obtain

$$\operatorname{dist}(f^{-k}(x), f^{-k}(y)) \leq \sigma^{-k/2} \operatorname{dist}(x, y),$$

for all $x, y \in D^u$ and all $k \geq 0$. We conclude that every f^k is an $\sigma^{k/2}$-contraction on $D(x)$, and $D(x)$ is tangent to the center-unstable subbundle at every point in $\Lambda \cap D(x)$ (including x).

In particular, we have shown that the subspace E^{cu}_x is indeed uniformly expanding for Df. The domination property means that any expansion Df may exhibit along the complementary direction is weaker than this. Then, there exists a unique strong-unstable manifold $W^u_{\text{loc}}(x)$ tangent to E^{cu} which is contracted by negative iterates of f by a rate of at least $\sigma^{k/2}$, when k becomes large, see [**17**]. Moreover, $D(x)$ is contained in $W^u(x)$ because it is contracted by every f^{-k}, $k \geq 1$, and all of its negative iterates are tangent to the center-unstable cone field. □

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LEMMA 3.2. *The union K^*_∞ of all disks in $\mathcal{K}^*_\infty$ has positive μ^*-measure.*

Proof. Recall from §3.1 that there exists $\alpha > 0$ such that, for all $n \geq 0$, we have $\mu_n(K^n_\infty) \geq \alpha > 0$. Let us fix $\delta > 0$. Then there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $K^n_\infty \subseteq B(K^*_\infty, \delta)$. On the other hand, we know that $K^*_\infty = \bigcap_{\delta>0} B(K^*_\infty, \delta)$. Choosing $\delta > 0$ such that $\mu^*(\partial B(K^*_\infty, \delta)) = 0$, we have

$$\mu^*(B(K^*_\infty, \delta)) = \lim_{n\to\infty} \mu_n(B(K^*_\infty, \delta)) \geq \alpha > 0,$$

and so

$$\mu^*(K^*_\infty) = \liminf_{\delta\to 0} \mu^*(B(K^*_\infty, \delta)) \geq \alpha > 0. \qquad \square$$

LEMMA 3.3. *There exist a constant $C_1 > 0$ and a family of conditional measures $(\mu^*_D)_D$ of $\mu^*|K^*_\infty$ along the disks $D \in \mathcal{K}^*_\infty$ such that μ^*_D is absolutely continuous with respect to Lebesgue measure m_D on D, with*

$$\frac{1}{C_3} m_D(B) \leq \mu^*_D(B) \leq C_3 m_D(B)$$

for every Borel set $B \subseteq D$.

Proof. We start by taking the cylinder $\mathcal{C}^*$ from the construction proving Lemma 3.1. By the compactness of $\mathcal{C}^*$, for any $\xi \in D$ where D is any disk in $\mathcal{K}^*_\infty$, the exponential map is well defined in a ball of radius $\tilde{r} > 0$ around ξ. For any Borel set $B \subset D$ we define the set $\tilde{B}$ as the tubular neighborhood of B, that is, the union of the images under the exponential map at each point $\xi \in B$ of all vectors orthogonal to D at ξ.

We fix a sequence of partitions $\mathcal{P}_k$ of $\mathcal{K}^*_\infty$ constructed as follows. Let V be the image of $\{0\} \times B^s_*$ under the diffeomorphism between $B^u_* \times B^s_*$ and $\mathcal{C}^*$. Fix a sequence $\mathcal{V}_k$, $k \geq 1$, of increasing partitions of V with positive diameter less than $\tilde{r}$ and going to zero. Then, we say that two points $x, y \in \hat{K}$ are in the same atom of the partition $\mathcal{P}_k$ if the disk D_1 containing x and the disk D_2 containing y intersect the same element of $\mathcal{V}_k$. It is clear from the construction that for any point $\xi \in K^*_\infty$,

$$\mathcal{P}_1(\xi) \supset \cdots \supset \mathcal{P}_k(\xi) \supset \cdots$$

and $\bigcap_{k=1}^\infty \mathcal{P}_k(\xi)$ coincides with the disk in D that contains ξ.

For any Borelean set $B \subseteq D$ we have, from Proposition 3.4,

$$\frac{1}{C_3} m(B_n)\mu_n(\mathcal{P}_k(\xi)) \leq \mu_n(\tilde{B} \cap \mathcal{P}_k(\xi)) \leq C_3 m(B_n)\mu_n(\mathcal{P}_k(\xi))$$

where C_3 does not depend on n, $B_n = \tilde{B} \cap D_n$ and D_n is a disk in $\mathcal{K}^n_\infty$ near D, $n \geq 1$. By construction $m(B_n)$ converges to $m(B)$ and so passing to the limit when $n \to \infty$ we have

$$\frac{1}{C_3} m(B)\mu^*(\mathcal{P}_k(\xi)) \leq \mu^*(\tilde{B} \cap \mathcal{P}_k(\xi)) \leq C_3 m(B)\mu^*(\mathcal{P}_k(\xi)).$$

Now, by the Radon–Nikodym theorem, we have that the disintegration of μ^* along the disk $\bigcap_{k=1}^\infty \mathcal{P}_k(\xi)$ is absolutely continuous with respect to Lebesgue measure in this disk, and the densities are almost everywhere bounded from above by C_3 and from below by $1/C_3$. $\square$

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Remark 6. The densities of μ_D^* are uniformly (with respect to D) bounded away from zero and infinity.

Fix B, a measurable subset of M, such that

$$m_\gamma(B \cap \gamma) = 0 \quad \text{for every } \gamma \in \mathcal{K}_\infty^*,$$

and $\mu^*(B)$ is maximal among all measurable sets with this property. Observe that $\mu^*(B) = 0$, because μ^* is absolutely continuous along the leaves on $\mathcal{K}_\infty^*$ (cf. Lemma 3.3). Let $Z_\infty = K_\infty^* \cap \Sigma(f) \cap R(f) \setminus B$. Then $\mu^*(Z_\infty) > 0$ and let $(\mu^*|Z_\infty)$ be the restriction of μ^* to Z_∞.

Let A be any measurable subset of Z_∞ such that $m_\gamma(A \cap \gamma) = 0$ for every $\gamma \in \mathcal{K}_\infty^*$. Then $\mu^*(A)$ must be zero, since we took $\mu^*(B)$ maximal. This means that $(\mu^*|Z_\infty)$ is absolutely continuous with respect to the product $m_\gamma \times \hat{\mu}^*$, where $\hat{\mu}^*$ stands for the quotient measure induced by $(\mu^*|Z_\infty)$ on $\mathcal{K}_\infty^*$. As a consequence, the conditional measures $\tilde{\mu}_\gamma^*$ of $(\mu^*|Z_\infty)$ on the disks $\gamma \in \mathcal{K}_\infty^*$ are absolutely continuous with respect to Lebesgue measure m_γ for $\hat{\mu}^*$-almost all $\gamma \in \mathcal{K}_\infty^*$. On the other hand, by the ergodic decomposition theorem (cf. [**14**]), for any measurable set $A \subseteq Z_\infty$,

$$\mu^*(A) = \int \mu_x^*(A)\, d\mu^*(x),$$

where the integral is taken over $\Sigma(f) \subseteq M$.

Let us denote by $\mathbf{1}_A$ the characteristic function of the measurable subset A. Then we have (as in §2.2)

$$\mu_x^*(A) = \int \mathbf{1}_A\, d\mu_x^* = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \mathbf{1}_A(f^j(x))$$

μ^*-almost everywhere. So $\mu_x^*(A) > 0$, and only if x has some iterate in $A \subseteq Z_\infty$, for μ^*-almost every point x. Let $k(z)$ denote the first backward return time to Z_∞ of a point $z \in Z_\infty$: $k(z)$ is the smallest positive integer such that $f^{-k(z)}(z) \in Z_\infty$. This is defined μ^*-almost everywhere, by Poincare's recurrence theorem. As in §2.2, we have

$$\mu^*(A) = \int \mu_x^*(A)\, d\mu^*(x) = \int_{Z_\infty} k(z)\mu_z^*(A)\, d\mu^*(z)$$

for any measurable subset A of Z_∞.

LEMMA 3.4. *The measure μ_x^* is a Gibbs cu-state, for μ^*-almost every point $x \in K_\infty^*$.*

Proof. Let $x \in K_\infty^* \cap \Sigma \cap R(f)$. Observe that $K_\infty^* \cap \Sigma \cap R(f)$ has full μ^*-measure on K_∞^* and that

$$\lim_{n\to\infty} \frac{1}{n} \log \|Df^{-n}|E^{cu}_{f^n(x)}\| \le -\frac{1}{2} \log \sigma < 0$$

by the uniform contraction of E_x^{cu} for Df^{-1} (see the proof of Lemma 3.1). This implies that the u largest Lyapunov exponents are positive.

Now, if we take $Z = Z_\infty$, $\lambda = (\mu^*|Z_\infty)$, $\mathcal{K} = \mathcal{K}_\infty^*$, $\lambda_z = (\mu_z^*|Z_\infty)$ and $l(z) = k(z)$ for each $z \in Z_\infty$ in Lemma 2.3, the conditional probability measures $\tilde{\mu}_{z,D}^*$ of $(\mu_z^*|Z_\infty)$ along the disks $D \in \mathcal{K}_\infty^*$ coincide almost everywhere with the corresponding conditional

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measures $\tilde{\mu}_D^*$ of $(\mu^*|Z_\infty)$. Recall that we had already shown that the latter are almost everywhere absolutely continuous with respect to Lebesgue measure on the corresponding disks $D \in \mathcal{K}_\infty^*$. □

Now, let K_0 be the set $\bigcup_{j=0}^{\infty} f^{-j}(K_\infty^*)$ and define G to be the set of all $x \in \Sigma(f) \cap R(f) \cap K_0$ such that μ_x^* is a Gibbs *cu*-state and set

$$\nu^* = \int_G \mu_x^* \, d\mu^*(x).$$

Since μ_x^* is ergodic for μ^*-almost every x and for all $j \in \mathbb{N}$, then $\mu_x^*(G) = 1$ if and only if, $x \in G$, so $\operatorname{supp} \nu^* = \overline{G}$ (see [**28**, Theorem 1.5]). Moreover, μ_x^* is a Gibbs *cu*-state for all $x \in G$, and so is $\nu^*/\nu^*(M)$, see Remark 2 (note that since $K_\infty^* \subseteq G$ we have $\nu^*(G) \geq \mu^*(K_\infty^*) \geq \alpha > 0$).

For $n \in \mathbb{N}$, each μ_n is ergodic and $\mu_n(K_\infty^n) \geq \alpha > 0$, so (see [**28**, Theorem 1.5])

$$\mu_n\left(\bigcup_{j=0}^{\infty} f_n^{-j}(K_\infty^n)\right) = 1. \tag{13}$$

Set $A_n^j = \{x \in K_\infty^n : j \text{ is the first return time of } x\}$. These sets are pairwise disjoint and $K_\infty^n = \bigcup_{j\in\mathbb{N}} A_n^j$ up to a zero μ_n-measure subset. So,

$$\mu_n(K_\infty^n) = \sum_{j=1}^{\infty} \mu_n(A_n^j).$$

LEMMA 3.5. *The series $\sum_{j=1}^{\infty} \mu_n(A_n^j)$ converge uniformly with respect to $n \in \mathbb{N}$.*

Proof. Let us fix $j \in \mathbb{N}$. By invariance of μ_n and disintegration, we have

$$\mu_n(A_n^j) = \mu_n(f_n^j(A_n^j)) = \int_{K_\infty^n} \int_D \mathbf{1}_{f_n^j(A_n^j)}(x) \rho_D^n(x) \, dm_D(x) \, d\hat{\mu}_n(D).$$

From Proposition 3.4, there is a uniform (with respect to n) constant $C_3 \geq 0$ such that, for $\hat{\mu}_n$-almost every $D \in K_\infty^n$, we have

$$\begin{aligned}\int_D \mathbf{1}_{f_n^j(A_n^j)}(x) \rho_D^n(x) \, dm_D(x) &= \int_{f_n^{-j}(D)} \mathbf{1}_{A_n^j}(x) \rho^n_{f_n^{-j}(D)}(x) \, dm_{f_n^{-j}(D)}(x) \\ &\leq C_3 m_{f_n^{-j}(D)}(A_n^j).\end{aligned}$$

However, D is an unstable disk for f_n which smallest Lyapunov exponent is bounded below by $\lambda = -\log\sigma > 0$ (see Theorem 3.2(4)). This together with property (PT2) stated in §2.1.2 implies that there exists a constant C_4 such that

$$m_{f_n^{-j}(D)}(A_n^j) \leq C_4 e^{-\lambda j}$$

for all $n \geq 0$. Hence, we conclude that

$$\int_D \mathbf{1}_{f_n^j(A_n^j)}(x) \rho_D^n(x) \, dm_D(x) \leq C_3 C_4 e^{-\lambda j}$$

for every $n \geq 0$ and so we may conclude that the series

$$\sum_{j=1}^{\infty} \mu_n(A_n^j) \leq \sum_{j=1}^{\infty} C_3 C_4 e^{-\lambda j}$$

converge uniformly with respect to $n \in \mathbb{N}$. □

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Remark 7. If we define $A^j = \{x \in K^*_\infty : j \text{ is the first return time of } x\}$ as above, the series $\sum_{j=1}^{\infty} \mu^*(A^j)$ converges. Moreover, from Lemmas 3.3 and 3.4 and property (PT2), we have

$$\sum_{j=1}^{\infty} \mu^*(A^j) \leq \sum_{j=1}^{\infty} C_3 C_4 e^{-\lambda j}.$$

Let $k(x,n)$ the smallest positive integer such that $f_n^{k(x,n)}(x) \in K^n_\infty$. From (13), this number is defined for μ_n-almost every point x. For $N \in \mathbb{N}$, let us denote by K_n^N the set

$$K_n^N = \{x \in M : k(x,n) = N\}.$$

For all $N \in \mathbb{N}$, we define $\mu_n^N = (\mu_n | K_n^N)$. It follows from Lemma 3.5 that, for any $n \geq 0$ fixed, μ_n^N converge in the weak* topology to μ_n as N goes to infinity, uniformly from n.

In the same way, let $k(x)$ be the smallest positive integer such that $f^{k(x)}(x) \in K^n_\infty$. For $N \in \mathbb{N}$, let us denote by K_0^N the set

$$K_0^N = \{x \in M : k(x,0) = N\}.$$

For all $N \in \mathbb{N}$, we define $\nu_N^* = (\nu^* | K_0^N)$ and clearly we have $\nu_N^* = (\mu^* | K_0^N)$. As above, ν_N^* converge in the weak* topology to ν^* as N goes to infinity.

On the other hand, observing that $K_n^N \subseteq f_n^{-N}(K^n_\infty)$ and $K^N \subseteq f^{-N}(K^*_\infty)$, it is clear that, for any $N \in \mathbb{N}$ fixed, μ_n^N converges to ν_N^* when n goes to infinity.

LEMMA 3.6. *We have* $\mu^* = \nu^*$.

Proof. Let $\varphi : M \to \mathbb{R}$ be continuous and let $\varepsilon > 0$. Fix $N \in \mathbb{N}$ such that

$$\left| \int \varphi \, d\nu^* - \int \varphi \, d\nu_N^* \right| < \varepsilon \quad \text{and} \quad \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu_n^N \right| < \varepsilon,$$

for every $n \geq 1$. Let N be fixed and let $n \in \mathbb{N}$ such that

$$\left| \int \varphi \, d\mu_n - \int \varphi \, d\mu^* \right| < \varepsilon \quad \text{and} \quad \left| \int \varphi \, d\nu_N^* - \int \varphi \, d\mu_n^N \right| < \varepsilon$$

and, finally,

$$\begin{aligned} \left| \int \varphi \, d\nu^* - \int \varphi \, d\mu^* \right| &\leq \left| \int \varphi \, d\nu^* - \int \varphi \, d\nu_N^* \right| + \left| \int \varphi \, d\nu_N^* - \int \varphi \, d\mu_n^N \right| \\ &\quad + \left| \int \varphi \, d\mu_n^N - \int \varphi \, d\mu_n \right| + \left| \int \varphi \, d\mu_n - \int \varphi \, d\mu^* \right| \\ &< 4\varepsilon. \end{aligned}$$

□

We conclude that $\mu^* = \nu^*$ and so μ^* is a Gibbs *cu*-state, concluding the proof of Theorem 3.4.

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3.2.1. *Convergence of the densities.* Let D_n and D be disks in K^n_∞ and K^*_∞, respectively, such that D is accumulated by D_n as n goes to infinity. Let ρ_n and ρ be the densities of μ_n and μ^* defined in D_n and D, respectively. We may assume, for n large enough, that $D_n = D$.

From Proposition 2.1 we know that ρ_n and ρ are Hölder continuous functions with the same Hölder constants and uniformly bounded away from zero and infinity.

PROPOSITION 3.5. *ρ_n converges to ρ uniformly.*

Proof. Since ρ_n and ρ are uniformly bounded away from zero and infinity, then $\log \rho_n$ and $\log \rho$ are uniformly bounded.

Let us use the notation $J^u_{k,n}(x) = |\det Df_n^{-1}|E^{cu}_{f_n^k(x)}|$ and $J^u_k(x) = |\det Df^{-1}|E^{cu}_{f^k(x)}|$, $k \geq 0$. From Proposition 2.1 we have

$$\frac{\rho_n(x)}{\rho_n(y)} = \prod_{k=0}^{\infty} \frac{J^u_{k,n}(x)}{J^u_{k,n}(y)} \quad \text{and} \quad \frac{\rho(x)}{\rho(y)} = \prod_{k=0}^{\infty} \frac{J^u_k(x)}{J^u_k(x)},$$

for all $x, y \in D$, and for $n \geq 1$. Then, repeating the proof of Lemma 2.5, we have, for every $n \geq 1$,

$$\begin{aligned} |\log \rho_n(x) - \log \rho_n(y)| &\leq \sum_{k=0}^{\infty} |\log J^u_{k,n}(x) - \log J^u_{k,n}(y)| \\ &\leq \sum_{k=0}^{\infty} C_1 \mathrm{dist}_{f_n^k(D)}(f_n^k(x), f_n^k(y))^{\xi} \\ &\leq \sum_{k=0}^{\infty} C_1 C^{\xi} e^{-k\lambda\xi} \mathrm{dist}_D(x, y)^{\xi}. \end{aligned}$$

The same inequality holds for $|\log \rho(x) - \log \rho(y)|$. In consequence, the set $\{\rho_n, \rho\}$ is equicontinuous. The Arzela–Ascoli theorem implies the result. □

4. *Gibbs cu-states and SRB measures*

In this section we study the relationship between Gibbs *cu*-states and SRB measures and conclude with some applications of our result to the study of statistical stability for partially hyperbolic systems.

We assume that $f \in \mathrm{Diff}^r(M)$, $r > 1$, has an attracting set Λ with dominated splitting and non-uniformly expanding along the center-unstable direction. Consider a disk D transverse to the center-stable direction. First, we prove that the constructions of the previous section can be done by replacing the disk D by a positive Lebesgue measure subset E.

PROPOSITION 4.1. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, having an attracting set Λ with dominated splitting and non-uniformly expanding along the center-unstable direction. Given a center-unstable domain D and any positive Lebesgue measure set $E \subseteq D$, every weak* accumulation point of*

$$\mu_{n,E} = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j \left(\frac{m_E}{m_D(E)} \right)$$

has an ergodic component which is a Gibbs cu-state of f.

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Proof. Given any $\delta > 0$ we may find pairwise disjoint domains $D_1, \dots, D_s$ in D such that the relative Lebesgue measure on E inside each D_i is larger than $1 - \delta$, and the total measure of E outside the union of the D_i is less than $\delta m_D(E)$. Then, for any $j \geq 1$, we have

$$f_*^j\left(\frac{m_E}{m_D(E)}\right) = \sum_{i=1}^{s} \frac{m_D(D_i)}{m_D(E)} f_*^j\left(\frac{m_{D_i}}{m_D(D_i)}\right) + \frac{1}{m_D(E)} f_*^j m_{(E\setminus\bigcup_{i=1}^{s} D_i)} - \frac{1}{m_D(E)} \sum_{i=1}^{s} f_*^j m_{D_i\setminus E}.$$

The total masses of both the second and third terms do not depend on j, and are less than δ. Therefore, every accumulation point of $\mu_{n,E}$ differs from an accumulation point of

$$\sum_{i=1}^{s} \frac{m_D(D_i)}{m_D(E)} \frac{1}{n} \sum_{j=0}^{n-1} f_*^j\left(\frac{m_{D_i}}{m_D(D_i)}\right)$$

by a measure whose total mass is less than δ. Applying Theorem 3.1 to each domain D_i, every point of accumulation of this last sequence has an ergodic component which is a Gibbs *cu*-state whose densities are uniformly bounded and satisfy the ratio relation of Proposition 2.1. Making δ go to zero and applying Theorems 2.1, 3.3 and 3.4, we get that every weak* accumulation point of $\mu_{n,E}$ has an ergodic component which is a Gibbs *cu*-state. □

COROLLARY 4.1. *Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, having an attracting set Λ with dominated splitting and non-uniformly expanding along the center-unstable direction. Then every ergodic SRB measure is a Gibbs cu-state.*

Proof. Let μ be an ergodic SRB measure for f supported in Λ. Consider any disk D inside U, where U is a neighborhood of Λ as in §1.1. Let us suppose that D is transverse to the center-stable subbundle and intersecting the basin of μ on a positive Lebesgue measure subset D_0. On the one hand,

$$\frac{1}{n} \sum_{j=0}^{n-1} f_*^j\left(\frac{m_{D_0}}{m_D(D_0)}\right) = \frac{1}{m_D(D_0)} \int_{D_0} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)}\, dm_D(x)$$

converges to μ when $n \to \infty$. On the other hand, from Proposition 4.1 and the hypothesis of ergodicity it follows that μ must be a Gibbs *cu*-state. □

4.1. *Statistical stability.* Now we present several applications of our results to the study of statistical stability for systems with a weak form of hyperbolicity.

4.1.1. *The partially hyperbolic case.* Let $f \in \mathrm{Diff}^r(M)$, $r > 1$, having a partially hyperbolic attracting set Λ with splitting $T_\Lambda M = E^{cu} \oplus E^s$. The measure μ constructed in Section 3 has an ergodic component μ_*, with support contained in Λ, which is a Gibbs *cu*-state. Then there exists a disk $D_\infty \in \mathcal{K}_\infty$ such that $m_{D_\infty}(B(\mu_*)) > 0$ ([**3**, Lemma 4.5]). Because the strong-stable foliation is absolutely continuous [**8**], $m(B(\mu_*))$

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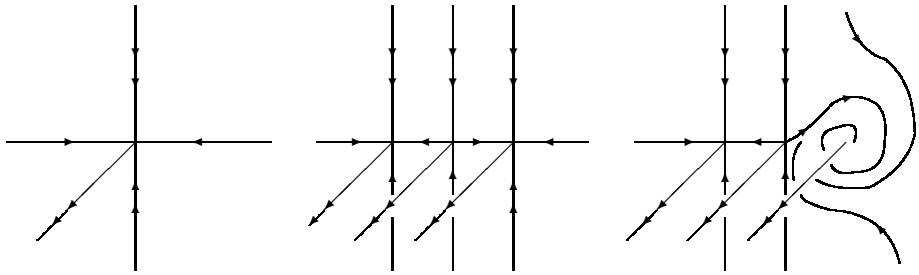


FIGURE 1. Deformation of an Anosov diffeomorphism f_0 at a fixed point p_1.

must be positive, so μ_* is an SRB measure. Moreover, a full Lebesgue measure subset of U is contained in the union of finitely many SRB measures supported in Λ (see [**3**, Corollary 4.6]).

Proof of Theorem A. Let $(\mu_n)_n$ be a sequence of ergodic SRB probability measures for f_n, converging to μ in the weak* topology. By Corollary D, each μ_n is a Gibbs *cu*-state for f_n and, by Theorem C, μ is also a Gibbs *cu*-state for f. By Theorem A, there exist a cylinder $\mathcal{C}^*$ and a family $\mathcal{K}^*_\infty$ of disjoint disks contained in $\mathcal{C}^*$ which are graphs over B^u and local uniformly expanding manifolds. Moreover, $\mu(\mathcal{K}^*_\infty) \geq \alpha > 0$ and μ is absolutely continuous with respect to Lebesgue measure along these disks.

Now if we take a tubular neighborhood of $D_\infty \in \mathcal{K}_\infty$ given by Lemma 2.6 using the stable foliation, then since this foliation is Hölder continuous we have a positive Lebesgue measure set of points in $B(\mu)$, so μ is an SRB measure. By Theorem 1.1 there are finitely many ergodic SRB measures and the union of their basins covers a full Lebesgue measure subset of U, so μ must be in the convex hull of such measures. □

4.1.2. *The dominated splitting case.* As we said in the introduction, Bonatti and Viana [**5**] constructed an open class of C^r-diffeomorphisms $\mathcal{N}$, $r > 1$, defined on $\mathbb{T}^4$. The outline of that construction is the following. We start with a linear Anosov diffeomorphism f_0 induced in $\mathbb{T}^4$ by a linear map of $\mathbb{R}^4$ with eigenvalues

$$0 < \lambda_1 \leq \lambda_2 < 1 < \lambda_3 \leq \lambda_4.$$

Up to replacing it by some iterate, we may suppose that f_0 has as least two fixed points p_1 and p_2. Let $\mathcal{F}^u_0$ and $\mathcal{F}^s_0$ be the unstable and stable foliations of f_0. Let $0 < \delta \ll 1$ be fixed. Let $V = B(p_1, \delta) \cup B(p_2, \delta)$ be a small neighborhood containing p_1 and p_2. The idea is to deform the Anosov diffeomorphism f_0 inside V, passing first through a flip bifurcation along the $\mathcal{F}^s_0$ inside $B(p_1, \delta)$ and then we obtain three fixed points, two of them with stability index 2 and one with stability index 1. Then with another perturbation we make the eigenvalues complex for one of the fixed points with stability index 2 as we show in Figure 1.

We make a similar perturbation on the other fixed point p_2 but now along the $\mathcal{F}^u_0$ inside $B(p_2, \delta)$. Then we obtain three fixed points, one of them with stability index 3 and the

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others with stability index 2. One of the fixed points with stability index 2 has complex eigenvalues with norm greater than 1.

So we obtain an open set $\mathcal{N} \subseteq \mathrm{Diff}^r(M)$, $r > 1$, such that every $f \in \mathcal{N}$ has dominated splitting but is not partially hyperbolic and it is robustly non-uniformly expanding in the center-unstable direction.

The reader can see [**5**] for more details of the construction in $\mathbb{T}^4$ and [**26**] to see how that construction can be made on $\mathbb{T}^n$, for $n \geq 4$.

Proof of Theorem B. Alves *et al* proved that there exist SRB measures for such f. After this, Tahzibi [**26**] proved that the SRB measure is unique.

In this case, the SRB measure corresponds to a unique Gibbs *cu*-state and, by Theorem 3.4, this SRB measure is C^r-statistically stable, $r > 1$. □

When f has an attracting set Λ with dominated splitting $E^{cs} \oplus E^{cu}$, $s = \dim E^{cs}$ and all Lyapunov exponents of μ negative, μ is in fact an SRB measure. This is a consequence of the absolute continuity property of μ and the absolute continuity of the stable lamination [**17**]: the union of the stable manifolds through the point whose time averages are given by μ is a positive Lebesgue measure subset of M.

Proof of Theorem C. First we prove the existence of SRB measures. Let μ be an ergodic Gibbs *cu*-state for f. It exists by Theorem 3.1. Let D_∞ be a disk such that $D_\infty \cap B(\mu) \cap R(f)$ has full Lebesgue measure on D_∞. Such a disk exists by Lemma 2.6, and is contained in some local unstable manifold. By hypothesis, a positive Lebesgue measure subset of D_∞ satisfies (3). So the set A of points in $D_\infty \cap B(\mu) \cap R(f)$ satisfying (3) has positive Lebesgue measure on D_∞.

For $\varepsilon > 0$, we denote by $D_\infty(\varepsilon)$ the tubular neighborhood of radius ε around D_∞, defined as the image under the exponential map of M of all of the vectors of norm less that $\varepsilon > 0$ in the orthogonal complement of E^{cu}_x, for all $x \in D_\infty$. If $\varepsilon > 0$ is small enough, then $D_\infty(\varepsilon)$ is a cylinder.

For every point in $x \in A$, there exists a C^1 embedded disk $W^s_{\mathrm{loc}}(x)$ tangent to E^{cs}_x at x such that the diameter of $f^n(W^s_{\mathrm{loc}}(x))$ converges exponentially fast to zero as $n \to \infty$. These disks $W^s_{\mathrm{loc}}(x)$ depend in a measurable way on the point x, and the lamination $\{W^s_{\mathrm{loc}}(x) : x \in A\}$ is absolutely continuous. Since $A \subseteq B(\mu)$, every $y \in W^s_{\mathrm{loc}}(x)$ is in $B(\mu)$ also.

The domination condition on the splitting together with the absolute continuity of the stable lamination implies that every disk D tangent to the E^{cu} direction crossing $D_\infty(\varepsilon)$, and close enough to D_∞, intersects the lamination $\{W^s_{\mathrm{loc}}(x) : x \in A\}$ in a positive Lebesgue measure subset. Finally, Fubini's theorem implies that the Lebesgue measure of $B(\mu)$ is positive.

Now we prove that there are finitely many ergodic SRB measures. Suppose otherwise. Let (μ_n) be a sequence of ergodic SRB measures of f converging in the weak* topology to a measure μ. By Corollary 4.1, each μ_n must be a Gibbs *cu*-state for f. Theorem 3.4 implies that μ is a Gibbs *cu*-state. By the argument used above, μ must also be an SRB measure.

Observe from Theorem 3.3 that $\mu_n \in \mathcal{G}(f)$ for each n, so there is a sequence $\mathcal{C}_n$ of hyperbolic blocks associated to μ_n converging to $\mathcal{C}$, a hyperbolic block associated to μ.

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Moreover, the size of the disks crossing the cylinder is uniformly bounded from below.

Let D_∞ crossing $\mathcal{C}$, $D_\infty(\varepsilon)$ and A be the sets defined above for μ. Let D_n be the corresponding disk defined for μ_n in the block $\mathcal{C}_n$, given by Lemma 2.2. For $n \geq 1$ large enough, the disk D_n crosses $D_\infty(\varepsilon)$. The argument above implies that D_n intersects the lamination $\{W^s_{\mathrm{loc}}(x) : x \in A\}$ in a subset with positive Lebesgue measure on D_n. Each D_n is contained in some local unstable manifold, so if there exists some point in the basin of μ then every point in these manifolds is in the basin too. However, there exists a positive Lebesgue measure subset of D_n contained in the basin of μ_n, so $B(\mu) = B(\mu_n)$ for all $n > 1$ large enough, and then $\mu = \mu_n$.

Let $\mu_1, \ldots, \mu_n$ be the finitely many SRB measures for f supported in Λ. Now we prove that $m\big(B(\Lambda) \setminus \bigcup_{i=1}^n B(\mu_i)\big) = 0$. Suppose that $m\big(U \setminus \bigcup_{i=1}^n B(\mu_i)\big) > 0$. Then there exists a C^2-disk D tangent to the center-unstable cone field such that conditions (1) and (2) of §3.1 hold and $m_D\big(D \cap \bigcup_{i=1}^n B(\mu_i)\big) = 0$. Let $\mu = \mu_i$ be a Gibbs cu-state constructed from the iterates of Lebesgue measure on D as in §3.1.

From this construction, given $n \geq 1$ large enough, the Lebesgue measure of $f^n(D) \cap B(\mu)$ on $f^n(D)$ is bounded from below away from zero, thus the Lebesgue measure of $D \cap B(\mu)$ on D is also bounded from below and away from zero. This is a contradiction.

In order to prove statistical stability, consider $(f_n)_n$ a sequence of C^r-diffeomorphisms converging to f in the C^r-topology, $r > 1$. Assume that $(\mu_n)_n$ is a sequence of ergodic SRB measures for $(f_n)_{n\geq 1}$ and that μ is a weak* accumulation measure of this sequence. By Corollary 4.1, each μ_n must be a Gibbs cu-state for f_n. Theorem 3.4 implies that μ is a Gibbs cu-state. By Theorem 2.1 every ergodic component of μ is a Gibbs cu-state. Applying to each ergodic component of μ the argument used above, every ergodic component of μ must also be an SRB measure, and so it must be in the convex hull of finitely many ergodic SRB measures. □

Remark 8. However, it is not known whether there are general conditions ensuring the uniqueness of the SRB measure for diffeomorphisms with dominated splitting.

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