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CONSTRUCTIBLE FUNCTIONS ON ARTIN STACKS

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Abstract

Let $\mathbb{K}$ be an algebraically closed field, let X be a $\mathbb{K}$-variety, and let $X(\mathbb{K})$ be the set of closed points in X. A *constructible set* C in $X(\mathbb{K})$ is a finite union of subsets $Y(\mathbb{K})$ for subvarieties Y in X. A *constructible function* $f : X(\mathbb{K}) \to \mathbb{Q}$ has $f(X(\mathbb{K}))$ finite and $f^{-1}(c)$ constructible for all $c \neq 0$. Write $\mathrm{CF}(X)$ for the vector space of such f. Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms of $\mathbb{C}$-varieties. MacPherson defined a linear *pushforward* $\mathrm{CF}(\phi) : \mathrm{CF}(X) \to \mathrm{CF}(Y)$ by 'integration' with respect to the topological Euler characteristic. It is functorial, that is, $\mathrm{CF}(\psi \circ \phi) = \mathrm{CF}(\psi) \circ \mathrm{CF}(\phi)$. This was extended to $\mathbb{K}$ of characteristic zero by Kennedy.

This paper generalizes these results to $\mathbb{K}$-*schemes* and *Artin* $\mathbb{K}$-*stacks* with affine stabilizer groups. We define the notions of Euler characteristic for constructible sets in $\mathbb{K}$-schemes and $\mathbb{K}$-stacks, and pushforwards and pullbacks of constructible functions, with functorial behaviour. Pushforwards and pullbacks commute in Cartesian squares. We also define *pseudomorphisms*, a generalization of morphisms well suited to constructible functions problems.

1. *Introduction*

Let $\mathbb{K}$ be an algebraically closed field, let X be a $\mathbb{K}$-scheme, and let $X(\mathbb{K})$ be the set of closed points in X. A *constructible set* $C \subseteq X(\mathbb{K})$ is a finite union of subsets $Y(\mathbb{K})$ for finite type $\mathbb{K}$-subschemes Y in X. A *constructible function* $f : X(\mathbb{K}) \to \mathbb{Q}$ has $f(X(\mathbb{K}))$ finite and $f^{-1}(c)$ constructible for all $0 \neq c \in f(X(\mathbb{K}))$. Write $\mathrm{CF}(X)$ for the $\mathbb{Q}$-vector space of constructible functions on X.

Let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms of $\mathbb{C}$-varieties. MacPherson [**18**, Proposition 1] defined a $\mathbb{Q}$-linear *pushforward* $\mathrm{CF}(\phi) : \mathrm{CF}(X) \to \mathrm{CF}(Y)$ with $(\mathrm{CF}(\phi)\delta_W)(y) = \chi_{\mathrm{an}}(\phi^{-1}(y) \cap W)$ for subvarieties W in X and $y \in Y(\mathbb{C})$, where χ_{an} is the topological Euler characteristic in compactly-supported cohomology with the analytic topology, and δ_W the characteristic function of $W(\mathbb{C})$ in $X(\mathbb{C})$. It satisfies $\mathrm{CF}(\psi \circ \phi) = \mathrm{CF}(\psi) \circ \mathrm{CF}(\phi)$, so that CF is a *functor* from the category of $\mathbb{C}$-varieties to the category of $\mathbb{Q}$-vector spaces. This was extended to other fields $\mathbb{K}$ of characteristic zero by Kennedy [**15**].

This paper generalizes these results to $\mathbb{K}$-*schemes* and *algebraic* $\mathbb{K}$-*stacks* in the sense of Artin, for $\mathbb{K}$ of characteristic zero. We introduce a notion of *pseudomorphism* Φ between locally constructible sets in $\mathbb{K}$-schemes or $\mathbb{K}$-stacks, generalizing morphisms. Pushforwards $\mathrm{CF}(\Phi)$ exist, and pseudomorphisms seem very natural for constructible functions problems.

The motivation for this work is my series of papers [**10**–**13**]. Let $\mathrm{coh}(P)$ be the abelian category of *coherent sheaves* on a projective $\mathbb{K}$-scheme P, and let $(\tau, T, \leqslant)$ be a *stability condition* on $\mathrm{coh}(P)$. Then the moduli space $\mathfrak{Obj}_{\mathrm{coh}(P)}$ of sheaves in $\mathrm{coh}(P)$ is an Artin $\mathbb{K}$-stack, and the set $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$ of τ-semistable sheaves in class α is a constructible subset in $\mathfrak{Obj}_{\mathrm{coh}(P)}$. We shall define invariants of $P, (\tau, T, \leqslant)$

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as generalized Euler characteristics of $\mathrm{Obj}_{\mathrm{ss}}^{\alpha}(\tau)$, and study the identities that they satisfy, and transformation laws under a change of stability condition.

To carry out this programme requires a theory of constructible sets and functions in algebraic $\mathbb{K}$-stacks, and compatible notions of the Euler characteristic and pushforward. As we could not find these tools in the literature, we develop them here. It seemed better to write a stand-alone paper that others could use, rather than include the material in the series [**10**–**13**].

Section 2 gives some background on schemes, varieties and stacks. In § 3 we recall MacPherson's constructible functions theory for $\mathbb{C}$-varieties, extend it to $\mathbb{K}$-schemes using *l-adic cohomology* in place of cohomology with the analytic topology for $\mathbb{C}$-varieties when $\mathbb{K}$ has characteristic zero, and define and study *pseudomorphisms* between locally constructible sets in $\mathbb{K}$-schemes. We also explain why the theory *cannot* be extended to $\mathbb{K}$-schemes for $\mathbb{K}$ of positive characteristic.

Sections 4 and 5 extend these ideas to stacks. An important difference between stacks and schemes is that in an algebraic $\mathbb{K}$-stack $\mathfrak{F}$ points $x \in \mathfrak{F}(\mathbb{K})$ have *stabilizer groups* $\mathrm{Iso}_{\mathbb{K}}(x)$, which are algebraic $\mathbb{K}$-groups, trivial if $\mathfrak{F}$ is a $\mathbb{K}$-scheme. It turns out that there are *many different ways* of including stabilizer groups when extending Euler characteristics χ and pushforwards CF to stacks.

We highlight three interesting cases, the *naïve pushforward* $\mathrm{CF}^{\mathrm{na}}$, which ignores stabilizer groups, the *stack pushforward* $\mathrm{CF}^{\mathrm{stk}}$, which is most natural in many stack problems, and the *orbifold pushforward* $\mathrm{CF}^{\mathrm{orb}}$, related to Deligne–Mumford stacks and their crepant resolutions. Each is associated with a notion of *Euler characteristic* $\chi^{\mathrm{na}}, \chi^{\mathrm{stk}}, \chi^{\mathrm{orb}}$ of constructible sets in $\mathbb{K}$-stacks.

As $\chi^{\mathrm{stk}}, \mathrm{CF}^{\mathrm{stk}}$ involve weighting by $1/\chi(\mathrm{Iso}_{\mathbb{K}}(x))$, the obvious definitions fail when $\chi(\mathrm{Iso}_{\mathbb{K}}(x)) = 0$. However, for *representable* 1-morphisms $\phi : \mathfrak{F} \to \mathfrak{G}$ we give a more subtle definition of $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ in § 5.1, which is always well-defined, and suffices for the applications in [**11**–**13**]. We also define *pullbacks* ψ^* by *finite type* 1-morphisms $\psi : \mathfrak{F} \to \mathfrak{G}$, and show that pullbacks ψ^* and pushforwards $\mathrm{CF}^{\mathrm{stk}}(\phi)$ commute in *Cartesian squares*.

In a companion paper [**9**] we study 'stack functions' on Artin stacks, which are a universal generalization of constructible functions containing more information, and discuss how 'motivic' invariants of $\mathbb{K}$-varieties such as Euler characteristics and virtual Poincaré polynomials are best extended to Artin stacks.

All $\mathbb{K}$-schemes and $\mathbb{K}$-stacks in this paper are assumed *locally of finite type*.

2. *Schemes, varieties and stacks*

Fix an algebraically closed field $\mathbb{K}$ throughout. There are four main classes of 'spaces' over $\mathbb{K}$ used in algebraic geometry, in increasing order of generality:

$$\mathbb{K}\text{-varieties} \subset \mathbb{K}\text{-schemes} \subset \text{algebraic } \mathbb{K}\text{-spaces} \subset \text{algebraic } \mathbb{K}\text{-stacks}.$$

Section 2.1 gives a few definitions and facts on $\mathbb{K}$-schemes and $\mathbb{K}$-varieties, and § 2.2 introduces algebraic $\mathbb{K}$-stacks. Some good references for § 2.1 are Hartshorne [**6**], and for § 2.2 are Gómez [**5**] and Laumon and Moret-Bailly [**17**].

2.1. *Schemes and varieties*

We assume a good knowledge of $\mathbb{K}$-schemes and their morphisms, following Hartshorne [**6**]. We make the conventions that:

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- all $\mathbb{K}$-schemes in this paper are *locally of finite type*;
- all $\mathbb{K}$-*subschemes* are *locally closed*, but not necessarily closed;
- a $\mathbb{K}$-*variety* is a reduced, irreducible, separated $\mathbb{K}$-scheme of finite type.

DEFINITION 2.1. For a $\mathbb{K}$-scheme X, write $X(\mathbb{K})$ for the set $\mathrm{Hom}(\mathrm{Spec}\,\mathbb{K}, X)$ of morphisms of $\mathbb{K}$-schemes $\mathrm{Spec}\,\mathbb{K} \to X$. Then $X(\mathbb{K})$ is naturally identified with the subset of *closed points* of the underlying topological space of X. Elements of $X(\mathbb{K})$ are also called *geometric points* or $\mathbb{K}$-*points* of X.

There is a natural identification $(X \times Y)(\mathbb{K}) \cong X(\mathbb{K}) \times Y(\mathbb{K})$. If $\phi : X \to Y$ is a morphism of $\mathbb{K}$-schemes, composition $\mathrm{Spec}\,\mathbb{K} \to X \xrightarrow{\phi} Y$ gives a natural map $\phi_* : X(\mathbb{K}) \to Y(\mathbb{K})$. If X is a $\mathbb{K}$-*subscheme* of Y, then $X(\mathbb{K}) \subseteq Y(\mathbb{K})$.

Much of the paper will involve cutting schemes or stacks into pieces. To do this we shall use two different notions of *disjoint union*.

DEFINITION 2.2. Let X be a $\mathbb{K}$ scheme and let $\{X_i : i \in I\}$ be a family of $\mathbb{K}$-subschemes of X. We say that X is the *set-theoretic disjoint union* of the X_i for $i \in I$ if $X(\mathbb{K}) = \coprod_{i \in I} X_i(\mathbb{K})$.

If $\{X_i : i \in I\}$ is a family of $\mathbb{K}$-schemes, we define the *abstract disjoint union* of the X_i to be the $\mathbb{K}$-scheme $(X, \mathcal{O}_X)$, where X is the disjoint union of the topological spaces X_i, and $\mathcal{O}_X|_{X_i} = \mathcal{O}_{X_i}$. Then X exists and is unique up to isomorphism, and the X_i are open and closed $\mathbb{K}$-subschemes of X. Clearly, an abstract disjoint union is a set-theoretic disjoint union, but not necessarily *vice versa*. When we just say 'disjoint union' we mean set-theoretic disjoint union.

Here is a useful result of Rosenlicht [**23**], on the existence of quotients of varieties by algebraic groups.

THEOREM 2.3. *Let $\mathbb{K}$ be an algebraically closed field, let X be a $\mathbb{K}$-variety, let G be an algebraic $\mathbb{K}$-group, and let $\rho : G \times X \to X$ be an algebraic action of G on X. Then there exists a dense, Zariski open subset X' of X, a $\mathbb{K}$-variety Y, and a surjective morphism $\pi : X' \to Y$ inducing a bijection between G-orbits in X' and $\mathbb{K}$-points in Y, such that any G-invariant rational function on X' defined at $x \in X'$ is the pull-back of a rational function on Y defined at $\pi(x)$.*

2.2. *Algebraic stacks*

Algebraic stacks (also known as Artin stacks) were introduced by Artin, generalizing *Deligne–Mumford stacks*. For a good introduction to algebraic stacks see Gómez [**5**], and for a thorough treatment see Laumon and Moret-Bailly [**17**]. As for schemes, we make the convention that all algebraic $\mathbb{K}$-stacks in this paper are *locally of finite type*, and $\mathbb{K}$-substacks are *locally closed*.

Algebraic $\mathbb{K}$-stacks form a 2-*category*. That is, we have *objects* which are $\mathbb{K}$-stacks $\mathfrak{F}, \mathfrak{G}$, and also two kinds of morphisms: 1-*morphisms* $\phi, \psi : \mathfrak{F} \to \mathfrak{G}$ between $\mathbb{K}$-stacks; and 2-*morphisms* $A : \phi \to \psi$ between 1-morphisms. An analogy to keep in mind is a 2-category of categories, where objects are categories, 1-morphisms are functors between the categories, and 2-morphisms are isomorphisms (natural transformations) between functors.

We define the set of $\mathbb{K}$-*points* of a stack.

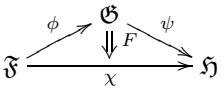
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DEFINITION 2.4. Let $\mathfrak{F}$ be a $\mathbb{K}$-stack. Write $\mathfrak{F}(\mathbb{K})$ for the set of 2-isomorphism classes $[x]$ of 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$. Elements of $\mathfrak{F}(\mathbb{K})$ are called $\mathbb{K}$-*points*, or *geometric points*, of $\mathfrak{F}$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism then composition with ϕ induces a map of sets $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$.

For a 1-morphism $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$, the *stabilizer group* $\mathrm{Iso}_{\mathbb{K}}(x)$ is the group of 2-morphisms $x \to x$. When $\mathfrak{F}$ is an algebraic $\mathbb{K}$-stack, $\mathrm{Iso}_{\mathbb{K}}(x)$ is an *algebraic* $\mathbb{K}$-*group*. We say that $\mathfrak{F}$ *has affine geometric stabilizers* if $\mathrm{Iso}_{\mathbb{K}}(x)$ is an affine algebraic $\mathbb{K}$-group for all 1-morphisms $x : \operatorname{Spec}\mathbb{K} \to \mathfrak{F}$.

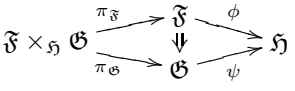
As an algebraic $\mathbb{K}$-group up to isomorphism, $\mathrm{Iso}_{\mathbb{K}}(x)$ depends only on the isomorphism class $[x] \in \mathfrak{F}(\mathbb{K})$ of x in $\mathrm{Hom}(\operatorname{Spec}\mathbb{K}, \mathfrak{F})$. If $\phi : \mathfrak{F} \to \mathfrak{G}$ is a 1-morphism, composition induces a morphism of algebraic $\mathbb{K}$-groups $\phi_* : \mathrm{Iso}_{\mathbb{K}}([x]) \to \mathrm{Iso}_{\mathbb{K}}(\phi_*([x]))$, for $[x] \in \mathfrak{F}(\mathbb{K})$.

One important difference in working with 2-categories rather than ordinary categories is that in diagram-chasing one only requires 1-morphisms to be 2-*isomorphic* rather than *equal*. The simplest kind of *commutative diagram* is



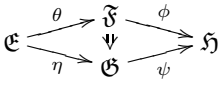
by which we mean that $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ are $\mathbb{K}$-stacks, ϕ, ψ, χ are 1-morphisms, and $F : \psi \circ \phi \to \chi$ is a 2-isomorphism. Usually we omit F, and mean that $\psi \circ \phi \cong \chi$.

DEFINITION 2.5. Let $\phi : \mathfrak{F} \to \mathfrak{H}$, $\psi : \mathfrak{G} \to \mathfrak{H}$ be 1-morphisms of $\mathbb{K}$-stacks. Then one can define the *fibre product stack* $\mathfrak{F} \times_{\phi,\mathfrak{H},\psi} \mathfrak{G}$, or $\mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$ for short, with 1-morphisms $\pi_{\mathfrak{F}}, \pi_{\mathfrak{G}}$ fitting into the following commutative diagram.



(1)

A commutative diagram



is a *Cartesian square* if it is isomorphic to (1), so there is a 1-isomorphism $\mathfrak{E} \cong \mathfrak{F} \times_{\mathfrak{H}} \mathfrak{G}$. Cartesian squares may also be characterized by a universal property.

The following is a definition from Kresch [**16**, Definition 3.5.3], slightly modified.

DEFINITION 2.6. Let $\mathfrak{F}$ be a finite type algebraic $\mathbb{K}$-stack, and let $\mathfrak{F}^{\mathrm{red}}$ be the associated reduced stack. We say that $\mathfrak{F}$ *can be stratified by global quotient stacks* if $\mathfrak{F}^{\mathrm{red}}$ is the disjoint union of finitely many locally closed substacks U_i with each U_i 1-isomorphic to a stack of the form $[X_i/G_i]$, where X_i is a $\mathbb{K}$-variety and G_i a smooth, connected, affine algebraic $\mathbb{K}$-group acting linearly on X_i. For a stack to be the *disjoint union* of a family of locally closed substacks is defined in [**17**, p. 22]. It implies that $\mathfrak{F}(\mathbb{K}) = \mathfrak{F}^{\mathrm{red}}(\mathbb{K}) = \coprod_i U_i(\mathbb{K})$.

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Kresch [**16**, Proposition 3.5.2(ii)] takes the X_i to be *quasiprojective schemes*, rather than varieties, but this is equivalent to our definition. Kresch [**16**, Proposition 3.5.9] characterizes stacks stratified by global quotients.

THEOREM 2.7. *Let $\mathfrak{F}$ be a finite type algebraic $\mathbb{K}$-stack. Then $\mathfrak{F}$ can be stratified by global quotient stacks if and only if $\mathfrak{F}$ has affine geometric stabilizers.*

3. *Constructible functions on $\mathbb{K}$-schemes*

We now introduce *constructible sets* and *functions* on $\mathbb{K}$-schemes, and the *pushforward* of constructible functions by morphisms. Section 3.1 defines (locally) constructible sets and functions on $\mathbb{K}$-schemes. We explain the Euler characteristic and pushforwards over $\mathbb{C}$ in § 3.2, and over other fields $\mathbb{K}$ in § 3.3. In § 3.4 we define *pseudomorphisms*, a notion of morphism for (locally) constructible sets, and pushforwards along pseudomorphisms.

Some references are Mumford [**21**, p. 51] and Hartshorne [**6**, p. 94] for constructible sets, and MacPherson [**18**], Viro [**24**] and Kennedy [**15**] for constructible functions and the pushforward. As far as the author can tell the ideas of §§ 3.3 and 3.4 are new, although elementary and probably obvious to experts.

3.1. *Constructible sets and functions on $\mathbb{K}$-schemes*

We define *constructible* and *locally constructible sets*.

DEFINITION 3.1. Let $\mathbb{K}$ be an algebraically closed field, and let X be a $\mathbb{K}$-scheme. A subset $C \subseteq X(\mathbb{K})$ is called *constructible* if $C = \bigcup_{i\in I} X_i(\mathbb{K})$, where $\{X_i : i \in I\}$ is a finite collection of finite type $\mathbb{K}$-subschemes X_i of X. We call $S \subseteq X(\mathbb{K})$ *locally constructible* if $S \cap C$ is constructible for all constructible $C \subseteq X(\mathbb{K})$.

This is easily seen to be equivalent to a stronger definition, where we take the union $C = \bigcup_{i\in I} X_i(\mathbb{K})$ to be *disjoint*, and the X_i to be *separated*.

PROPOSITION 3.2. *Let X be a $\mathbb{K}$-scheme, and let $C \subseteq X(\mathbb{K})$ be a constructible subset. Then we may write $C = \coprod_{i\in I} X_i(\mathbb{K})$, where $\{X_i : i \in I\}$ is a finite collection of separated, finite type $\mathbb{K}$-subschemes X_i of X.*

The following properties of constructible sets in $\mathbb{K}$-*varieties* are well known [**6**, p. 94; **21**, p. 51]. Our extension to $\mathbb{K}$-schemes is straightforward.

PROPOSITION 3.3. *Let X, Y be $\mathbb{K}$-schemes, let $\phi : X \to Y$ be a morphism, and let $A, B \subseteq X(\mathbb{K})$ be constructible subsets. Then $A \cup B$, $A \cap B$ and $A \setminus B$ are constructible in $X(\mathbb{K})$, and $\phi_*(A)$ is constructible in $Y(\mathbb{K})$.*

Note that showing that $\phi_*(A)$ is constructible, and the stack analogue in Proposition 4.5, are the only places we use the convention that $\mathbb{K}$-schemes and $\mathbb{K}$-stacks are *locally of finite type*. Next we define (*locally*) *constructible functions*.

DEFINITION 3.4. Let X be a $\mathbb{K}$-scheme and let $S \subseteq X(\mathbb{K})$ be locally constructible. A *constructible function* on S is a function $f : S \to \mathbb{Q}$ such that $f(S)$ is

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finite and $f^{-1}(c)$ is a constructible set in $S \subseteq X(\mathbb{K})$ for each $c \in f(S) \setminus \{0\}$. Note that we do *not* require $f^{-1}(0)$ to be constructible. Write $\mathrm{CF}(S)$ for the $\mathbb{Q}$-vector space of constructible functions on S, and for brevity write $\mathrm{CF}(X)$ for $\mathrm{CF}(X(\mathbb{K}))$.

A *locally constructible function* on S is a function $f : S \to \mathbb{Q}$ such that $f|_C$ is constructible for all constructible $C \subseteq S$. Equivalently, f is locally constructible if $f^{-1}(c)$ is locally constructible for all $c \in \mathbb{Q}$, and $f(C)$ is finite for all constructible $C \subseteq S$. Write $\mathrm{LCF}(S)$ for the $\mathbb{Q}$-vector space of locally constructible functions on S, and $\mathrm{LCF}(X)$ for $\mathrm{LCF}(X(\mathbb{K}))$.

Using Proposition 3.3 we see that products of (locally) constructible functions are (locally) constructible, so $\mathrm{CF}(S)$ and $\mathrm{LCF}(S)$ are *commutative* $\mathbb{Q}$*-algebras*, with $\mathrm{CF}(S)$ an *ideal* in $\mathrm{LCF}(S)$. Note that $1 \in \mathrm{CF}(S)$ if and only if S is constructible, so if it is not, then $\mathrm{CF}(S)$ is an *algebra without identity*.

We now give some remarks on this material.

- To define constructible functions $f : X(\mathbb{K}) \to \mathbb{Q}$ on $\mathbb{K}$-schemes X which are *not of finite type*, or $f : S \to \mathbb{Q}$ for S *not constructible*, we must allow $f^{-1}(0)$ to be non-constructible. If we did not there would be *no* constructible functions on X or S, not even 0.
  For X not of finite type we can think of $X(\mathbb{K})$ as being 'large', or 'unbounded'. Constructible functions $f : X(\mathbb{K}) \to \mathbb{Q}$ are nonzero only on small, bounded subsets of $X(\mathbb{K})$, and $f^{-1}(0)$ is the remaining, large, unbounded part of $X(\mathbb{K})$.
- We can also consider constructible functions with values in $\mathbb{Z}$, or any other abelian group, ring or field. However, for simplicity we restrict to $\mathbb{Q}$.

3.2. *Euler characteristics and pushforward for* $\mathbb{C}$*-schemes*

We define the *analytic Euler characteristic* χ_{an}.

Definition 3.5. Let X be a separated $\mathbb{C}$-scheme of finite type. Then $X(\mathbb{C})$ is a Hausdorff topological space with the *analytic topology*. Write $\chi_{\mathrm{an}}(X)$ for the *Euler characteristic* of $X(\mathbb{C})$, in *compactly-supported cohomology*.

The following properties of χ_{an} are well known.

Proposition 3.6. *Let X, Y be separated $\mathbb{C}$-schemes of finite type. Then:*

(i) *if Z is a closed subscheme of X, then $\chi_{\mathrm{an}}(X) = \chi_{\mathrm{an}}(Z) + \chi_{\mathrm{an}}(X \setminus Z)$;*
(ii) *if X is the disjoint union of subschemes $U_1, \ldots, U_m$, then*

$$\chi_{\mathrm{an}}(X) = \sum_{i=1}^{m} \chi_{\mathrm{an}}(U_m);$$

(iii) $\chi_{\mathrm{an}}(X \times Y) = \chi_{\mathrm{an}}(X)\chi_{\mathrm{an}}(Y)$;
(iv) *if $\phi : X \to Y$ is a morphism which is a locally trivial fibration in the analytic topology with fibre F, then $\chi_{\mathrm{an}}(X) = \chi_{\mathrm{an}}(F)\chi_{\mathrm{an}}(Y)$;*
(v) $\chi_{\mathrm{an}}(\mathbb{C}^m) = 1$ *and* $\chi_{\mathrm{an}}(\mathbb{CP}^m) = m + 1$ *for all* $m \geqslant 0$.

Now we can define *pushforwards* on $\mathbb{C}$-schemes.

Definition 3.7. Let X be a $\mathbb{C}$-scheme and $C \subseteq X(\mathbb{C})$ a constructible subset. Proposition 3.2 gives $C = \coprod_{i \in I} X_i(\mathbb{C})$ for $\{X_i : i \in I\}$ finitely many separated,

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finite type subschemes of X. Define $\chi_{\rm an}(C)=\sum_{i\in I}\chi_{\rm an}(X_i)$. If $\{Y_j:j\in J\}$ is another choice from Proposition 3.2, then X_i is the set-theoretic union of $X_i\cap Y_j$ for $j\in J$, so Proposition 3.6(ii) gives $\chi_{\rm an}(X_i)=\sum_{j\in J}\chi_{\rm an}(X_i\cap Y_j)$. Hence,

$$\sum_{i\in I}\chi_{\rm an}(X_i)=\sum_{i\in I}\Bigl[\sum_{j\in J}\chi_{\rm an}(X_i\cap Y_j)\Bigr]=\sum_{j\in J}\Bigl[\sum_{i\in I}\chi_{\rm an}(X_i\cap Y_j)\Bigr]=\sum_{j\in J}\chi_{\rm an}(Y_j),$$

and $\chi_{\rm an}(C)$ is well-defined.

For $f\in{\rm CF}(X)$, define the *weighted Euler characteristic* $\chi_{\rm an}(X,f)\in\mathbb{Q}$ by

$$\chi_{\rm an}(X,f)=\sum_{c\in f(X(\mathbb{C}))\setminus\{0\}}c\chi_{\rm an}(f^{-1}(c)). \tag{2}$$

This is well-defined as $f(X(\mathbb{C}))$ is finite and $f^{-1}(c)\subseteq X(\mathbb{C})$ is constructible for each $c\in f(X(\mathbb{C}))\setminus\{0\}$. Clearly, $f\mapsto\chi_{\rm an}(X,f)$ is a linear map ${\rm CF}(X)\to\mathbb{Q}$.

Now let $\phi:X\to Y$ be a morphism of $\mathbb{C}$-schemes, and let $f\in{\rm CF}(X)$. Define the *pushforward* ${\rm CF}(\phi)f:Y(\mathbb{C})\to\mathbb{Q}$ of f to Y by

$${\rm CF}(\phi)f(y)=\chi_{\rm an}(X,f\cdot\delta_{\phi_*^{-1}(y)})\quad\text{for } y\in Y(\mathbb{C}). \tag{3}$$

Here $\phi_*:X(\mathbb{C})\to Y(\mathbb{C})$ is the induced map, $\phi_*^{-1}(y)\subseteq X(\mathbb{C})$ is the inverse image of $\{y\}$ under ϕ_*, and $\delta_{\phi_*^{-1}(y)}$ is its characteristic function. It is a locally constructible function, so $f\cdot\delta_{\phi_*^{-1}(y)}\in{\rm CF}(X)$, and (3) is well-defined.

MacPherson [**18**, Proposition 1] gives an important property of the pushforward for algebraic $\mathbb{C}$-varieties. The extension to $\mathbb{C}$-schemes is straightforward. One can prove it by dividing X,Y into pieces upon which ϕ is a locally trivial fibration in the analytic topology, and using Proposition 3.6(ii) and (iv).

THEOREM 3.8. *Let X,Y,Z be $\mathbb{C}$-schemes, let $\phi:X\to Y$ and $\psi:Y\to Z$ be morphisms, and let $f\in{\rm CF}(X)$. Then ${\rm CF}(\phi)f$ is constructible, so ${\rm CF}(\phi):{\rm CF}(X)\to{\rm CF}(Y)$ is a $\mathbb{Q}$-linear map. Also ${\rm CF}(\psi\circ\phi)={\rm CF}(\psi)\circ{\rm CF}(\phi)$ as linear maps ${\rm CF}(X)\to{\rm CF}(Z)$. Hence,* CF *is a functor from the category of $\mathbb{C}$-schemes to the category of $\mathbb{Q}$-vector spaces.*

Viro [**24**] gives an interesting point of view on constructible functions. One can regard the Euler characteristic as a *measure*, defined on constructible sets. Then $\chi_{\rm an}(X,f)$ in (2) is the *integral* of f with respect to this measure, and the pushforward ${\rm CF}(\phi)f$ integrates f over the fibres of ϕ.

3.3. *Extension to other fields* $\mathbb{K}$

To extend § 3.2 to other fields $\mathbb{K}$, we need a good notion of Euler characteristic $\chi(X)$ for a separated $\mathbb{K}$-scheme X of finite type.

DEFINITION 3.9. Let $\mathbb{K}$ be an algebraically closed field of characteristic p, which may be zero, and fix a prime number $l\neq p$. Write $\mathbb{Q}_l$ for the field of *l-adic rationals*. Let X be a separated $\mathbb{K}$-scheme of finite type. Then one may define the *compactly-supported l-adic cohomology groups* $H^i_{\rm cs}(X,\mathbb{Q}_l)$ of X, for $i\geqslant 0$. The original references for étale and l-adic cohomology are from Grothendieck *et al.* [**1**, **2**, **7**], and a good book is Milne [**20**]. Define the *Euler characteristic* $\chi(X)$ of X

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to be

$$\chi(X) = \sum_{i=0}^{2\dim X} (-1)^i \dim_{\mathbb{Q}_l} H^i_{\rm cs}(X, \mathbb{Q}_l). \tag{4}$$

We now give some properties of χ, generalizing Proposition 3.6.

THEOREM 3.10. *Let $\mathbb{K}$ be an algebraically closed field and let X, Y be separated $\mathbb{K}$-schemes of finite type. Then:*

(i) *if Z is a closed subscheme of X, then $\chi(X) = \chi(Z) + \chi(X \setminus Z)$;*
(ii) *if X is the disjoint union of subschemes $U_1, \ldots, U_m$, then*

$$\chi(X) = \sum_{i=1}^{m} \chi(U_i);$$

(iii) $\chi(X \times Y) = \chi(X)\chi(Y)$;
(iv) *$\chi(X)$ is independent of the choice of l in Definition 3.9;*
(v) *when $\mathbb{K} = \mathbb{C}$ we have $\chi(X) = \chi_{\rm an}(X)$;*
(vi) *$\chi(\mathbb{K}^m) = 1$ and $\chi(\mathbb{KP}^m) = m + 1$ for all $m \geqslant 0$.*

Proof. Part (i) comes from the long exact sequence [**1**, (XVII.5.1.16.3)]:

$$\cdots \to H^i_{\rm cs}(X \setminus Z, \mathbb{Q}_l) \to H^i_{\rm cs}(X, \mathbb{Q}_l) \to H^i_{\rm cs}(Z, \mathbb{Q}_l) \to H^{i+1}_{\rm cs}(X \setminus Z, \mathbb{Q}_l) \to \cdots,$$

and part (ii) follows from part (i) by standard arguments. Part (iii) is a consequence of the *Künneth formula* [**1**, Théorème XVII.5.4.3]. Part (iv) is proved for X *proper* in [**7**, § VII.4.10]. The general case follows from part (i) as we may write $X \cong \bar{X} \setminus Z$ for $\bar{X}$ a proper separated $\mathbb{K}$-scheme of finite type, and Z a closed subscheme. Part (v) follows from the *comparison theorem* [**1**, Théorème XVI.4.1]. For part (vi), calculation shows $H^i_{\rm cs}(\mathbb{K}^m, \mathbb{Q}_l)$ is $\mathbb{Q}_l$ if $i = 2m$ and 0 otherwise, so $\chi(\mathbb{K}^m) = 1$. Then $\chi(\mathbb{KP}^m) = m + 1$ by part (ii) and $\mathbb{KP}^m = \coprod_{n=0}^{m} \mathbb{K}^n$. □

We now generalize of Definition 3.7 and Theorem 3.8 to $\mathbb{K}$.

DEFINITION 3.11. Let X be a $\mathbb{K}$-scheme and let $C \subseteq X(\mathbb{K})$ be a constructible subset. Write $C = \coprod_{i \in I} X_i(\mathbb{K})$ as in Proposition 3.2, and define $\chi(C) = \sum_{i \in I} \chi(X_i)$. This is well-defined as in Definition 3.7, using Theorem 3.10(ii). For $f \in {\rm CF}(X)$, define the *weighted Euler characteristic* $\chi(X, f) \in \mathbb{Q}$ by

$$\chi(X, f) = \sum_{c \in f(X(\mathbb{K})) \setminus \{0\}} c\, \chi(f^{-1}(c)). \tag{5}$$

Then $f \mapsto \chi(X, f)$ is a linear map ${\rm CF}(X) \to \mathbb{Q}$. Let $\phi : X \to Y$ be a morphism of $\mathbb{K}$-schemes. Define the *pushforward* ${\rm CF}(\phi) f : Y(\mathbb{K}) \to \mathbb{Q}$ of f to Y by

$${\rm CF}(\phi) f(y) = \chi(X, f \cdot \delta_{\phi_*^{-1}(y)}) \quad \text{for } y \in Y(\mathbb{K}). \tag{6}$$

When $\mathbb{K} = \mathbb{C}$ these definitions agree with Definition 3.7 by Theorem 3.10(v).

There are several ways to prove the next theorem. One is to use results of Kennedy [**15**]. He defines pushforwards implicitly using intersections of Lagrangian cycles, but one can show, using base change and comparison theorems for l-adic cohomology, that his definition of ${\rm CF}(\phi)$ agrees with ours. Another is to use Katz

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and Laumon [**14**, Theorem 3.1.2], which in characteristic zero relates pushforwards of constructible sheaves and functions, so the functoriality of CF follows from that for sheaf pushforwards.

THEOREM 3.12. *Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, let X, Y, Z be $\mathbb{K}$-schemes, let $\phi : X \to Y$ and $\psi : Y \to Z$ be morphisms, and let $f \in \mathrm{CF}(X)$. Then $\mathrm{CF}(\phi)f$ is constructible, so $\mathrm{CF}(\phi) : \mathrm{CF}(X) \to \mathrm{CF}(Y)$ is a $\mathbb{Q}$-linear map. Also $\mathrm{CF}(\psi \circ \phi) = \mathrm{CF}(\psi) \circ \mathrm{CF}(\phi)$, so CF is a functor.*

The last part of the theorem is *false* for $\mathbb{K}$ of characteristic $p > 0$, and we are grateful to Jörg Schürmann for the following explanation. The proof using constructible sheaves fails because if $\mathcal{L}$ is a locally constant $\mathbb{Q}_l$-sheaf of rank r on a non-proper $\mathbb{K}$-scheme X for $l \neq p$, we need the fact that

$$\chi(H^*_{\mathrm{cs}}(X, \mathcal{L})) = r \cdot \chi(H^*_{\mathrm{cs}}(X, \mathbb{Q}_l)).$$

This holds in characteristic zero, but not in characteristic $p > 0$ without extra conditions on $\mathcal{L}$, which are studied in Illusie [**8**].

We now give a counterexample to Theorem 3.12 in positive characteristic. Let $\mathbb{K}$ have characteristic $p \geqslant 2$, and let $\phi : \mathbb{K} \to \mathbb{K}$ be the Artin–Schreier morphism $\phi : x \mapsto x^p - x$. It is a p-fold étale covering of $\mathbb{K}$ by itself, so $\mathrm{CF}(\phi)1 = p$ in $\mathrm{CF}(\mathbb{K})$. Thus, taking $\psi : \mathbb{K} \to \operatorname{Spec} \mathbb{K}$ to be the projection we have $\mathrm{CF}(\psi \circ \phi)1 = 1$, but $\mathrm{CF}(\psi) \circ \mathrm{CF}(\phi)1 = p$ in $\mathrm{CF}(\operatorname{Spec} \mathbb{K}) = \mathbb{Q}$, so $\mathrm{CF}(\psi \circ \phi) \neq \mathrm{CF}(\psi) \circ \mathrm{CF}(\phi)$.

When $Z = \operatorname{Spec} \mathbb{K}$ and $\psi : Y \to \operatorname{Spec} \mathbb{K}$ is the projection we have $\mathrm{CF}(Z) = \mathbb{Q}$ and $\mathrm{CF}(\psi)g = \chi(Y, g)$ for $g \in \mathrm{CF}(Y)$. So $\mathrm{CF}(\psi \circ \phi) = \mathrm{CF}(\psi) \circ \mathrm{CF}(\phi)$ gives a relation between pushforwards and weighted Euler characteristics.

COROLLARY 3.13. *Let $\mathbb{K}$ have characteristic zero, let X, Y be $\mathbb{K}$-schemes, let $\phi : X \to Y$ be a morphism, and let $f \in \mathrm{CF}(X)$. Then $\chi(X, f) = \chi(Y, \mathrm{CF}(\phi)f)$.*

3.4. *Extension to pseudomorphisms*

We define *pseudomorphisms*, a notion of morphism between locally constructible sets that generalizes morphisms of schemes.

DEFINITION 3.14. Let $\mathbb{K}$ be an algebraically closed field, let X, Y be $\mathbb{K}$-schemes, let $S \subseteq X(\mathbb{K})$, $T \subseteq Y(\mathbb{K})$ be locally constructible and let $\Phi : S \to T$ be a map. Define the *graph* $\Gamma_\Phi = \{(s, \Phi(s)) : s \in S\}$ in $X(\mathbb{K}) \times Y(\mathbb{K}) = (X \times Y)(\mathbb{K})$. We call Φ a *pseudomorphism* if $\Gamma_\Phi \cap (C \times Y(\mathbb{K}))$ is constructible for all constructible $C \subseteq X(\mathbb{K})$. This implies that Γ_Φ is locally constructible.

A pseudomorphism Φ is a *pseudoisomorphism* if Φ is bijective and $\Phi^{-1} : T \to S$ is a pseudomorphism. When $S = X(\mathbb{K})$ and $T = Y(\mathbb{K})$ we shall also call $\Phi : X \to Y$ a pseudomorphism (pseudoisomorphism) from X to Y.

When X, Y are $\mathbb{K}$-*varieties*, pseudomorphisms $\Phi : X \to Y$ coincide with *definable functions* in the model theory of algebraic geometry. See, for instance, Marker [**19**, § 7.4], in particular [**19**, Lemma 7.4.7] which shows that Φ restricts to a *quasimorphism* of varieties on a nonempty open affine subset X_0 of X, and a *morphism* if $\mathbb{K}$ has characteristic zero. Here are some basic properties of pseudomorphisms.

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They are easily proved using Proposition 3.3 and the projection morphisms $X\times Y\to Y$, $X\times Y\times Z\to X\times Z$.

PROPOSITION 3.15. *Let $\mathbb{K}$ be an algebraically closed field.*

(a) *Let $\phi:X\to Y$ be a morphism (isomorphism) of $\mathbb{K}$-schemes. Then $\phi_*:X(\mathbb{K})\to Y(\mathbb{K})$ is a pseudomorphism (pseudoisomorphism).*

(b) *Let X,Y be $\mathbb{K}$-schemes, let $S\subseteq X(\mathbb{K})$, $T\subseteq Y(\mathbb{K})$ be locally constructible, let $\Phi:S\to T$ be a pseudomorphism, and let $C\subseteq S$ be constructible. Then $\Phi(C)$ is constructible in $Y(\mathbb{K})$. Also, if $t\in T$, then $C\cap\Phi^{-1}(t)$ is constructible in $X(\mathbb{K})$. Hence, $\Phi^{-1}(t)$ is locally constructible in $X(\mathbb{K})$.*

(c) *Let X,Y,Z be $\mathbb{K}$-schemes, let $S\subseteq X(\mathbb{K})$, $T\subseteq Y(\mathbb{K})$, $U\subseteq Z(\mathbb{K})$ be locally constructible, and let $\Phi:S\to T$, $\Psi:T\to U$ be pseudo(iso)morphisms. Then $\Psi\circ\Phi:S\to U$ is a pseudo(iso)morphism.*

We define *pushforwards* $\mathrm{CF}(\Phi):\mathrm{CF}(S)\to\mathrm{CF}(T)$ along pseudomorphisms.

DEFINITION 3.16. Let X,Y be $\mathbb{K}$-schemes, let $S\subseteq X(\mathbb{K})$, $T\subseteq Y(\mathbb{K})$ be locally constructible, let $\Phi:S\to T$ be a pseudomorphism, and let $f\in\mathrm{CF}(S)$. Define the *pushforward* $\mathrm{CF}(\Phi)f:T\to\mathbb{Q}$ by

$$\mathrm{CF}(\Phi)f(t)=\chi(S,f\cdot\delta_{\Phi^{-1}(t)})\quad\text{for } t\in T. \tag{7}$$

Here $\delta_{\Phi^{-1}(t)}$ is the characteristic function of $\Phi^{-1}(t)\subseteq S$ on S. By Proposition 3.15(b) $\delta_{\Phi^{-1}(t)}\in\mathrm{LCF}(S)$, and $f\in\mathrm{CF}(S)$, so $f\cdot\delta_{\Phi^{-1}(t)}\in\mathrm{CF}(S)$. Thus, (7) is well-defined, by Definition 3.11. If $\phi:X\to Y$ is a morphism of $\mathbb{K}$-schemes, then $\phi_*:X(\mathbb{K})\to Y(\mathbb{K})$ is a pseudomorphism by Proposition 3.15(a), and $\mathrm{CF}(\phi)$ in Definition 3.11 coincides with $\mathrm{CF}(\phi_*)$ above.

We now generalize Theorems 3.8 and 3.12 to pseudomorphisms.

THEOREM 3.17. *Let $\mathbb{K}$ have characteristic zero, let X,Y,Z be $\mathbb{K}$-schemes, let $S\subseteq X(\mathbb{K})$, $T\subseteq Y(\mathbb{K})$, $U\subseteq Z(\mathbb{K})$ be locally constructible, let $\Phi:S\to T$, $\Psi:T\to U$ be pseudomorphisms, and let $f\in\mathrm{CF}(S)$. Then $\mathrm{CF}(\Phi)f$ is constructible, so $\mathrm{CF}(\Phi):\mathrm{CF}(S)\to\mathrm{CF}(T)$ is $\mathbb{Q}$-linear, and $\mathrm{CF}(\Psi\circ\Phi)=\mathrm{CF}(\Psi)\circ\mathrm{CF}(\Phi)$.*

Proof. Define $F_{XY}:X(\mathbb{K})\times Y(\mathbb{K})\to\mathbb{Q}$, $F_{XZ}:X(\mathbb{K})\times Z(\mathbb{K})\to\mathbb{Q}$, $F_{YZ}:Y(\mathbb{K})\times Z(\mathbb{K})\to\mathbb{Q}$ and $F_{XYZ}:X(\mathbb{K})\times Y(\mathbb{K})\times Z(\mathbb{K})\to\mathbb{Q}$ by

$$\begin{aligned}
F_{XY}(x,y)&=\begin{cases}f(x), & x\in S\text{ and }y=\Phi(x),\\ 0, & \text{otherwise,}\end{cases}\\
F_{XZ}(x,z)&=\begin{cases}f(x), & x\in S\text{ and }z=\Psi\circ\Phi(x),\\ 0, & \text{otherwise,}\end{cases}\\
F_{YZ}(y,z)&=\begin{cases}(\mathrm{CF}(\Phi)f)(y), & y\in T\text{ and }z=\Psi(y),\\ 0, & \text{otherwise.}\end{cases}\\
F_{XYZ}(x,y,z)&=\begin{cases}f(x), & x\in S,\ y=\Phi(x)\text{ and }z=\Psi(y)\\ 0, & \text{otherwise.}\end{cases}
\end{aligned}\tag{8}$$

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Write $\Pi_Y^{XY} : X \times Y \to Y$ for the projection morphism, and so on. It is easy to show $F_{XY} \in \mathrm{CF}(X \times Y)$, so $\mathrm{CF}(\Pi_Y^{XY})F_{XY} \in \mathrm{CF}(Y)$ by Theorem 3.12. However, comparing (6), (7) and (8) shows $\mathrm{CF}(\Phi)f = (\mathrm{CF}(\Pi_Y^{XY})F_{XY})|_T$. Therefore, $\mathrm{CF}(\Phi)f \in \mathrm{CF}(T)$, proving the first part. For the second part, F_{XZ}, F_{YZ}, F_{XYZ} are constructible on $X \times Z, Y \times Z$ and $X \times Y \times Z$ in the same way, and it is easy to prove that

$$\mathrm{CF}(\Psi \circ \Phi)f = (\mathrm{CF}(\Pi_Z^{XZ})F_{XZ})|_U, \quad \mathrm{CF}(\Psi) \circ \mathrm{CF}(\Phi)f = (\mathrm{CF}(\Pi_Z^{YZ})F_{YZ})|_U,$$
$$\mathrm{CF}(\Pi_{XZ}^{XYZ})F_{XYZ} = F_{XZ} \quad \text{and} \quad \mathrm{CF}(\Pi_{YZ}^{XYZ})F_{XYZ} = F_{YZ}.$$

However, as $\Pi_Z^{XZ} \circ \Pi_{XZ}^{XYZ} = \Pi_Z^{XYZ} = \Pi_Z^{YZ} \circ \Pi_{YZ}^{XYZ}$, Theorem 3.12 gives $\mathrm{CF}(\Pi_Z^{XZ}) \circ \mathrm{CF}(\Pi_{XZ}^{XYZ}) = \mathrm{CF}(\Pi_Z^{XYZ}) = \mathrm{CF}(\Pi_Z^{YZ}) \circ \mathrm{CF}(\Pi_{YZ}^{XYZ})$, and the result follows. □

If $\Phi : S \to T$ is a *pseudoisomorphism*, then $\Phi^{-1}(t)$ is a single point in (7), giving $\mathrm{CF}(\Phi)f(t) = f \circ \Phi^{-1}(t)$. We deduce the following.

COROLLARY 3.18. *Let X, Y be $\mathbb{K}$-schemes, let $S \subseteq X(\mathbb{K})$, $T \subseteq Y(\mathbb{K})$ be locally constructible, and let $\Phi : S \to T$ be a pseudoisomorphism. Then $\mathrm{CF}(\Phi) : \mathrm{CF}(S) \to \mathrm{CF}(T)$ is an isomorphism, with $\mathrm{CF}(\Phi)f = f \circ \Phi^{-1}$ and $\mathrm{CF}(\Phi^{-1})g = g \circ \Phi$.*

The moral is that pseudoisomorphic (locally) constructible sets are essentially *the same* from the point of view of constructible functions. So, in problems involving constructible functions, we can work with (*locally*) *constructible sets up to pseudoisomorphism*, and pseudomorphisms between them.

4. *Constructible functions on stacks*

We now generalize § 3 to stacks. §§ 4.1 and 4.2 develop the basic definitions and properties of constructible sets and functions, and show that any finite type algebraic $\mathbb{K}$-stack $\mathfrak{F}$ with affine geometric stabilizers is pseudoisomorphic to a finite type $\mathbb{K}$-scheme. This enables us to reduce to the scheme case of § 3.

An important difference between stacks and schemes is that points $x \in \mathfrak{F}(\mathbb{K})$ in a $\mathbb{K}$-stack $\mathfrak{F}$ have *stabilizer groups* $\mathrm{Iso}_{\mathbb{K}}(x)$, which are trivial if $\mathfrak{F}$ is a $\mathbb{K}$-scheme. There are *many different ways* of including stabilizer groups when extending Euler characteristics χ and pushforwards CF to stacks. § 4.3 studies the simplest of these, the *naïve* versions $\chi^{\mathrm{na}}, \mathrm{CF}^{\mathrm{na}}$, which just ignore stabilizer groups.

Given an *allowable weight function* w upon affine algebraic $\mathbb{K}$-groups, in § 4.4 we modify $\chi^{\mathrm{na}}, \mathrm{CF}^{\mathrm{na}}$ to get χ_w, CF_w by weighting by $w_{\mathfrak{F}} : x \mapsto w(\mathrm{Iso}_{\mathbb{K}}(x))$ on $\mathfrak{F}(\mathbb{K})$. Two special cases are the *stack* versions $\chi^{\mathrm{stk}}, \mathrm{CF}^{\mathrm{stk}}$, which are most natural in many problems, and the *orbifold* versions $\chi^{\mathrm{orb}}, \mathrm{CF}^{\mathrm{orb}}$, related to Deligne–Mumford stacks and their crepant resolutions.

4.1. *Basic definitions*

We begin by giving analogues for stacks of the major definitions of § 3.

DEFINITION 4.1. Let $\mathbb{K}$ be an algebraically closed field, and let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack. We call $C \subseteq \mathfrak{F}(\mathbb{K})$ *constructible* if $C = \bigcup_{i \in I} \mathfrak{F}_i(\mathbb{K})$, where $\{\mathfrak{F}_i : i \in I\}$ is a finite collection of finite type algebraic $\mathbb{K}$-substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. We call $S \subseteq \mathfrak{F}(\mathbb{K})$ *locally constructible* if $S \cap C$ is constructible for all constructible $C \subseteq \mathfrak{F}(\mathbb{K})$.

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We now give a partial analogue of Proposition 3.3, proved in the same way.

LEMMA 4.2. *Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack and $A, B \subseteq \mathfrak{F}(\mathbb{K})$ constructible subsets. Then $A \cup B$, $A \cap B$ and $A \setminus B$ are constructible in $\mathfrak{F}(\mathbb{K})$.*

DEFINITION 4.3. Let $\mathbb{K}$ be an algebraically closed field, $\mathfrak{F}$ an algebraic $\mathbb{K}$-stack, and $S \subseteq \mathfrak{F}(\mathbb{K})$ be locally constructible. Call a function $f : S \to \mathbb{Q}$ *constructible* if $f(S)$ is finite and $f^{-1}(c)$ is a constructible set for each $c \in f(S) \setminus \{0\}$. Call $f : S \to \mathbb{Q}$ *locally constructible* if $f|_C$ is constructible for all constructible $C \subseteq S \subseteq \mathfrak{F}(\mathbb{K})$. Write $\mathrm{CF}(S), \mathrm{LCF}(S)$ for the sets of (locally) constructible functions on S. Using Lemma 4.2 we see that $\mathrm{CF}(S), \mathrm{LCF}(S)$ are $\mathbb{Q}$-vector spaces. For brevity write $\mathrm{CF}(\mathfrak{F}), \mathrm{LCF}(\mathfrak{F})$ rather than $\mathrm{CF}(\mathfrak{F}(\mathbb{K})), \mathrm{LCF}(\mathfrak{F}(\mathbb{K}))$. Note that if $f \in \mathrm{CF}(S)$ and $g \in \mathrm{LCF}(S)$, then $fg \in \mathrm{CF}(S)$.

Now let $\mathfrak{F}, \mathfrak{G}$ be algebraic $\mathbb{K}$-stacks, and let $S \subseteq \mathfrak{F}(\mathbb{K})$, $T \subseteq \mathfrak{G}(\mathbb{K})$ be locally constructible. Then $\mathfrak{F} \times \mathfrak{G}$ is an algebraic $\mathbb{K}$-stack with $(\mathfrak{F} \times \mathfrak{G})(\mathbb{K}) = \mathfrak{F}(\mathbb{K}) \times \mathfrak{G}(\mathbb{K})$. Let $\Phi : S \to T$ be a map, and define the *graph* $\Gamma_\Phi = \{(s, \Phi(s)) : s \in S\}$. We call Φ a *pseudomorphism* if $\Gamma_\Phi \cap (C \times \mathfrak{G}(\mathbb{K}))$ is constructible in $(\mathfrak{F} \times \mathfrak{G})(\mathbb{K})$ for all constructible $C \subseteq \mathfrak{F}(\mathbb{K})$. A pseudomorphism Φ is a *pseudoisomorphism* if Φ is bijective and $\Phi^{-1} : T \to S$ is a pseudomorphism.

These definitions agree with those of § 3 when $\mathfrak{F}, \mathfrak{G}$ are $\mathbb{K}$-schemes.

4.2. *Constructible sets and pseudomorphisms in stacks*

We now extend properties of constructible sets and pseudomorphisms in $\mathbb{K}$-schemes to algebraic $\mathbb{K}$-stacks with affine geometric stabilizers.

PROPOSITION 4.4. *Let $\mathfrak{F}$ be a finite type algebraic $\mathbb{K}$-stack with affine geometric stabilizers. Then there exist substacks $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ of $\mathfrak{F}$ with $\mathfrak{F}(\mathbb{K}) = \coprod_{a=1}^n \mathfrak{F}_a(\mathbb{K})$, $\mathbb{K}$-varieties $Y_1, \ldots, Y_n$, and 1-morphisms $\phi_a : \mathfrak{F}_a \to Y_a$ with $(\phi_a)_* : \mathfrak{F}_a(\mathbb{K}) \to Y_a(\mathbb{K})$ bijective for $a = 1, \ldots, n$.*

Proof. By Theorem 2.7, $\mathfrak{F}$ can be stratified by global quotient stacks. Thus, there exist finitely many substacks U_i of $\mathfrak{F}$ with $\mathfrak{F}(\mathbb{K}) = \coprod_i U_i(\mathbb{K})$, and each U_i 1-isomorphic to a quotient stack $[X_i/G_i]$, for X_i a $\mathbb{K}$-variety and G_i an affine algebraic $\mathbb{K}$-group acting on X_i. Theorem 2.3 gives a dense open G_i-invariant $X_i' \subseteq X_i$, and a morphism of $\mathbb{K}$-varieties $\pi_i : X_i' \to Y_i$ inducing a bijection $X_i'(\mathbb{K})/G_i \to Y_i(\mathbb{K})$. We may write $X_i \setminus X_i'$ as a disjoint union of finitely many G_i-invariant $\mathbb{K}$-subvarieties X_{ij} with $\dim X_{ij} < \dim X_i$. Applying Theorem 2.3 again gives a dense open G_i-invariant $X_{ij}' \subseteq X_{ij}$ and a morphism of $\mathbb{K}$-varieties $\pi_{ij} : X_{ij}' \to Y_{ij}$ inducing a bijection $X_{ij}'(\mathbb{K})/G_{ij} \to Y_{ij}(\mathbb{K})$.

As the dimension decreases at each stage, this process eventually yields finitely many substacks $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ of $\mathfrak{F}$ with $\mathfrak{F}(\mathbb{K}) = \coprod_{a=1}^n \mathfrak{F}_a(\mathbb{K})$, such that $\mathfrak{F}_a$ is 1-isomorphic to $[X_a/G_a]$ for $\mathbb{K}$-varieties X_a, Y_a and G_a an algebraic $\mathbb{K}$-group acting on X_a, and morphisms $\pi_a : X_a \to Y_a$ inducing a bijection between $X_a(\mathbb{K})/G_a$ and $Y_a(\mathbb{K})$. The 1-isomorphisms $\mathfrak{F}_a \cong [X_a/G_a]$ and π_a combine to give a 1-morphism $\phi_a : \mathfrak{F}_a \to Y_a$ with the properties we want. □

We extend the last part of Proposition 3.3 to stacks.

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PROPOSITION 4.5. *Let $\mathbb{K}$ be an algebraically closed field, let $\mathfrak{F},\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $\phi:\mathfrak{F}\to\mathfrak{G}$ be a 1-morphism, and let $C\subseteq\mathfrak{F}(\mathbb{K})$ be constructible. Then $\phi_*(C)$ is constructible in $\mathfrak{G}(\mathbb{K})$.*

Proof. By Definition 4.1, $C=\bigcup_{i\in I}\mathfrak{F}_i(\mathbb{K})$, where $\{\mathfrak{F}_i:i\in I\}$ are finitely many finite type substacks $\mathfrak{F}_i$ of $\mathfrak{F}$. So by Lemma 4.2 it is enough to show each $\phi_*(\mathfrak{F}_i(\mathbb{K}))$ is constructible. As by convention $\mathfrak{G}$ is locally of finite type it admits an open cover $\{\mathfrak{G}_j:j\in J\}$ of finite type substacks $\mathfrak{G}_j$. By Proposition 4.4, for $a=1,\ldots,n_j$ there exist substacks $\mathfrak{G}_{ja}$ of $\mathfrak{G}_j$, $\mathbb{K}$-varieties Y_{ja} and 1-morphisms $\psi_{ja}:\mathfrak{G}_{ja}\to Y_{ja}$ with $(\psi_{ja})_*$ bijective, such that $\mathfrak{G}_j(\mathbb{K})=\coprod_{a=1}^{n_j}\mathfrak{G}_{ja}(\mathbb{K})$.

Now $\{\phi^{-1}(\mathfrak{G}_j):j\in J\}$ covers $\mathfrak{F}_i$, which is *quasicompact* as it is of finite type. So there exists a finite subset $J_i\subseteq J$ such that $\{\phi^{-1}(\mathfrak{G}_j):j\in J_i\}$ covers $\mathfrak{F}_i$. Set $\mathfrak{F}_{ija}=\mathfrak{F}_i\cap\phi^{-1}(\mathfrak{G}_{ja})$ for $j\in J_i$ and $a=1,\ldots,n_j$. Then $\mathfrak{F}_{ija}$ is a finite type $\mathbb{K}$-substack, and $\phi_{ija}=\phi|_{\mathfrak{F}_{ija}}:\mathfrak{F}_{ija}\to\mathfrak{G}_{ja}$ a 1-morphism with $\phi_*(\mathfrak{F}_i(\mathbb{K}))=\bigcup_{j\in J_i}\bigcup_{a=1}^{n_j}(\phi_{ija})_*(\mathfrak{F}_{ija}(\mathbb{K}))$. Hence, by Lemma 4.2 it suffices to show that $(\phi_{ija})_*(\mathfrak{F}_{ija}(\mathbb{K}))$ is constructible in $\mathfrak{G}_{ja}(\mathbb{K})$ for all i,j,a.

As $\mathfrak{F}_{ija}$ is finite type it has an atlas $u_{ija}:U_{ija}\to\mathfrak{F}_{ija}$ with U_{ija} a finite type $\mathbb{K}$-scheme. Then $\psi_{ja}\circ\phi_{ija}\circ u_{ija}:U_{ija}\to Y_{ja}$ is a morphism of $\mathbb{K}$-schemes. However, $U_{ija}(\mathbb{K})$ is constructible as U_{ija} is of finite type, so Proposition 3.3 shows that

$$\begin{aligned}(\psi_{ja}\circ\phi_{ija}\circ u_{ija})_*(U_{ija}(\mathbb{K}))&=(\psi_{ja})_*\circ(\phi_{ija})_*\circ(u_{ija})_*(U_{ija}(\mathbb{K}))\\&=(\psi_{ja})_*\circ(\phi_{ija})_*(\mathfrak{F}_{ija}(\mathbb{K}))\end{aligned}$$

is constructible in $Y_{ja}(\mathbb{K})$, where the second line follows since $(u_{ija})_*$ is surjective as u_{ija} is an atlas. Now $\psi_{ja}:\mathfrak{G}_{ja}\to Y_{ja}$ is a finite type 1-morphism, so it pulls back constructible subsets to constructible subsets. Therefore,

$$(\psi_{ja})_*^{-1}((\psi_{ja})_*\circ(\phi_{ija})_*(\mathfrak{F}_{ija}(\mathbb{K})))=(\phi_{ija})_*(\mathfrak{F}_{ija}(\mathbb{K}))$$

is constructible in $\mathfrak{G}_{ja}(\mathbb{K})$, using $(\psi_{ja})_*$ a bijection in the second step. □

Lemma 4.2 and Proposition 4.5 extend Proposition 3.3 to algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. As the proof of Proposition 3.15 depends only on Proposition 3.3, it extends to such stacks.

PROPOSITION 4.6. *Let $\mathbb{K}$ be an algebraically closed field, and let $\mathfrak{F},\mathfrak{G},\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers.*

(a) *Let $\phi:\mathfrak{F}\to\mathfrak{G}$ be a 1-morphism (1-isomorphism). Then $\phi_*:\mathfrak{F}(\mathbb{K})\to\mathfrak{G}(\mathbb{K})$ is a pseudomorphism (pseudoisomorphism).*

(b) *Let $S\subseteq\mathfrak{F}(\mathbb{K})$, $T\subseteq\mathfrak{G}(\mathbb{K})$ be locally constructible, let $\Phi:S\to T$ be a pseudomorphism, and let $C\subseteq S$ be constructible. Then $\Phi(C)$ is constructible in $\mathfrak{G}(\mathbb{K})$. Also, if $t\in T$, then $C\cap\Phi^{-1}(t)$ is constructible in $\mathfrak{F}(\mathbb{K})$. Hence, $\Phi^{-1}(t)$ is locally constructible in $\mathfrak{F}(\mathbb{K})$.*

(c) *Let $S\subseteq\mathfrak{F}(\mathbb{K})$, $T\subseteq\mathfrak{G}(\mathbb{K})$, $U\subseteq\mathfrak{H}(\mathbb{K})$ be locally constructible, and let $\Phi:S\to T$, $\Psi:T\to U$ be pseudo(iso)morphisms. Then $\Psi\circ\Phi:S\to U$ is a pseudo(iso)morphism.*

The following proposition allows results about constructible sets and functions on schemes to be easily extended to stacks.

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PROPOSITION 4.7. *Let $\mathbb{K}$ be an algebraically closed field, let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and let $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible. Then C is pseudoisomorphic to $Y(\mathbb{K})$ for a separated, finite type $\mathbb{K}$-scheme Y.*

Proof. Write $C = \coprod_{i\in I} \mathfrak{F}_i(\mathbb{K})$ for $\mathfrak{F}_i, i \in I$ finitely many finite type substacks in $\mathfrak{F}$. Proposition 4.4 gives substacks $\mathfrak{F}_{ia}$ in $\mathfrak{F}_i$, $\mathbb{K}$-varieties Y_{ia}, and 1-morphisms $\phi_{ia} : \mathfrak{F}_{ia} \to Y_{ia}$ for $a = 1, \ldots, n_i$, with $\mathfrak{F}_i(\mathbb{K}) = \coprod_{a=1}^{n_i} \mathfrak{F}_{ia}(\mathbb{K})$, and $(\phi_{ia})_*$ bijective. Let Y be the abstract disjoint union of the Y_{ia} for $i \in I$ and $a = 1, \ldots, n_i$, as in Definition 2.2. It is a separated, finite type $\mathbb{K}$-scheme. Define $\Phi : C \to Y(\mathbb{K})$ by $\Phi|_{\mathfrak{F}_{ia}(\mathbb{K})} = (\phi_{ia})_*$ for all i, a. Then Φ is bijective, as $(\phi_{ia})_*$ is. Proposition 4.6(a) shows that $(\phi_{ia})_*$ is a pseudomorphism, so Φ is a pseudomorphism. As Φ is bijective and $C, Y(\mathbb{K})$ are constructible, Φ is a pseudoisomorphism. □

4.3. *The naïve Euler characteristic and pushforward*

Fix an algebraically closed field $\mathbb{K}$ of *characteristic zero* for the rest of the section. We consider the simplest generalization of $\chi(X), \chi(S, f), \mathrm{CF}(\Phi)$ to $\mathbb{K}$-stacks $\mathfrak{F}$, which we call *naïve* as it ignores the *stabilizer groups* $\mathrm{Iso}_{\mathbb{K}}(x)$ for $x \in \mathfrak{F}(\mathbb{K})$. We now give the analogue of Definitions 3.11 and 3.16.

DEFINITION 4.8. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, and let $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible. Then C is pseudoisomorphic to $Y(\mathbb{K})$ for a separated, finite type $\mathbb{K}$-scheme Y by Proposition 4.7. Define the *naïve Euler characteristic* $\chi^{\mathrm{na}}(C)$ by $\chi^{\mathrm{na}}(C) = \chi(Y)$, where $\chi(Y)$ is as in Definition 3.9. If Y' is another choice for Y, then $Y(\mathbb{K})$ is pseudoisomorphic to $Y'(\mathbb{K})$ by Proposition 4.6(c), so $\chi(Y) = \chi(Y')$. Thus, $\chi^{\mathrm{na}}(C)$ is well-defined. Now let $S \subseteq \mathfrak{F}(\mathbb{K})$ be locally constructible. For $f \in \mathrm{CF}(S)$, define the *naïve weighted Euler characteristic* $\chi^{\mathrm{na}}(S, f) \in \mathbb{Q}$ by

$$\chi^{\mathrm{na}}(S, f) = \sum_{c\in f(S)\setminus\{0\}} c\,\chi^{\mathrm{na}}(f^{-1}(c)).$$

Suppose that $\mathfrak{F}, \mathfrak{G}$ are algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, $S \subseteq \mathfrak{F}(\mathbb{K})$, $T \subseteq \mathfrak{G}(\mathbb{K})$ are locally constructible, $\Phi : S \to T$ is a pseudomorphism, and $f \in \mathrm{CF}(S)$. Define the *naïve pushforward* $\mathrm{CF}^{\mathrm{na}}(\Phi)f : T \to \mathbb{Q}$ of f to T by

$$\mathrm{CF}^{\mathrm{na}}(\Phi)f(t) = \chi^{\mathrm{na}}(S, f \cdot \delta_{\Phi^{-1}(t)}) \quad \text{for } t \in T. \tag{9}$$

Here $\delta_{\Phi^{-1}(t)}$ is the characteristic function of $\Phi^{-1}(t) \subseteq S$ on S. As $\Phi^{-1}(t)$ is locally constructible by Proposition 4.6(b) we have $\delta_{\Phi^{-1}(t)} \in \mathrm{LCF}(S)$, and $f \in \mathrm{CF}(S)$, so $f \cdot \delta_{\Phi^{-1}(t)} \in \mathrm{CF}(S)$. Thus, (9) is well-defined.

We now give the naïve generalizations of Theorem 3.17 and Corollary 3.13.

THEOREM 4.9. *Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $S \subseteq \mathfrak{F}(\mathbb{K})$, $T \subseteq \mathfrak{G}(\mathbb{K})$, $U \subseteq \mathfrak{H}(\mathbb{K})$ be locally constructible, and let $\Phi : S \to T$, $\Psi : T \to U$ be pseudomorphisms. If $f \in \mathrm{CF}(S)$, then $\mathrm{CF}^{\mathrm{na}}(\Phi)f$ is a constructible function on T. Thus, $\mathrm{CF}^{\mathrm{na}}(\Phi) : \mathrm{CF}(S) \to \mathrm{CF}(T)$ is a $\mathbb{Q}$-linear map. Also $\mathrm{CF}^{\mathrm{na}}(\Psi \circ \Phi) = \mathrm{CF}^{\mathrm{na}}(\Psi) \circ \mathrm{CF}^{\mathrm{na}}(\Phi)$ as linear maps $\mathrm{CF}(S) \to \mathrm{CF}(U)$.*

Proof. Define $A = \mathrm{supp}(f) \subseteq S$, $B = \Phi(A) \subseteq T$ and $C = \Psi(B) \subseteq U$. Then A is constructible by Definition 4.3, so B, C are constructible by Proposition 4.6(b).

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By Proposition 4.7, there exist separated, finite type $\mathbb{K}$-schemes X, Y, Z and pseudo-isomorphisms $\alpha : A \to X(\mathbb{K})$, $\beta : B \to Y(\mathbb{K})$, and $\gamma : C \to Z(\mathbb{K})$. Then $f \circ \alpha^{-1} \in \mathrm{CF}(X)$ as $f|_A \in \mathrm{CF}(A)$. By Proposition 4.6(c), $\beta \circ \Phi \circ \alpha^{-1} : X(\mathbb{K}) \to Y(\mathbb{K})$ and $\gamma \circ \Psi \circ \beta^{-1} : Y(\mathbb{K}) \to Z(\mathbb{K})$ are pseudomorphisms of $\mathbb{K}$-schemes, so Theorem 3.17 gives $\mathrm{CF}(\beta \circ \Phi \circ \alpha^{-1})(f \circ \alpha^{-1}) \in \mathrm{CF}(Y)$ and

$$\mathrm{CF}(\gamma \circ \Psi \circ \Phi \circ \alpha^{-1})(f \circ \alpha^{-1}) = \mathrm{CF}(\gamma \circ \Psi \circ \beta^{-1}) \circ \mathrm{CF}(\beta \circ \Phi \circ \alpha^{-1})(f \circ \alpha^{-1}).$$

Since β, γ identify constructible sets and functions with constructible sets and functions, these easily imply $(\mathrm{CF}^{\mathrm{na}}(\Phi)f)|_B \in \mathrm{CF}(B)$ and

$$(\mathrm{CF}^{\mathrm{na}}(\Psi \circ \Phi)f)|_C = (\mathrm{CF}^{\mathrm{na}}(\Psi) \circ \mathrm{CF}^{\mathrm{na}}(\Phi)f)|_C.$$

As the unrestricted functions are zero outside B, C, the theorem follows. □

COROLLARY 4.10. *Let $\mathfrak{F}, \mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $S \subseteq \mathfrak{F}(\mathbb{K})$, $T \subseteq \mathfrak{G}(\mathbb{K})$ be locally constructible, let $\Phi : S \to T$ be a pseudomorphism, and let $f \in \mathrm{CF}(S)$. Then $\chi^{\mathrm{na}}(S, f) = \chi^{\mathrm{na}}(T, \mathrm{CF}^{\mathrm{na}}(\Phi)f)$.*

4.4. *Stabilizers* $\mathrm{Iso}_{\mathbb{K}}(x)$ *and weight functions*

We now discuss how to modify the naïve Euler characteristic χ^{na} and pushforward $\mathrm{CF}^{\mathrm{na}}$ of § 4.3 to take account of *stabilizer groups* $\mathrm{Iso}_{\mathbb{K}}(x)$ for $x \in \mathfrak{F}(\mathbb{K})$. We do this by inserting a *weight* $w_{\mathfrak{F}}$ depending on $\mathrm{Iso}_{\mathbb{K}}(x)$. We continue to fix $\mathbb{K}$ algebraically closed of characteristic zero.

DEFINITION 4.11. Let $w : \{\text{affine algebraic } \mathbb{K}\text{-groups } G\} \to \mathbb{Q} \cup \{\infty\}, G \mapsto w(G)$ be a map with $w(G) = w(G')$ if $G \cong G'$. If $\mathfrak{F}$ is an algebraic $\mathbb{K}$-stack with affine geometric stabilizers, define $w_{\mathfrak{F}} : \mathfrak{F}(\mathbb{K}) \to \mathbb{Q} \cup \{\infty\}$ by $w_{\mathfrak{F}}(x) = w(\mathrm{Iso}_{\mathbb{K}}(x))$. We call w an *allowable weight function* if $w_{\mathfrak{F}}$ is a *locally constructible function* on $\mathfrak{F}$ with values in $\mathbb{Q} \cup \{\infty\}$ for all $\mathfrak{F}$. We also call w *multiplicative* if $w(G \times H) = w(G)w(H)$ for all affine algebraic $\mathbb{K}$-groups G, H.

We now give the weighted analogues $\chi_w(C)$, $\chi_w(S, f)$, $\mathrm{CF}_w(\Phi)f$ of $\chi^{\mathrm{na}}(C)$, $\chi^{\mathrm{na}}(S, f)$, $\mathrm{CF}^{\mathrm{na}}(\Phi)f$. We allow w to take the values $0, \infty$ to accommodate the examples below. This means that $\chi_w(C), \chi_w(S, f), \mathrm{CF}_w(\Phi)f$ are not always defined.

DEFINITION 4.12. Let w be an allowable weight function, let $\mathfrak{F}, \mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible, let $S \subseteq \mathfrak{F}(\mathbb{K})$, $T \subseteq \mathfrak{G}(\mathbb{K})$ be locally constructible, and let $\Phi : S \to T$ be a pseudomorphism.

If $w_{\mathfrak{F}} \neq \infty$ on C, define the *w-Euler characteristic* $\chi_w(C) = \chi^{\mathrm{na}}(C, w_{\mathfrak{F}}|_C)$. If $w_{\mathfrak{F}}(c) = \infty$ for some $c \in C$, we say that $\chi_w(C)$ is *undefined*. For $f \in \mathrm{CF}(S)$ with $w_{\mathfrak{F}} \neq \infty$ on $\mathrm{supp}\, f$, define the *weighted w-Euler characteristic* $\chi_w(S, f)$ by $\chi_w(S, f) = \chi^{\mathrm{na}}(S, w_{\mathfrak{F}} f) = \chi^{\mathrm{na}}(\mathrm{supp}\, f, w_{\mathfrak{F}} f)$, taking $w_{\mathfrak{F}} f = 0$ outside $\mathrm{supp}\, f$ even where $w_{\mathfrak{F}} = \infty$. If $w_{\mathfrak{F}}(s) = \infty$ for some $s \in \mathrm{supp}\, f$, we say that $\chi_w(S, f)$ is *undefined*. If $w_{\mathfrak{F}} \neq \infty$ on S and $w_{\mathfrak{G}} \neq 0$ on T, then $w_{\mathfrak{F}} \in \mathrm{LCF}(S)$ and $w_{\mathfrak{G}}^{-1} \in \mathrm{LCF}(T)$ by Definition 4.11. Define

$$\mathrm{CF}_w(\Phi)f = w_{\mathfrak{G}}^{-1} \cdot \mathrm{CF}^{\mathrm{na}}(\Phi)(w_{\mathfrak{F}} f) \quad \text{for } f \in \mathrm{CF}(S). \tag{10}$$

This is well-defined in $\mathrm{CF}(T)$ as $w_{\mathfrak{F}} f \in \mathrm{CF}(S)$, so $\mathrm{CF}^{\mathrm{na}}(\Phi)(w_{\mathfrak{F}} f) \in \mathrm{CF}(T)$. Therefore, $\mathrm{CF}_w(\Phi) : \mathrm{CF}(S) \to \mathrm{CF}(T)$ is a $\mathbb{Q}$-linear map. If $w_{\mathfrak{F}}(s) = \infty$ for some $s \in S$ or $w_{\mathfrak{G}}(t) = 0$ for some $t \in T$, we say that $\mathrm{CF}_w(\Phi)$ is *undefined*.

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Then χ_w satisfies the following analogues of Theorem 3.10(ii) and (iii).

LEMMA 4.13. *Let w be an allowable weight function, and $\mathfrak{F},\mathfrak{G}$ algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. Then we have the following.*

(i) *Suppose that $C,D_1,\ldots,D_m\subseteq\mathfrak{F}(\mathbb{K})$ are constructible with $C=\coprod_{i=1}^m D_i$. Then $\chi_w(C)=\sum_{i=1}^m\chi_w(D_i)$ if either side is defined.*

(ii) *If w is multiplicative and $C\subseteq\mathfrak{F}(\mathbb{K})$, $D\subseteq\mathfrak{G}(\mathbb{K})$ are constructible, then $\chi_w(C\times D)=\chi_w(C)\chi_w(D)$ if both sides are defined.*

For the analogue of Theorem 4.9, from (10) we have

$$\begin{aligned}\mathrm{CF}_w(\Psi\circ\Phi)f&=w_{\mathfrak{H}}^{-1}\cdot\mathrm{CF}^{\mathrm{na}}(\Psi\circ\Phi)(w_{\mathfrak{F}}f)=w_{\mathfrak{H}}^{-1}\cdot\mathrm{CF}^{\mathrm{na}}(\Psi)\circ\mathrm{CF}^{\mathrm{na}}(\Phi)(w_{\mathfrak{F}}f)\\&=w_{\mathfrak{H}}^{-1}\cdot\mathrm{CF}^{\mathrm{na}}(\Psi)[w_{\mathfrak{G}}\cdot\mathrm{CF}_w(\Phi)f]=\mathrm{CF}_w(\Psi)\circ\mathrm{CF}_w(\Phi)f\end{aligned}$$

by Theorem 4.9, provided everything is defined. So we deduce the following.

COROLLARY 4.14. *Let w be an allowable weight function, let $\mathfrak{F},\mathfrak{G},\mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $S\subseteq\mathfrak{F}(\mathbb{K})$, $T\subseteq\mathfrak{G}(\mathbb{K})$, $U\subseteq\mathfrak{H}(\mathbb{K})$ be locally constructible with $w_{\mathfrak{F}}\neq\infty$ on S, $w_{\mathfrak{G}}\neq0,\infty$ on T, $w_{\mathfrak{H}}\neq0$ on U, and let $\Phi:S\to T$, $\Psi:T\to U$ be pseudomorphisms. Then $\mathrm{CF}_w(\Psi\circ\Phi)=\mathrm{CF}_w(\Psi)\circ\mathrm{CF}_w(\Phi)$.*

As in Corollary 4.10, we have the following.

COROLLARY 4.15. *Let w be an allowable weight function, let $\mathfrak{F},\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $S\subseteq\mathfrak{F}(\mathbb{K})$, $T\subseteq\mathfrak{G}(\mathbb{K})$ be locally constructible with $w_{\mathfrak{F}}\neq\infty$ on S, $w_{\mathfrak{G}}\neq0,\infty$ on T, let $\Phi:S\to T$ be a pseudomorphism, and let $f\in\mathrm{CF}(S)$. Then $\chi_w(S,f)=\chi_w(T,\mathrm{CF}_w(\Phi)f)$.*

The following are two examples of multiplicative allowable weight functions.

PROPOSITION 4.16. *We have the following.*

(a) *Define $e:\{$affine algebraic $\mathbb{K}$-groups$\}\to\mathbb{Z}$ by $e(G)=\chi(G)$. Then e is a multiplicative allowable weight function.*

(b) *If G is an affine algebraic $\mathbb{K}$-group, define the adjoint action of G on itself by $\mathrm{Ad}(g)h=ghg^{-1}$. Then the quotient $[G/\mathrm{Ad}(G)]$ is an algebraic $\mathbb{K}$-stack of finite type. Define $o(G)=\chi^{\mathrm{na}}([G/\mathrm{Ad}(G)])$. Then $o:\{$affine algebraic $\mathbb{K}$-groups$\}\to\mathbb{Z}$ is a multiplicative allowable weight function.*

Proof. Clearly e and o are well-defined and multiplicative. Let $\mathfrak{F}$ be an algebraic $\mathbb{K}$-stack with affine geometric stabilizers. We must show $e_{\mathfrak{F}},o_{\mathfrak{F}}\in\mathrm{LCF}(\mathfrak{F})$, which holds provided that $e_{\mathfrak{G}}=e_{\mathfrak{F}}|_{\mathfrak{G}(\mathbb{K})}$ and $o_{\mathfrak{G}}=o_{\mathfrak{F}}|_{\mathfrak{G}(\mathbb{K})}$ lie in $\mathrm{CF}(\mathfrak{G})$ for all finite type $\mathbb{K}$-substacks $\mathfrak{G}$ in $\mathfrak{F}$. Theorem 2.7 gives $\mathfrak{G}(\mathbb{K})=\coprod_{i\in I}U_i(\mathbb{K})$, where $\{U_i:i\in I\}$ are finitely many substacks of $\mathfrak{G}$ with U_i 1-isomorphic to $[X_i/G_i]$ for X_i a $\mathbb{K}$-variety and G_i an affine algebraic $\mathbb{K}$-group, acting on X_i by $\rho_i:X_i\times G_i\to X_i$.

Write $\pi_i:X_i\to U_i$ for the projection 1-morphism. Let Y_i be the inverse image under $\mathrm{id}_{X_i}\times\rho_i:X_i\times G_i\to X_i\times X_i$ of the diagonal in $X_i\times X_i$. Then Y_i is a finite type closed subscheme of $X_i\times G_i$. Let $\sigma_i:Y_i\to X_i$ be the restriction of the projection $X_i\times G_i\to X_i$. Then for each $x\in X_i(\mathbb{K})$, $\sigma_i^{-1}(x)=\{x\}\times\mathrm{Stab}_x(G_i)$, where $\mathrm{Stab}_x(G_i)$ is the stabilizer subgroup of x in G_i.

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Theorem 3.17 gives $\mathrm{CF}(\sigma_i)1 \in \mathrm{CF}(X_i)$, as $1 \in \mathrm{CF}(Y_i)$. However, for $x \in X_i(\mathbb{K})$

$$(\mathrm{CF}(\sigma_i)1)(x) = \chi[\mathrm{Stab}_x(G_i)] = \chi[\mathrm{Iso}_{\mathbb{K}}((\pi_i)_*(x))] = e_{\mathfrak{G}}((\pi_i)_*(x)),$$

as $\sigma_i^{-1}(x) = \{x\} \times \mathrm{Stab}_x(G_i)$, and the stabilizer group $\mathrm{Iso}_{\mathbb{K}}((\pi_i)_*(x))$ in $U_i \cong [X_i/G_i]$ is $\mathrm{Stab}_x(G_i)$. Therefore, $\mathrm{CF}(\pi_i)1 = e_{\mathfrak{G}} \circ (\pi_i)_*$ as maps $X_i(\mathbb{K}) \to \mathbb{Q}$. Since $(\pi_i)_*$ is surjective, this implies that $e_{\mathfrak{G}}|_{U_i(\mathbb{K})} \in \mathrm{CF}(U_i)$, as $(\pi_i)_*$ takes constructible sets to constructible sets by Proposition 4.5. However $\mathfrak{G}(\mathbb{K}) = \coprod_{i \in I} U_i(\mathbb{K})$ and I is finite, so $e_{\mathfrak{G}} \in \mathrm{CF}(\mathfrak{G})$. This proves part (a).

For part (b), we form an algebraic $\mathbb{K}$-stack $\mathfrak{H}_i$ with 1-morphisms $Y_i \xrightarrow{\alpha_i} \mathfrak{H}_i \xrightarrow{\beta_i} X_i$ with $\pi_i = \beta_i \circ \alpha_i$, such that if $x \in X_i(\mathbb{K})$ with $\mathrm{Stab}_x(G_i) = H$, so that $\pi_i^{-1}(x) = \{x\} \times H$, then $\beta_i^{-1}(x) = \{x\} \times [H/\mathrm{Ad}(H)]$, and $\alpha_i : \pi_i^{-1}(x) \to \beta_i^{-1}(x)$ is the projection $H \to [H/\mathrm{Ad}(H)]$. Then α_i is an atlas, so $\mathfrak{H}_i$ is of finite type. Thus, $1 \in \mathrm{CF}(\mathfrak{H}_i)$, so $\mathrm{CF}^{\mathrm{na}}(\beta_i)1 \in \mathrm{CF}(X_i)$ by Theorem 4.9. However, for $x \in X_i(\mathbb{K})$,

$$\begin{aligned}(\mathrm{CF}^{\mathrm{na}}(\beta_i)1)(x) &= \chi^{\mathrm{na}}([\mathrm{Stab}_x(G_i)/\mathrm{Ad}(\mathrm{Stab}_x(G_i))]) \\ &= \chi^{\mathrm{na}}([\mathrm{Iso}_{\mathbb{K}}((\pi_i)_*(x))/\mathrm{Ad}(\mathrm{Iso}_{\mathbb{K}}((\pi_i)_*(x)))]) = o_{\mathfrak{G}}((\pi_i)_*(x)).\end{aligned}$$

The rest of the proof is as for part (a). □

Other weight functions constructed from e, o in a multiplicative way are also multiplicative and allowable, such as $e^k, o^k, |e|^k, |o|^k, \mathrm{sign}(e), \mathrm{sign}(o)$, and $e^k o^l$ for $k, l \in \mathbb{Z}$ with $k\,l > 0$. We give special names to two interesting cases.

DEFINITION 4.17. Let w be an allowable weight function, let $\mathfrak{F}, \mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, let $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible, let $S \subseteq \mathfrak{F}(\mathbb{K})$, $T \subseteq \mathfrak{G}(\mathbb{K})$ be locally constructible, let $\Phi : S \to T$ be a pseudomorphism, and let $f \in \mathrm{CF}(S)$.

(a) Define the *stack* (*weighted*) *Euler characteristic* by $\chi^{\mathrm{stk}}(C) = \chi_{1/e}(C)$ and $\chi^{\mathrm{stk}}(S, f) = \chi_{1/e}(S, f)$, and the *stack pushforward* $\mathrm{CF}^{\mathrm{stk}}(\Phi) = \mathrm{CF}_{1/e}(\Phi)$, in the sense of Definition 4.11, where e is the weight function of Proposition 4.16(a), and $1/e$ is allowable. Here $\chi^{\mathrm{stk}}(C)$ is only defined if $\chi(\mathrm{Iso}_{\mathbb{K}}(c)) \neq 0$ for all $c \in C$, and $\chi^{\mathrm{stk}}(S, f)$ is only defined if $\chi(\mathrm{Iso}_{\mathbb{K}}(s)) \neq 0$ for all $s \in \mathrm{supp}\, f$, and $\mathrm{CF}^{\mathrm{stk}}(\Phi)$ is only defined if $\chi(\mathrm{Iso}_{\mathbb{K}}(s)) \neq 0$ for all $s \in S$.

(b) Define the *orbifold* (*weighted*) *Euler characteristic* by $\chi^{\mathrm{orb}}(C) = \chi_o(C)$ and $\chi^{\mathrm{orb}}(S, f) = \chi_o(S, f)$, and the *orbifold pushforward* $\mathrm{CF}^{\mathrm{orb}}(\Phi) = \mathrm{CF}_o(\Phi)$, where o is the allowable weight function of Proposition 4.16(b). As o takes values in $\mathbb{Z}$, $\chi^{\mathrm{orb}}(C)$ and $\chi^{\mathrm{orb}}(S, f)$ are always defined, and $\mathrm{CF}^{\mathrm{orb}}(\Phi)$ is defined if $o(\mathrm{Iso}_{\mathbb{K}}(t)) \neq 0$ for all $t \in T$.

The *stack Euler characteristic* χ^{stk} and its pushforward $\mathrm{CF}^{\mathrm{stk}}$ turn out to be the natural notions for the problems in [**10**–**13**]. If X is a $\mathbb{K}$-variety and G is an algebraic $\mathbb{K}$-group acting on X with $\chi(G) \neq 0$, then $\chi^{\mathrm{stk}}([X/G]) = \chi(X)/\chi(G)$. It also has a universal property in *Cartesian squares*, in § 5.2.

Unfortunately, as $\chi(G) = 0$ for any algebraic $\mathbb{K}$-group G with $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$ as a $\mathbb{K}$-subgroup, $\chi^{\mathrm{stk}}(C), \chi^{\mathrm{stk}}(S, f)$, and $\mathrm{CF}^{\mathrm{stk}}(\Phi)$ above are *undefined* in many interesting situations, including everything in [**10**–**13**]. However, in § 5.1 we will extend the definition of $\mathrm{CF}^{\mathrm{stk}}(\Phi)$ to $\mathrm{CF}^{\mathrm{stk}}(\phi)$ for $\phi : \mathfrak{F} \to \mathfrak{G}$ a *representable* 1-morphism, and this will be sufficient for the applications of [**10**–**13**].

For *Deligne–Mumford* stacks all stabilizer groups are finite, and for G finite $\chi(G) = |G| > 0$, so that $\chi^{\mathrm{stk}}, \mathrm{CF}^{\mathrm{stk}}$ are always defined. It is well-established that

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for enumerative problems on Deligne–Mumford stacks one counts a point $x \in \mathfrak{F}(\mathbb{K})$ with weight $1/|\mathrm{Iso}_{\mathbb{K}}(x)|$, and χ^{stk} generalizes this approach.

The *orbifold Euler characteristic* is the author's attempt to generalize something that is already well-understood for orbifolds to stacks. Let G be a finite group acting on a compact manifold M, so that M/G is an *orbifold*. Dixon *et al.* [**4**, p. 684] observed that the correct Euler characteristic of M/G in string theory is

$$\chi(M,G) = \frac{1}{|G|} \sum_{g,h \in G : gh=hg} \chi(M^{g,h}), \tag{11}$$

where $M^{g,h} = \{x \in M : g \cdot x = h \cdot x = x\}$.

Atiyah and Segal [**3**] later interpreted $\chi(M,G)$ as the Euler characteristic of *equivariant K-theory* $K_G(M)$. For a survey and further references on orbifold Euler characteristics, see Roan [**22**]. In particular, it is believed and in many cases known that for a complex orbifold M/G, $\chi(M,G)$ coincides with the Euler characteristic $\chi(X)$ of any *crepant resolution* of M/G.

Let M be a $\mathbb{K}$-scheme acted on by a finite group G. Then $M^{g,h}$ is a subscheme of M, and (11) makes sense. An easy calculation shows $\chi(M,G) = \chi^{\mathrm{na}}([M/G], f)$, where $f \in \mathrm{CF}([M/G])$ is given by

$$\begin{aligned} f(x) &= \frac{|\{(g,h) \in \mathrm{Iso}_{\mathbb{K}}(x)^2 : gh = hg\}|}{|\mathrm{Iso}_{\mathbb{K}}(x)|} = \sum_{g \in \mathrm{Iso}_{\mathbb{K}}(x)} \frac{|\{h \in \mathrm{Iso}_{\mathbb{K}}(x) : gh = hg\}|}{|\mathrm{Iso}_{\mathbb{K}}(x)|} \\ &= \sum_{g \in \mathrm{Iso}_{\mathbb{K}}(x)} \frac{1}{|\mathrm{Ad}(\mathrm{Iso}_{\mathbb{K}}(x))g|} = |\mathrm{Iso}_{\mathbb{K}}(x)/\mathrm{Ad}(\mathrm{Iso}_{\mathbb{K}}(x))| = o_{[M/G]}(x) \end{aligned}$$

for $x \in [M/G](\mathbb{K})$. Hence, $\chi(M,G) = \chi^{\mathrm{orb}}([M/G])$ by Definitions 4.11 and 4.17.

Thus, our orbifold Euler characteristic $\chi^{\mathrm{orb}}([M/G])$ of the Deligne–Mumford stack $[M/G]$ agrees with the physicists' orbifold Euler characteristic $\chi(M,G)$ of the complex orbifold M/G when $\mathbb{K} = \mathbb{C}$, but our notion χ^{orb} is also defined over other fields $\mathbb{K}$ and for more general stacks $\mathfrak{F}$. It would be interesting to know whether $\chi(M,G)$ being the Euler characteristic of any crepant resolution over $\mathbb{C}$ extends using χ^{orb} to other fields, or to more general stacks.

5. *Representable and finite type* 1*-morphisms*

Next we study stack pushforwards $\mathrm{CF}^{\mathrm{stk}}(\phi)$ by 1-*morphisms* $\phi : \mathfrak{F} \to \mathfrak{G}$. Then $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$ is a pseudomorphism, so the obvious definition is $\mathrm{CF}^{\mathrm{stk}}(\phi) = \mathrm{CF}^{\mathrm{stk}}(\phi_*)$. However, $\mathrm{CF}^{\mathrm{stk}}(\phi_*)$ is *undefined* if $x \in \mathfrak{F}(\mathbb{K})$ with $\chi(\mathrm{Iso}_{\mathbb{K}}(x)) = 0$. Since $\chi(G) = 0$ for many affine algebraic $\mathbb{K}$-groups G, this is a serious drawback. Instead, by using the *extra data* of the homomorphisms $\phi_* : \mathrm{Iso}_{\mathbb{K}}(x) \to \mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$, in § 5.1 we define $\mathrm{CF}^{\mathrm{stk}}(\phi)$ in many cases when $\mathrm{CF}^{\mathrm{stk}}(\phi_*)$ is undefined, in particular for all *representable* ϕ.

Section 5.2 defines the *pullback* $\psi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ for a *finite type* 1-morphism $\psi : \mathfrak{F} \to \mathfrak{G}$, and proves that pullbacks ψ^* and pushforwards $\mathrm{CF}^{\mathrm{stk}}(\phi)$ commute in *Cartesian squares*. This will be an important tool in [**10**–**13**]. In § 5.3, for finite type $\phi : \mathfrak{F} \to \mathfrak{G}$ we extend $\mathrm{CF}^{\mathrm{na}}(\phi_*), \mathrm{CF}^{\mathrm{stk}}(\phi)$ to *locally constructible* functions, with the usual functorial property.

Fix an algebraically closed field $\mathbb{K}$ of characteristic zero for all of this section.

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5.1. *Pushforwards by representable* 1*-morphisms*

The following is our definition of the stack pushforward $\mathrm{CF}^{\mathrm{stk}}(\phi)$ for a 1-morphism ϕ.

DEFINITION 5.1. Let $\mathfrak{F},\mathfrak{G}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers and let $\phi:\mathfrak{F}\to\mathfrak{G}$ be a 1-morphism. Then for any $x\in\mathfrak{F}(\mathbb{K})$ we have a morphism $\phi_*:\mathrm{Iso}_{\mathbb{K}}(x)\to\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$. The kernel $\mathrm{Ker}\,\phi_*$ is an affine algebraic $\mathbb{K}$-group in $\mathrm{Iso}_{\mathbb{K}}(x)$, so that $\chi(\mathrm{Ker}\,\phi_*)$ is defined. The image $\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$ is an affine algebraic $\mathbb{K}$-group closed in $\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$, so the quotient $\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$ is a quasiprojective $\mathbb{K}$-variety. Thus, $\chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x)))$ is also defined.

Suppose that $\chi(\mathrm{Ker}\,\phi_*)\neq 0$ for all $x\in\mathfrak{F}(\mathbb{K})$. Define $m_\phi:\mathfrak{F}(\mathbb{K})\to\mathbb{Q}$ by

$$m_\phi(x)=\frac{\chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x)))}{\chi(\mathrm{Ker}(\phi_*:\mathrm{Iso}_{\mathbb{K}}(x)\to\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))))}\quad\text{for } x\in\mathfrak{F}(\mathbb{K}).\tag{12}$$

An argument similar to Proposition 4.16 shows $m_\phi\in\mathrm{LCF}(\mathfrak{F})$. Define the *stack pushforward* $\mathrm{CF}^{\mathrm{stk}}(\phi):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ by

$$\mathrm{CF}^{\mathrm{stk}}(\phi)f=\mathrm{CF}^{\mathrm{na}}(\phi_*)(m_\phi\cdot f).\tag{13}$$

Here $m_\phi\cdot f\in\mathrm{CF}(\mathfrak{F})$ as $m_\phi\in\mathrm{LCF}(\mathfrak{F})$ and $f\in\mathrm{CF}(\mathfrak{F})$, so (13) is well-defined, and $\mathrm{CF}^{\mathrm{stk}}(\phi):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ is $\mathbb{Q}$-linear.

This agrees with the previous definition of $\mathrm{CF}^{\mathrm{stk}}(\phi_*)$ when it is defined, regarding $\phi_*:\mathfrak{F}(\mathbb{K})\to\mathfrak{G}(\mathbb{K})$ as a *pseudomorphism* by Proposition 4.6(a).

LEMMA 5.2. *In Definition* 5.1*, if* $\mathrm{CF}^{\mathrm{stk}}(\phi_*)$ *is defined in Definition* 4.17*, then* $\mathrm{CF}^{\mathrm{stk}}(\phi):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ *is defined and* $\mathrm{CF}^{\mathrm{stk}}(\phi_*)=\mathrm{CF}^{\mathrm{stk}}(\phi)$.

Proof. Suppose that $\mathrm{CF}^{\mathrm{stk}}(\phi_*)$ is defined. Then $1/e_{\mathfrak{F}}\neq\infty$ in $\mathfrak{F}(\mathbb{K})$, so $\chi(\mathrm{Iso}_{\mathbb{K}}(x))\neq 0$ for $x\in\mathfrak{F}(\mathbb{K})$. Now $\mathrm{Ker}\,\phi_*$ is normal in $\mathrm{Iso}_{\mathbb{K}}(x)$, with quotient $\mathrm{Iso}_{\mathbb{K}}(x)/\mathrm{Ker}\,\phi_*$ naturally isomorphic to $\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))$. Hence,

$$\chi(\mathrm{Iso}_{\mathbb{K}}(x))=\chi(\mathrm{Ker}\,\phi_*)\cdot\chi(\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))),$$

by general properties of χ. As $\chi(\mathrm{Iso}_{\mathbb{K}}(x))\neq 0$ this implies $\chi(\mathrm{Ker}\,\phi_*)\neq 0$ for $x\in\mathfrak{F}(\mathbb{K})$, so $\mathrm{CF}^{\mathrm{stk}}(\phi):\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{G})$ is defined. Similarly, we have

$$\chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x)))=\chi(\phi_*(\mathrm{Iso}_{\mathbb{K}}(x)))\cdot\chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\phi_*(\mathrm{Iso}_{\mathbb{K}}(x))).$$

Dividing this equation by the previous equation for $x\in\mathfrak{F}(\mathbb{K})$, which is valid as $\chi(\mathrm{Iso}_{\mathbb{K}}(x))\neq 0$, and using (12) gives

$$m_\phi(x)=\chi(\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))/\chi(\mathrm{Iso}_{\mathbb{K}}(x))=e_{\mathfrak{G}}(\phi_*(x))/e_{\mathfrak{F}}(x)\quad\text{for } x\in\mathfrak{F}(\mathbb{K}).$$

It follows immediately from Definition 4.17 that $\mathrm{CF}^{\mathrm{stk}}(\phi_*)=\mathrm{CF}^{\mathrm{stk}}(\phi)$. □

The functorial behaviour of Theorem 4.9 holds for $\mathrm{CF}^{\mathrm{stk}}(\phi)$.

THEOREM 5.3. *Let* $\mathfrak{F},\mathfrak{G},\mathfrak{H}$ *be algebraic* $\mathbb{K}$*-stacks with affine geometric stabilizers, and* $\phi:\mathfrak{F}\to\mathfrak{G}$, $\psi:\mathfrak{G}\to\mathfrak{H}$ 1*-morphisms. Suppose the kernels of* $\phi_*:\mathrm{Iso}_{\mathbb{K}}(x)\to\mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$ *for* $x\in\mathfrak{F}(\mathbb{K})$ *and* $\psi_*:\mathrm{Iso}_{\mathbb{K}}(y)\to\mathrm{Iso}_{\mathbb{K}}(\psi_*(y))$ *for* $y\in\mathfrak{G}(\mathbb{K})$ *have nonzero Euler characteristics. Then* $\mathrm{CF}^{\mathrm{stk}}(\psi\circ\phi)=\mathrm{CF}^{\mathrm{stk}}(\psi)\circ\mathrm{CF}^{\mathrm{stk}}(\phi)$ *as well-defined linear maps* $\mathrm{CF}(\mathfrak{F})\to\mathrm{CF}(\mathfrak{H})$.

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Proof. Let $x \in \mathfrak{F}(\mathbb{K})$, and set $y = \phi_*(x)$ and $z = \psi_*(y)$. Write

$$G_x = \mathrm{Iso}_{\mathbb{K}}(x), \quad G_y = \mathrm{Iso}_{\mathbb{K}}(y), \quad G_z = \mathrm{Iso}_{\mathbb{K}}(z), \quad \phi_x = \phi_* : G_x \to G_y,$$
$$\psi_y = \psi_* : G_y \to G_z, \quad K_{\phi,x} = \mathrm{Ker}(\phi_x),$$
$$K_{\psi,y} = \mathrm{Ker}(\psi_y), \quad K_{\psi\circ\phi,x} = \mathrm{Ker}(\psi_y \circ \phi_x),$$
$$I_{\phi,x} = \mathrm{Image}(\phi_x), \quad I_{\psi,y} = \mathrm{Image}(\psi_y), \quad I_{\psi\circ\phi,x} = \mathrm{Image}(\psi_y \circ \phi_x).$$

Then $K_{\phi,x}$ is normal in $K_{\psi\circ\phi,x}$, and the quotient $K_{\psi\circ\phi,x}/K_{\phi,x}$ is isomorphic to $I_{\phi,x} \cap K_{\psi,y}$. So general properties of χ give

$$\chi(K_{\psi\circ\phi,x}) = \chi(K_{\phi,x}) \cdot \chi(I_{\phi,x} \cap K_{\psi,y}). \tag{14}$$

The inclusions $I_{\phi,x} \cap K_{\psi,y} \subseteq K_{\psi,y}$ and $I_{\psi\circ\phi,x} \subseteq I_{\psi,y} \subseteq G_z$ imply that

$$\chi(K_{\psi,y}) = \chi(K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y})) \cdot \chi(I_{\phi,x} \cap K_{\psi,y}) \tag{15}$$

and

$$\chi(G_z/I_{\psi\circ\phi,x}) = \chi(G_z/I_{\psi,y}) \cdot \chi(I_{\psi,y}/I_{\psi\circ\phi,x}). \tag{16}$$

By assumption $\chi(K_{\phi,x}), \chi(K_{\psi,y}) \neq 0$ for $x \in \mathfrak{F}(\mathbb{K})$ and $y \in \mathfrak{G}(\mathbb{K})$, so $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$ and $\mathrm{CF}^{\mathrm{stk}}(\psi) : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{H})$ are defined. As $\chi(K_{\psi,y}) \neq 0$, (15) gives $\chi(I_{\phi,x} \cap K_{\psi,y}) \neq 0$, and this, $\chi(K_{\phi,x}) \neq 0$, and (14) show that $\chi(K_{\psi\circ\phi,x}) \neq 0$, which holds for all $x \in \mathfrak{F}(\mathbb{K})$. Hence, $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{H})$ is defined.

Now $\psi_x^{-1}(I_{\psi\circ\phi,x})$ is an algebraic group with $I_{\phi,x} \subseteq \psi_x^{-1}(I_{\psi\circ\phi,x}) \subseteq G_y$, so

$$\chi(G_y/I_{\phi,x}) = \chi(G_y/\psi_x^{-1}(I_{\psi\circ\phi,x})) \cdot \chi(\psi_x^{-1}(I_{\psi\circ\phi,x})/I_{\phi,x}).$$

However, ψ_x and $\gamma(I_{\phi,x} \cap K_{\psi,y}) \mapsto \gamma\, I_{\phi,x}$ induce isomorphisms of homogeneous spaces

$$G_y/\psi_x^{-1}(I_{\psi\circ\phi,x}) \cong I_{\psi,y}/I_{\psi\circ\phi,x}, \quad K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y}) \cong \psi_x^{-1}(I_{\psi\circ\phi,x})/I_{\phi,x}.$$

Therefore, the last two equations give

$$\chi(G_y/I_{\phi,x}) = \chi(I_{\psi,y}/I_{\psi\circ\phi,x}) \cdot \chi(K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y})). \tag{17}$$

Combining (12) and (14)–(17) yields

$$\begin{aligned} m_{\psi\circ\phi}(x) &= \frac{\chi(G_z/I_{\psi\circ\phi,x})}{\chi(K_{\psi\circ\phi,x})} = \frac{\chi(G_z/I_{\psi,y})\chi(I_{\psi,y}/I_{\psi\circ\phi,x})}{\chi(K_{\phi,x})\chi(I_{\phi,x} \cap K_{\psi,y})} \cdot \frac{\chi(K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y}))}{\chi(K_{\psi,y}/(I_{\phi,x} \cap K_{\psi,y}))} \\ &= \frac{\chi(G_z/I_{\psi,y})}{\chi(K_{\psi,y})} \cdot \frac{\chi(G_y/I_{\phi,x})}{\chi(K_{\phi,x})} = m_\psi(y) \cdot m_\phi(x). \end{aligned}$$

This identity is easily seen to be the extra ingredient needed to modify the proof of Theorem 4.9 to prove that $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) = \mathrm{CF}^{\mathrm{stk}}(\psi) \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$. □

If $\phi : \mathfrak{F} \to \mathfrak{G}$ is *representable*, then $\phi_* : \mathrm{Iso}_{\mathbb{K}}(x) \to \mathrm{Iso}_{\mathbb{K}}(\phi_*(x))$ is an *injective* morphism of algebraic $\mathbb{K}$-groups for all $x \in \mathfrak{F}(\mathbb{K})$. Thus, $\mathrm{Ker}\,\phi_* = \{1\}$, so $\chi(\mathrm{Ker}\,\phi_*) = 1 \neq 0$ for all $x \in \mathfrak{F}(\mathbb{K})$, and $\mathrm{CF}^{\mathrm{stk}}(\phi)$ is defined. This gives the following.

THEOREM 5.4. *Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and $\phi : \mathfrak{F} \to \mathfrak{G}$, $\psi : \mathfrak{G} \to \mathfrak{H}$ representable 1-morphisms. Then $\psi \circ \phi : \mathfrak{F} \to \mathfrak{H}$ is representable, and $\mathrm{CF}^{\mathrm{stk}}(\phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{G})$, $\mathrm{CF}^{\mathrm{stk}}(\psi) : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{H})$ and $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) : \mathrm{CF}(\mathfrak{F}) \to \mathrm{CF}(\mathfrak{H})$ are well-defined linear maps with $\mathrm{CF}^{\mathrm{stk}}(\psi \circ \phi) = \mathrm{CF}^{\mathrm{stk}}(\psi) \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$.*

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Also, for ϕ representable m_ϕ in (12) takes values in $\mathbb{Z}$, so $\mathrm{CF}^{\mathrm{stk}}(\phi)$ maps $\mathbb{Z}$-valued functions $\mathrm{CF}(\mathfrak{F})_{\mathbb{Z}} \subset \mathrm{CF}(\mathfrak{F})$ to $\mathbb{Z}$-valued functions $\mathrm{CF}(\mathfrak{G})_{\mathbb{Z}} \subset \mathrm{CF}(\mathfrak{G})$.

5.2. *Pullbacks by finite type* 1*-morphisms*

For *finite type* $\phi : \mathfrak{F} \to \mathfrak{G}$ we can pull back constructible functions from $\mathfrak{G}$ to $\mathfrak{F}$.

DEFINITION 5.5. Suppose that $\phi : \mathfrak{F} \to \mathfrak{G}$ is a finite type 1-morphism of algebraic $\mathbb{K}$-stacks and $C \subseteq \mathfrak{G}(\mathbb{K})$ is constructible. Then $C = \bigcup_{i\in I} \mathfrak{G}_i(\mathbb{K})$, where $\{\mathfrak{G}_i : i \in I\}$ are finitely many substacks of $\mathfrak{G}$. Set $\mathfrak{F}_i = \phi^*(\mathfrak{G}_i)$, a $\mathbb{K}$-substack of $\mathfrak{F}$. Then $\mathfrak{F}_i$ is of finite type, as $\phi, \mathfrak{G}_i$ are. Hence, $\phi_*^{-1}(C) = \bigcup_{i\in I} \phi_*^{-1}(\mathfrak{G}_i(\mathbb{K})) = \bigcup_{i\in I} \mathfrak{F}_i(\mathbb{K})$ is constructible in $\mathfrak{F}$. That is, $\phi_* : \mathfrak{F}(\mathbb{K}) \to \mathfrak{G}(\mathbb{K})$ *pulls back constructible sets to constructible sets*. Thus, if $f \in \mathrm{CF}(\mathfrak{G})$, then $f \circ \phi_*$ lies in $\mathrm{CF}(\mathfrak{F})$. Define the *pullback* $\phi^* : \mathrm{CF}(\mathfrak{G}) \to \mathrm{CF}(\mathfrak{F})$ by $\phi^*(f) = f \circ \phi_*$. Pullbacks commute with multiplication of functions, that is, $\phi^*(fg) = \phi^*(f)\phi^*(g)$. If also $\psi : \mathfrak{G} \to \mathfrak{H}$ is a finite type 1-morphism, it is immediate that $(\psi \circ \phi)^* = \phi^* \circ \psi^* : \mathrm{CF}(\mathfrak{H}) \to \mathrm{CF}(\mathfrak{F})$.

It is an interesting question how pullbacks ψ^* and pushforwards $\mathrm{CF}^{\mathrm{stk}}(\phi)$ are related. The following theorem shows that they commute in *Cartesian squares*, as in Definition 2.5. This will be an important tool in [**10**–**13**]. The theorem would *not* hold if we replaced $\mathrm{CF}^{\mathrm{stk}}(\eta), \mathrm{CF}^{\mathrm{stk}}(\phi)$ in (18) by $\mathrm{CF}^{\mathrm{na}}(\eta), \mathrm{CF}^{\mathrm{na}}(\phi)$, or pushforwards defined using some other weight function. This supports our claim that $\mathrm{CF}^{\mathrm{stk}}$ is the most natural pushforward in many stack problems.

THEOREM 5.6. *Let* $\mathfrak{E}, \mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ *be algebraic* $\mathbb{K}$*-stacks with affine geometric stabilizers. If*

$$\begin{array}{ccc} \mathfrak{E} & \xrightarrow{\eta} & \mathfrak{G} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\psi} \\ \mathfrak{F} & \xrightarrow{\phi} & \mathfrak{H} \end{array}$$

is a Cartesian square with η, ϕ *representable and* θ, ψ *of finite type, then the following diagram commutes.*

$$\begin{array}{ccc} \mathrm{CF}(\mathfrak{E}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\eta)} & \mathrm{CF}(\mathfrak{G}) \\ \uparrow{\scriptstyle\theta^*} & & \uparrow{\scriptstyle\psi^*} \\ \mathrm{CF}(\mathfrak{F}) & \xrightarrow{\mathrm{CF}^{\mathrm{stk}}(\phi)} & \mathrm{CF}(\mathfrak{H}) \end{array} \tag{18}$$

Proof. Let $C \subseteq \mathfrak{F}(\mathbb{K})$ be constructible, and let $\delta_C \in \mathrm{CF}(\mathfrak{F})$ be its characteristic function. We shall prove that

$$(\mathrm{CF}^{\mathrm{stk}}(\eta) \circ \theta^*)\delta_C = (\psi^* \circ \mathrm{CF}^{\mathrm{stk}}(\phi))\delta_C. \tag{19}$$

As $\mathrm{CF}^{\mathrm{stk}}(\eta) \circ \theta^*$, $\psi^* \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$ are linear and such δ_C generate $\mathrm{CF}(\mathfrak{F})$, this implies $\mathrm{CF}^{\mathrm{stk}}(\eta) \circ \theta^* = \psi^* \circ \mathrm{CF}^{\mathrm{stk}}(\phi)$, as we want.

Define $B = \theta_*^{-1}(C)$. Since θ is of finite type B is constructible, as in Definition 5.5, and $\theta^*(\delta_C) = \delta_B$, the characteristic function of B. Let $x \in \mathfrak{G}(\mathbb{K})$, and define $y = \psi_*(x)$, $B_x = B \cap \eta_*^{-1}(\{x\})$, and $C_y = C \cap \phi_*^{-1}(\{y\})$. Then B_x, C_y are constructible, as B, C are, and $\eta_*^{-1}(\{x\}), \phi_*^{-1}(\{y\})$ are locally constructible.

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Write $\delta_{B_x}, \delta_{C_y}, \delta_{\eta_*^{-1}(x)}, \delta_{\phi_*^{-1}(y)}$ for the characteristic functions. Then

$$\begin{aligned}((\mathrm{CF}^{\mathrm{stk}}(\eta)\circ\theta^*)\delta_C)(x) &= (\mathrm{CF}^{\mathrm{stk}}(\eta)(\delta_C\circ\theta_*))(x) = (\mathrm{CF}^{\mathrm{stk}}(\eta)\delta_B)(x)\\ &= (\mathrm{CF}^{\mathrm{na}}(\eta_*)(m_\eta\cdot\delta_B))(x) = \chi^{\mathrm{na}}(\mathfrak{E}, m_\eta\cdot\delta_{B_x})\\ &= \chi^{\mathrm{na}}(\mathfrak{F}, \mathrm{CF}^{\mathrm{na}}(\theta)(m_\eta\cdot\delta_{B_x})),\end{aligned} \tag{20}$$

by (9), (13), and Corollary 4.10, where m_η is defined in (12). Similarly we have

$$((\psi^*\circ\mathrm{CF}^{\mathrm{stk}}(\phi))\delta_C)(x) = \chi^{\mathrm{na}}(\mathfrak{F}, m_\phi\cdot\delta_{C_y}). \tag{21}$$

We shall prove that

$$\mathrm{CF}^{\mathrm{na}}(\theta)(m_\eta\cdot\delta_{B_x}) = m_\phi\cdot\delta_{C_y} \quad \text{in } \mathrm{CF}(\mathfrak{F}). \tag{22}$$

If $z\in\mathfrak{F}(\mathbb{K})\setminus C_y$, then both sides of (22) are zero at z. So let $z\in C_y$. Then $\theta_*^{-1}(\{z\})\cap B_x = \theta_*^{-1}(\{z\})\cap\eta_*^{-1}(\{x\})$, so by (9), (22) at z reduces to

$$\chi^{\mathrm{na}}(\mathfrak{E}, m_\eta\cdot\delta_{\eta_*^{-1}(x)}\cdot\delta_{\theta_*^{-1}(z)}) = m_\phi(z). \tag{23}$$

Define $G_x=\mathrm{Iso}_{\mathbb{K}}(x)$, $G_y=\mathrm{Iso}_{\mathbb{K}}(y)$ and $G_z=\mathrm{Iso}_{\mathbb{K}}(z)$, as algebraic $\mathbb{K}$-groups. Since $\psi_*(x)=y$ and $\phi_*(z)=y$ we have homomorphisms $\psi_*:\mathrm{Iso}_{\mathbb{K}}(x)\to\mathrm{Iso}_{\mathbb{K}}(y)$ and $\phi_*:\mathrm{Iso}_{\mathbb{K}}(z)\to\mathrm{Iso}_{\mathbb{K}}(y)$. Write these as $\psi_x:G_x\to G_y$ and $\phi_z:G_z\to G_y$. Then ϕ_z is injective, as ϕ is representable, so $\chi(\mathrm{Ker}\,\phi_z)=\{1\}$ and (12) gives

$$m_\phi(z) = \chi(G_y/\phi_z(G_z)). \tag{24}$$

As (18) is Cartesian $\mathfrak{E}$ is 1-isomorphic to $\mathfrak{F}\times_{\mathfrak{H}}\mathfrak{G}$ by Definition 2.5. By definition of the fibre products we find that $\eta_*^{-1}(\{x\})\cap\theta_*^{-1}(\{z\})$ is naturally isomorphic to $\psi_x(G_x)\backslash G_y/\phi_z(G_z)$, a *biquotient*. The stabilizer groups are given by

$$\mathrm{Iso}_{\mathbb{K}}(\psi_x(G_x)\beta\phi_z(G_z)) = \{(\alpha,\gamma)\in G_x\times G_y : \psi_x(\alpha)\beta=\beta\phi_z(\gamma)\}\quad \text{for } \beta\in G_y,$$

and the group homomorphism $\eta_*:\mathrm{Iso}_{\mathbb{K}}(\psi_x(G_x)\beta\phi_z(G_z))\to\mathrm{Iso}_{\mathbb{K}}(x)=G_x$ is given by $(\alpha,\gamma)\mapsto\alpha$. It is injective as ϕ_z is injective. Therefore, (12) yields

$$m_\eta(\psi_x(G_x)\beta\phi_z(G_z)) = \chi(G_x/\{\alpha\in G_x:\psi_x(\alpha)\beta\phi_z(G_z)=\beta\phi_z(G_z)\}). \tag{25}$$

Let $\Pi_{x,y,z}:G_y/\phi_z(G_z)\to\psi_x(G_x)\backslash G_y/\phi_z(G_z)$ be the projection. Then the fibre of $\Pi_{x,y,z}$ over $\psi_x(G_x)\beta\phi_z(G_z)$ is isomorphic to $G_x/\{\alpha\in G_x:\psi_x(\alpha)\beta\phi_z(G_z)=\beta\phi_z(G_z)\}$. So (25) implies $\mathrm{CF}^{\mathrm{na}}(\Pi_{x,y,z})1=m_\eta$ in $\mathrm{CF}(\psi_x(G_x)\backslash G_y/\phi_z(G_z))$. Thus,

$$\begin{aligned}\chi^{\mathrm{na}}(\mathfrak{E}, m_\eta\cdot\delta_{\eta_*^{-1}(x)}\cdot\delta_{\theta_*^{-1}(z)}) &= \chi^{\mathrm{na}}(\psi_x(G_x)\backslash G_y/\phi_z(G_z), m_\eta)\\ &= \chi^{\mathrm{na}}(\psi_x(G_x)\backslash G_y/\phi_z(G_z), \mathrm{CF}^{\mathrm{na}}(\Pi_{x,y,z})1)\\ &= \chi^{\mathrm{na}}(G_y/\phi_z(G_z), 1) = m_\phi(z),\end{aligned}$$

by Corollary 4.10 and (24). This proves (23), and hence (22). Equations (20)–(22) then give (19) at x, as we have to prove. □

Note that by [**17**, Remark 4.14.1, Lemma 3.11 and Remark 4.17(2)], in a Cartesian square (18) of algebraic $\mathbb{K}$-stacks, if ϕ is representable, then η is representable, and if ψ is of finite type, then θ is of finite type. Thus, it is enough to suppose only that ϕ is representable and ψ is of finite type in (18).

5.3. *Pushforwards of locally constructible functions*

Next we observe that if $\phi:\mathfrak{F}\to\mathfrak{G}$ is of *finite type*, then the definitions of $\mathrm{CF}^{\mathrm{na}}(\phi_*)f, \mathrm{CF}^{\mathrm{stk}}(\phi)f$ in (9), (13) make sense for f only *locally* constructible.

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DEFINITION 5.7. Let $\phi : \mathfrak{F} \to \mathfrak{G}$ be a finite type 1-morphism of algebraic $\mathbb{K}$-stacks with affine geometric stabilizers. For $f \in \mathrm{LCF}(\mathfrak{F})$, define $\mathrm{LCF}^{\mathrm{na}}(\phi)f$ by

$$\mathrm{LCF}^{\mathrm{na}}(\phi)f(x) = \chi^{\mathrm{na}}(\mathfrak{F}, f \cdot \delta_{\phi_*^{-1}(x)}) \quad \text{for } x \in \mathfrak{F}(\mathbb{K}), \tag{26}$$

following (9). This is well-defined as $\phi_*^{-1}(\{x\})$ is constructible since ϕ is of finite type. Thus, $\delta_{\phi_*^{-1}(x)} \in \mathrm{CF}(\mathfrak{F})$ and $f \in \mathrm{LCF}(\mathfrak{F})$, giving $f \cdot \delta_{\phi_*^{-1}(x)} \in \mathrm{CF}(\mathfrak{F})$. If ϕ is also representable, define $\mathrm{LCF}^{\mathrm{stk}}(\phi)f = \mathrm{LCF}^{\mathrm{na}}(\phi)(m_\phi \cdot f)$ as in (13).

Suppose that $C \subseteq \mathfrak{G}(\mathbb{K})$ is constructible, and let $B = \phi_*^{-1}(C)$. Then B is constructible as ϕ is of finite type, by Definition 5.5. Write δ_B, δ_C for the characteristic functions of B, C. Then $f \cdot \delta_B \in \mathrm{CF}(\mathfrak{F})$, and it follows easily that

$$(\mathrm{LCF}^{\mathrm{na}}(\phi)f)\delta_C = \mathrm{CF}^{\mathrm{na}}(\phi_*)(f \cdot \delta_B) \quad \text{and} \quad (\mathrm{LCF}^{\mathrm{stk}}(\phi)f)\delta_C = \mathrm{CF}^{\mathrm{stk}}(\phi)(f \cdot \delta_B).$$

Therefore, $(\mathrm{LCF}^{\mathrm{na}}(\phi)f)|_C$, $(\mathrm{LCF}^{\mathrm{stk}}(\phi)f)|_C$ are constructible by Theorems 4.9 and 5.4, for any constructible $C \subseteq \mathfrak{G}(\mathbb{K})$. Hence, $\mathrm{LCF}^{\mathrm{na}}(\phi)f$, $\mathrm{LCF}^{\mathrm{stk}}(\phi)f$ are *locally constructible*, and $\mathrm{LCF}^{\mathrm{na}}(\phi), \mathrm{LCF}^{\mathrm{stk}}(\phi)$ are linear maps $\mathrm{LCF}(\mathfrak{F}) \to \mathrm{LCF}(\mathfrak{G})$. From Theorems 4.9 and 5.4 we deduce the following.

THEOREM 5.8. *Let $\mathfrak{F}, \mathfrak{G}, \mathfrak{H}$ be algebraic $\mathbb{K}$-stacks with affine geometric stabilizers, and let $\phi : \mathfrak{F} \to \mathfrak{G}$, $\psi : \mathfrak{G} \to \mathfrak{H}$ be finite type 1-morphisms. Then so is $\psi \circ \phi$, and $\mathrm{LCF}^{\mathrm{na}}(\psi \circ \phi) = \mathrm{LCF}^{\mathrm{na}}(\psi) \circ \mathrm{LCF}^{\mathrm{na}}(\phi)$. If ϕ, ψ are representable, then so is $\psi \circ \phi$, and $\mathrm{LCF}^{\mathrm{stk}}(\psi \circ \phi) = \mathrm{LCF}^{\mathrm{stk}}(\psi) \circ \mathrm{LCF}^{\mathrm{stk}}(\phi)$.*

The locally constructible analogue of Theorem 5.6 also holds.

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