Proc. Camb. Phil. Soc. (1968), **64**, 317
PCPS 64–39
Printed in Great Britain

A quotient ordered space

S. D. McCARTAN
Queen's University, Belfast

(*Received* 16 *February* 1967)

Introduction. It is well known that, in the study of quotient spaces it suffices to consider a topological space $(X, \mathscr{T})$, an equivalence relation R on X and the projection mapping $p: X \to X/R$ (where X/R is the family of R-classes of X) defined by $p(x) = R^x$ (where R^x is the R-class to which x belongs) for each $x \in X$. A topology $\mathscr{T}'$ may be defined for the set X/R by agreeing that $U \subseteq X/R$ is $\mathscr{T}'$-open if and only if $p^{-1}(U)$ is $\mathscr{T}$-open in X. The topological space $(X/R, \mathscr{T}')$ is known as the *quotient space* relative to the space $(X, \mathscr{T})$ and projection p. If $(X, \mathscr{T}, \leqslant)$ (or simply $(X, \mathscr{T})$ since the symbol $\leqslant$ denotes all partial orders and no confusion arises) is a topological ordered space (that is, X is a set for which both a topology $\mathscr{T}$ and a partial order $\leqslant$ is defined) then, providing the projection p satisfies the property

$(*)$ $\{x < y, x^* \in p^{-1}(p(x)), \quad y^* \in p^{-1}(p(y))\}$ *implies* $x^* < y^*$ *in* X

a partial order may be defined in X/R by agreeing that $p(x) < p(y)$ if and only if $x < y$ in X. The topological ordered space $(X/R, \mathscr{T}')$ is known as the *quotient ordered space* relative to the ordered space $(X, \mathscr{T})$ and projection p.

Although in general, if the projection p is $(\mathscr{T}, \mathscr{T}')$-*closed* (that is, p maps $\mathscr{T}$-closed sets in X onto $\mathscr{T}'$-closed sets in X/R), the topology $\mathscr{T}'$ is entirely determined by p and the toplogy $\mathscr{T}$, the quotient space $(X/R, \mathscr{T}')$ seldom inherits any pleasant properties of the domain space $(X, \mathscr{T})$. The purpose of this paper is to study a particular quotient ordered space, of simple construction, which inherits some interesting properties of the domain ordered space, and to discuss the correlation between the quotient topology and a well-known intrinsic topology for the quotient ordered set.

For an elementary discussion on quotient spaces, the reader is referred to (1).

Preliminaries. Let X be a partially ordered set and let $x \in X$, then $(x, \rightarrow]$, $[\leftarrow, x)$ denote the sets $\{y \in X : x < y\}$ and $\{y \in X : y < x\}$ respectively while $[x, \rightarrow]$, $[\leftarrow, x]$ denote the sets $\{y \in X : x \leqslant y\}$ and $\{y \in X : y \leqslant x\}$ respectively. If $x, y \in X$, $x \nleqslant y$ is the negation of $x \leqslant y$ while $x \| y$ if and only if $x \nleqslant y$ and $y \nleqslant x$. If $A \subseteq X$, then $i(A)$ and $d(A)$ denote the sets $\cup\{[a, \rightarrow] : a \in A\}$ and $\cup\{[\leftarrow, a] : a \in A\}$ respectively and A is said to be *increasing* (*decreasing*) in X if and only if $A = i(A)$ $(A = d(A))$. An equivalence relation D is defined on X by agreeing that $x, y \in X$ are *D-equivalent* if and only if $(x, \rightarrow] = (y, \rightarrow]$ and $[\leftarrow, x) = [\leftarrow, y)$. As a consequence of ((2), Lemma 7) the projection p of X upon X/D (defined by $p(x) = D^x$) obeys property $(*)$ so that X/D is a well-defined partially ordered set for which $D^x < D^y$ if and only if $x < y$ in X. The *Interval Topology* $\mathscr{I}$ for X is that topology which has, as a sub-basis for the family of closed sets, the collection of closed intervals $[x, \rightarrow]$, $[\leftarrow, x]$ for all $x \in X$. In what follows we are mainly

https://doi.org/10.1017/S0305004100042870 Published online by Cambridge University Press

concerned with a discussion of the quotient ordered space $(X/D, \mathscr{I}')$ relative to domain ordered space $(X, \mathscr{I})$ and projection p upon X/D.

A topological ordered space $(X, \mathscr{T})$ is said to be *H-ordered* (or *Hausdorff-ordered*) if and only if for each pair of elements $x \not\leqslant y$ in X, there exists disjoint $\mathscr{T}$-neighbourhoods U, V containing x, y respectively such that U is increasing and V is decreasing in X. $(X, \mathscr{T})$ is said to be *upper* (*lower*) *regularly ordered* if and only if for each increasing (decreasing) $\mathscr{T}$-closed set F and element $a \notin F$ there exist disjoint $\mathscr{T}$-neighbourhoods U containing a and V containing F such that U is decreasing (increasing) and V is increasing (decreasing). $(X, \mathscr{T})$ is *regularly ordered* if and only if $(X, \mathscr{T})$ is both upper and lower regularly ordered. $(X, \mathscr{T})$ is *normally ordered* if and only if for each pair of disjoint $\mathscr{T}$-closed sets F_1, F_2, where F_1 is increasing and F_2 is decreasing in X, there exist disjoint $\mathscr{T}$-neighbourhoods U_1 of F_1 and U_2 of F_2 such that U_1 is increasing and U_2 is decreasing in X.

Together with showing the inheritance of the classical Hausdorff, regular and normal separation axioms (a full description of which may be found in (1)), the main result (Theorem 2) of this paper demonstrates that the quotient ordered space $(X/D, \mathscr{I}')$ inherits the (order-induced) separation axioms described above, when these axioms are satisfied by the domain ordered space $(X, \mathscr{I})$.

Let $\mathscr{T}$ and $\mathscr{F}$ be two topologies for a set X: $\mathscr{T}$ is said to be coarser than $\mathscr{F}$ (and this is denoted by $\mathscr{T} \subseteq \mathscr{F}$) if and only if each $\mathscr{T}$-closed ($\mathscr{T}$-open) subset of X is $\mathscr{F}$-closed ($\mathscr{F}$-open).

Remark. If $(X, \mathscr{T})$ is a topological ordered space and $\mathscr{I} \subseteq \mathscr{T}$ for X then $(X, \mathscr{T})$ is H-ordered (regularly ordered, normally ordered) implies $(X, \mathscr{T})$ is a Hausdorff space.

It is also worth remarking that the concept of H-order coincides with the concepts, '*continuous partial order*' of Ward (5) and '*closed partial order*' of Nachbin (3).

Let $(X/R, \mathscr{T}')$ be a quotient ordered space relative to ordered space $(X, \mathscr{T})$ and projection p, then $\mathscr{A} \subseteq X/R$ implies $p^{-1}(\mathscr{A}) \equiv \cup\{R^a : R^a \in \mathscr{A}\} \subseteq X$ and hence $\mathscr{A}$ is $\mathscr{T}'$-open ($\mathscr{T}'$-closed) in X/R if and only if $\cup\{R^a : R^a \in \mathscr{A}\}$ is $\mathscr{T}$-open ($\mathscr{T}$-closed) in X.

Lemma 1. *Y is increasing (decreasing) in X implies $p(Y)$ is increasing (decreasing) in X/R.*

Proof. Let $R^x \in p(Y)$ and suppose $R^x < R^y$ in X/R, then there exists $z \in Y$ such that $R^z = R^x$ and hence $R^z < R^y$ in X/R which implies $z < y$ in X. By hypothesis $y \in Y$ so that $R^y \in p(Y)$. Consequently $p(Y)$ is increasing in X/R.

The obvious dual argument suffices when Y is decreasing in X.

A net $\{x_\alpha : \alpha \in A\}$ in a partially ordered set X is called *cofinite* (*D-cofinite*) if and only if there exists $x \in X$ and a cofinal subset C of A such that $x_\gamma = x (x_\gamma \in D^x)$ for all $\gamma \in C$. For an elementary discussion on nets in topological spaces, the reader is referred to (1).

Lemma 2. *If non-D-cofinite net $\{x_\alpha : \alpha \in A\}$ in partially ordered set X, $\mathscr{I}$-converges to x and if, for each $\alpha \in A$, $y_\alpha D x_\alpha$ then $\{y_\alpha : \alpha \in A\}$ is a (non-cofinite) net $\mathscr{I}$-converging to x.*

Proof. Let $x \not\leqslant a$, then $U_a \equiv \{y \in X : y \not\leqslant a\}$ is $\mathscr{I}$-open and contains x; suppose there exists a cofinal subset B of A such that $y_\beta \leqslant a$, for all $\beta \in B$, then there exists cofinal subset C of B such that $y_\gamma < a$, for all $\gamma \in C$ (otherwise $\{y_\alpha : \alpha \in A\}$ is cofinite and hence

https://doi.org/10.1017/S0305004100042870 Published online by Cambridge University Press

$\{x_\alpha\colon \alpha\in A\}$ is D-cofinite) which implies $x_\gamma < a$, for all $\gamma\in C$, a contradiction (since $\{x_\alpha\colon \alpha\in A\}$ $\mathscr{I}$-converges to x). Hence $\{y_\alpha\colon \alpha\in A\}$ is residually contained in U_a.

Dually $\{y_\alpha\colon \alpha\in A\}$ is residually contained in $V_a \equiv \{y\in X\colon a \nleqslant y\}$ when $a \nleqslant x$.

Consequently the net $\{y_\alpha\colon \alpha\in A\}$ is residually contained in each sub-basic $\mathscr{I}$-open subset of X containing x and hence $\mathscr{I}$-converges to x.

THEOREM 1. *The following conditions are equivalent*:

(i) *The projection p of X upon X/D is $(\mathscr{I},\mathscr{I}')$-closed.*

(ii) *For each $\mathscr{I}$-closed $F\subseteq X$, $D(F)\equiv \cup\{D^a\colon a\in F\}$ is $\mathscr{I}$-closed in X.*

(iii) *D^x is $\mathscr{I}$-closed in X for each $x\in X$.*

(iv) *D^x is finite in X or is the intersection of closed intervals of X.*

(v) *$\mathscr{I}^*\subseteq\mathscr{I}'$ for X/D, where $\mathscr{I}^*$ is the interval topology for X/D.*

Proof. The equivalence of (i) and (ii) is shown in ((1), Theorem 16, p. 97).

(iii) implies (iv). By hypothesis, for each $x\in X$, D^x is the intersection of finite unions of closed intervals of X.

Let $\{I_i\colon 1\leqslant i\leqslant n\}$ be a finite sequence of closed-intervals whose union contains D^x; if D^x is infinite, there exists i', where $1\leqslant i'\leqslant n$, such that $I_{i'}$ contains an infinite number of members of D^x. Let $I_{i'}\equiv[c,\rightarrow]$ (a dual argument suffices when $I_{i'}\equiv[\leftarrow,c]$), then, in particular, there exists $y\in D^x$ such that $c<y$ and hence $c<z$ for each $z\in D^x$. That is, $D^x\subseteq[c,\rightarrow]$. Consequently in the expression of D^x as the intersection of finite unions of closed intervals, each finite union of closed intervals may be replaced by a single closed interval. (iv) implies (iii). This is immediate. (v) implies (iii). By hypothesis, for each $x\in X$, $\{D^x\}$ is $\mathscr{I}'$-closed in X/D so that $D^x = p^{-1}(\{D^x\})$ is $\mathscr{I}$-closed in X. (iii) implies (ii). Let F be $\mathscr{I}$-closed in X and let $\{x_\alpha\colon \alpha\in A\}$ be a net in $D(F)$ $\mathscr{I}$-converging to $y\in X$; if $\{x_\alpha\colon \alpha\in A\}$ is D-cofinite, there exists a cofinal subset C of A such that $x_\gamma\in D^z$, for all $\gamma\in C$, and for some $z\in F$. By hypothesis, $y\in D^z\subseteq D(F)$. On the other hand, if $\{x_\alpha\colon \alpha\in A\}$ is non D-cofinite, there exists $y_\alpha\in F$ such that $y_\alpha D x_\alpha$, for each $\alpha\in A$. By Lemma 2, the net $\{y_\alpha\colon \alpha\in A\}$ $\mathscr{I}$-converges to y so that, since F is $\mathscr{I}$-closed, $y\in F\subseteq D(F)$. Consequently each $\mathscr{I}$-convergent net in $D(F)$ has its $\mathscr{I}$-limits in $D(F)$ and hence $D(F)$ is $\mathscr{I}$-closed. (i) implies (v). By hypothesis, for each $x\in X$, $[D^x,\rightarrow]\equiv p([x,\rightarrow])$ in X/D and $[\leftarrow,D^x]\equiv p([\leftarrow,x])$ in X/D are $\mathscr{I}'$-closed in X/D. That is, each closed interval in X/D is $\mathscr{I}'$-closed so that $\mathscr{I}^*\subseteq\mathscr{I}'$ for X/D.

LEMMA 3. *If $(X,\mathscr{I})$ is Hausdorff, then, for each $x\in X$, D^x is finite (and hence $\mathscr{I}$-closed) in X.*

Proof. Suppose for some $x\in X$, D^x is infinite ; let $a,b,\in D^x$, with $a\neq b$, then, by hypothesis and ((4), Proposition 1), $X=\cup\{I_i\colon 1\leqslant i\leqslant n\}$ where each I_i is a closed interval in X and no I_i contains both a and b. There must exist r, $1\leqslant r\leqslant n$, such that I_r contains an infinite number of members of D^x; let $I_r\equiv[c,\rightarrow]$ (a dual argument holds when $I_r\equiv[\leftarrow,c]$) for some $c\in X$. In particular, there exists a member of D^x which is greater than c, whence each member of D^x is greater than c. Consequently $a\in[c,\rightarrow]$ and $b\in[c,\rightarrow]$ which is a contradiction.

THEOREM 2. (i) *If $(X,\mathscr{I})$ is Hausdorff (regular, normal) then $(X/D,\mathscr{I}')$ is Hausdorff (regular, normal).*

https://doi.org/10.1017/S0305004100042870 Published online by Cambridge University Press

(ii) *If* $(X,\mathscr{F})$ *is* H*-ordered (regularly ordered, normally ordered) then* $(X/D,\mathscr{F}')$ *is* H*-ordered (regularly ordered, normally ordered).*

Proof. (i) Suppose $(X,\mathscr{F})$ is Hausdorff and $D^x \neq D^y$ in X/D, then $a \neq b$ for each $a \in D^x$, $b \in D^y$ and there exist disjoint $\mathscr{F}$-open sets U_b containing a, V_b containing b. For a given $a \in D^x$, let $U(a) = \cap\{U_b : b \in D^y\}$ and $V(A) = \cup\{V_b : b \in D^y\}$; let

$$U = \cup\{U(a) : a \in D^x\} \quad \text{and let} \quad V = \cap\{V(A) : a \in D^x\}.$$

Then $U \cap V$ is empty and, by Lemma 3, U, V are $\mathscr{F}$-open in X so that

$$X = (X - U) \cup (X - V)$$

is the union of two $\mathscr{F}$-closed sets, with $D^x \subseteq U$, $D^y \subseteq V$. Hence

$$X/D = p(X) = p(X - U) \cup p(X - V)$$

and, by Lemma 3, and Theorem 1, $X/D - p(X - U)$, $X/D - p(X - V)$ are disjoint $\mathscr{F}'$-open sets containing elements D^x, D^y respectively. Consequently $(X/D,\mathscr{F}')$ is Hausdorff.

In view of the Remark earlier and Lemma 3, a proof, analogous to that above, shows that $(X/D,\mathscr{F}')$ inherits regularity and normality from the space $(X,\mathscr{F})$.

(ii) Suppose $(X,\mathscr{F})$ is H-ordered and $D^x \not\leqslant D^y$ in X/D then $a \not\leqslant b$ for each $a \in D^x$, $b \in D^y$ and there exist disjoint $\mathscr{F}$-neighbourhoods U_b of a, V_b of b such that U_b is increasing in X and V_b is decreasing. The sets U, V are constructed as in (i) above. Then $U \cap V$ is empty and U is an increasing $\mathscr{F}$-neighbourhood of a, V is a decreasing $\mathscr{F}$-neighbourhood of b, for each $a \in D^x$, $b \in D^y$. If U', V' are $\mathscr{F}$-open such that $D^x \subseteq U' \subseteq U$, $D^y \subseteq V' \subseteq V$ then, by using part of the argument of (i) above, applied to U', V', together with applying Lemma 1 to U, V, it follows that $X/D - p(X - U)$ is an increasing $\mathscr{F}'$neighbourhood of D^x disjoint from decreasing $\mathscr{F}'$-neighbourhood

$$X/D - p(X - V) \quad \text{of} \quad D^y.$$

Consequently, $(X/D,\mathscr{F}')$ is H-ordered.

In view of the Remark earlier and Lemma 3, a proof, analogous to that above, shows that $(X/D,\mathscr{F}')$ inherits regular order and normal order from the ordered space $(X,\mathscr{F})$.

In Theorem 1, conditions, necessary and sufficient for the validity of the relationship $\mathscr{F}^* \subseteq \mathscr{F}'$ for X/D, were exhibited. Before the study of the relationship between the topologies $\mathscr{F}^*$ and $\mathscr{F}'$ for X/D is continued, a preliminary result is needed.

LEMMA 4. *If* D^x *contains two distinct elements, then* $D_x \equiv \{y \in X : y \| x\} \cup \{x\}$ *is* $\mathscr{F}$*-open in* X.

Proof. Let $a \in D^x$, $a \neq x$, then

$$D_x = X - [\leftarrow, x) \cup (x, \rightarrow] = X - [\leftarrow, a) \cup (a, \rightarrow] = D_a$$

and $$D_x = (D_x - \{x\}) \cup (D_x - \{a\}) = X - \{[x, \rightarrow] \cup [\leftarrow, x]\} \cap \{[a, \rightarrow] \cup [\leftarrow, a]\}$$

is $\mathscr{F}$-open in X.

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Example 1. Let X be the unit square of the real plane partially ordered by agreeing that $(a, b) < (c, d)$ if and only if $b < d$ in the unit interval $[0, 1]$ of R, the real line.

Let $\mathbf{z} \equiv (z, z) \in X$, then $D^{\mathbf{z}} = \{(y, z): 0 \leqslant y \leqslant 1 \text{ in } R\} = D_{\mathbf{z}}$, so that, by Lemma 4, $D^{\mathbf{z}}$ is $\mathscr{I}$-open. Thus $(X/D, \mathscr{I}')$ is a discrete space, so that p is $(\mathscr{I}, \mathscr{I}')$-closed. But it is clear that X/D is order-isomorphic to the unit interval $I \equiv [0, 1]$ of R, and hence the spaces $(I, \mathscr{I})$, $(X/D, \mathscr{I}^*)$ are homeomorphic (under the mapping $x \to D^{\mathbf{x}}$) so that $(X/D, \mathscr{I}^*)$ is not a discrete space. Consequently $\mathscr{I}^* \subset \mathscr{I}'$ for X/D.

This example shows that, in Theorem 1, condition (v) cannot be replaced by '$\mathscr{I}^* = \mathscr{I}'$ for X/D'.

LEMMA 5. *If p is $(\mathscr{I}, \mathscr{I}^*)$-closed (that is, p maps $\mathscr{I}$-closed subsets of X upon $\mathscr{I}^*$-closed subsets of X/D) then $\mathscr{I}' \subseteq \mathscr{I}^*$ for X/D.*

Proof. Let $\mathscr{A}$ be $\mathscr{I}'$-closed in X/D, then $p^{-1}(\mathscr{A})$ is $\mathscr{I}$-closed in X, which implies, by hypothesis, that $\mathscr{A} = p(p^{-1}(\mathscr{A}))$ is $\mathscr{I}^*$-closed in X/D.

THEOREM 3. *$\mathscr{I}^* = \mathscr{I}'$ for X/D if and only if p is both $(\mathscr{I}, \mathscr{I}')$-closed and $(\mathscr{I}, \mathscr{I}^*)$-closed.*

Proof. If $\mathscr{I}^* = \mathscr{I}'$ for X/D, then by Theorem 1, p is $(\mathscr{I}, \mathscr{I}')$-closed and hence $(\mathscr{I}, \mathscr{I}^*)$-closed. On the other hand, when p is both $(\mathscr{I}, \mathscr{I}')$-closed and $(\mathscr{I}, \mathscr{I}^*)$-closed, $\mathscr{I}^* \subseteq \mathscr{I}'$ and $\mathscr{I}' \subseteq \mathscr{I}^*$, by Theorem 1 and Lemma 5 respectively.

The fact that there is no general relationship between $\mathscr{I}^*$ and $\mathscr{I}'$ for X/D is exhibited in some further examples.

Example 2. Let $X = A \cup \{u, v\}$, where $u \equiv (1, 1)$, $v \equiv (0, 0)$ and A consists of the elements $x_n \equiv (1 - 1/n, 1/n)$, $n \geqslant 1$, of the real plane. X is partially ordered by agreeing that $(a, b) \leqslant (c, d)$ if and only if $a \leqslant c$, $b \leqslant d$ in R, the real line. Hence $A = D^x$, for each $x \in A$ and $X/D \equiv \{\{D^v\}, \{D^x\}, \{D^u\}\}$ is finite, so that $(X/D, \mathscr{I}^*)$ is a discrete space. Since in particular, each sub-basic $\mathscr{I}$-open subset of X containing u is of the form

$$\{y \in X: y \nleqslant x_i\} \quad \text{or} \quad \{y \in X: y \nleqslant v\},$$

the net $\{x_n: n \in J^+\}$ (where J^+ denotes the set of positive integers) $\mathscr{I}$-converges to u and hence $A = D^x = p^{-1}(\{D^x\})$ is not $\mathscr{I}$-closed in X so that $\{D^x\}$ is not $\mathscr{I}'$-closed in X/D. Thus, $\mathscr{I}' \subset \mathscr{I}^*$ for X/D.

Example 3. Let $X = B \cup \{c\}$, where B is the partially ordered set of Example 1, and c is the element $(1, 2)$ of the real plane. X is partially ordered by agreeing that $z < c$, for each $z \in B$. Let $w = (1, 1)$, then $D^w = \{(x, 1): 0 \leqslant x \leqslant 1 \text{ in } R\} = D_w$, so that, as in Example 1, D^w is $\mathscr{I}$-open in X and hence $\{D^w\}$ is $\mathscr{I}'$-open in X/D. However, as in Example 1, $\{D^w\}$ is not $\mathscr{I}^*$-open in X/D which implies that $\mathscr{I}' \nsubseteq \mathscr{I}^*$ for X/D.

Further, each sub-basic $\mathscr{I}$-open set in X containing c, being of the form

$$U_z \equiv \{y \in X: y \nleqslant z, z \in B\},$$

contains members of D^w, so that the $\mathscr{I}$-closure of D^w in X contains c and hence D^w is not $\mathscr{I}$-closed in X. Consequently $\{D^w\}$, although obviously $\mathscr{I}^*$-closed in X/D, is not $\mathscr{I}'$-closed and $\mathscr{I}^* \nsubseteq \mathscr{I}'$ for X/D.

https://doi.org/10.1017/S0305004100042870 Published online by Cambridge University Press

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