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Maximum zero-dimensional compactifications

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All spaces considered in this paper are Hausdorff. θX is a compactification of a completely regular space X means that X is identified with a dense subspace of θX. We shall thus always regard X to be contained in θX. A compactification θX of X is zero-dimension (countable) if its 'outgrowth' $\theta X \sim X$ is zero-dimensional (countable).

We prove in (2·8) that if a space possesses a maximum countable compactification in accordance with Definition (1·1), then this is also its maximum zero-dimensional compactification. Also in (3·7) we show that a maximum zero-dimensional compactification of X is the minimum perfect compactification of Sklyarenko. As we should expect, if X has a maximum zero-dimensional compactification γX, then the maximum zero-dimensional compactification of any closed subset of X with compact boundary may be obtained by taking its closure in γX.

1. *Some results on compactifications.* We define an ordering on compactifications θX of X as follows:

1·1. Definition. *$\theta_1 X \succ \theta_2 X$ if there is a continuous map of $\theta_1 X$ onto $\theta_2 X$ which leaves the points of X fixed. We refer to this map as the 'natural map' of $\theta_1 X$ onto $\theta_2 X$.*

The ordering of (1·1) is in fact a partial ordering. Every compactification θX of X satisfies $\beta X \succ \theta X$, where βX is its Stone–Cech compactification.

We quote the following result, which may be easily proved (for example, as a direct corollary to Bourbaki (2), Chapter I, 3.4, Proposition 6).

1·2. Lemma. *If $\theta_1 X$, $\theta_2 X$, $\theta_3 X$ are compactifications of X such that $\theta_3 X \succ \theta_2 X$ and $\theta_3 X \succ \theta_1 X$ with natural maps h: $\theta_3 X \to \theta_2 X$ and g: $\theta_3 X \to \theta_1 X$, and if f is a function $\theta_2 X \to \theta_1 X$ such that $g = f \circ h$, then $\theta_2 X \succ \theta_1 X$ with natural map f.*

With respect to the ordering of (1·1) Engelking and Sklyarenko (3) have proved the following

1·3. Lemma. *Every non-empty family of compactifications $\{\theta_i X | i \in I\}$ has a least upper bound.*

Engelking and Sklyarenko prove this by taking the product space $P = \Pi\{\theta_i X | i \in I\}$, identifying the point $x \in X$ with the element of P, each of whose coordinates is x, and then taking the closure of X in P. This closure is the required least upper bound. The extension of the identity map on X to a continuous map of $\mathrm{cl}_P X$ onto $\theta_i X$ is simply the restriction of the projection p_i: $P \to \theta_i X$ to $\mathrm{cl}_P X$.

1·4. Definition. *A continuous map is perfect if it is closed and the inverse image of every point is compact.*

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The following result is Lemma 1 of Sklyarenko (12), and is there attributed to Ponomarev (9).

1·5. LEMMA. *Let f be a continuous map of a space X into a space Y, and $\bar{f}$ an extension of f to compactifications $\theta_1 X$ and $\theta_2 Y$. Then $\bar{f}(\theta_1 X \sim X) \subset \theta_2 Y \sim Y$ if and only if f is perfect.*

Thus if $\theta_1 X \succ \theta_2 X$ and g is the natural map of $\theta_1 X$ onto $\theta_2 X$, then

$$g(\theta_1 X \sim X) \subset \theta_2 X \sim X.$$

In fact $g(\theta_1 X \sim X) = \theta_2 X \sim X$, because g is onto.

In (1·3) if I is a directed set defined by $i \geqslant j$ if $\theta_i X \succ \theta_j X$, the corresponding mapping being ϕ_{ij}, then since p_i and p_j are the extensions of the identity on X to maps

$$\mathrm{cl}_P X \to \theta_i X \quad \text{and} \quad \mathrm{cl}_P X \to \theta_j X$$

respectively, we have $\phi_{ij} \circ p_i = p_j$.

Using the following result of Bourbaki (2) (Chapter I, 4.4)

1·6. LEMMA. $\lim_{\leftarrow} \{\theta_i X : \phi_{ij}, I\}$ *is the subspace of* $\Pi(\theta_2 X | i \in I\}$ *consisting of those elements* x *such that* $p_j(x) = \phi_{ij}(p_i(x))$ *for* $i \geqslant j$,

we obtain

1·7. PROPOSITION. $\text{l.u.b.}\{\theta_i X | i \in I\} = \lim_{\leftarrow} \{\theta_i X : \phi_{ij}, I\}$.

Proof. Clearly from the preceding paragraph and (1·6)

$$\text{l.u.b.}\{\theta_i X | i \in I\} = \mathrm{cl}_P X \subset \lim_{\leftarrow} \{\theta_i X : \phi_{ij}, I\}.$$

The natural imbedding here is a homeomorphism onto a subspace of $\lim_{\leftarrow} \{\theta_i X : \phi_{ij}, I\}$, which extends the identity on X. Since X is dense in $\mathrm{cl}_P X$ and in $\lim_{\leftarrow} \{\theta_i X : \phi_{ij}, I\}$, both of which are compact, the imbedding must be onto. Hence our result.

1·8. It is clear from (1·5) that $(\mathrm{cl}_P X) \sim X \subset \Pi\{\theta_i X \sim X | i \in I\}$, and, regarding I as a directed set, $(\mathrm{cl}_P X) \sim X$ is simply all elements $\{x_i : i \in I\}$ of $\Pi\{\theta_i X \sim X | i \in I\}$ such that $\phi_{ij}(x_i) = x_j$ if $i > j$.

2. *Zero-dimensional compactifications.*

2·1. DEFINITION. *ϵX is a zero-dimensional compactification of X if $\epsilon X \sim X$ has a base of open-and-closed sets.*

2·2. LEMMA. *If X possesses a zero-dimensional compactification, then it has a maximum one.*

Proof. Let $\{\epsilon_i X | i \in I\}$ be all the zero-dimensional compactifications of X, and let $\gamma X = \text{l.u.b.}\{\epsilon_i X | i \in I\}$, as given by (1·3). $\gamma X \sim X \subset \Pi\{\epsilon_i X \sim X | i \in I\}$ by (1·8), and the latter is a zero-dimensional space. Thus γX is a zero-dimensional compactification.

The following lemma is the offspring of A. K. and E. F. Steiner (14).

2·3. LEMMA. *If ϵX is any countable compactification of X, then ϵX has a base of open sets, whose boundaries are contained in X.*

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2·4. COROLLARY. *Every countable compactification of X is a zero-dimensional compactification.*

We now undertake a construction, which under the stated conditions enables us to obtain a compactification between two given ones.

Suppose $\theta_1 X$, $\theta_3 X$ are compactifications of X with $\theta_3 X \succ \theta_1 X$, such that the natural map $g\colon \theta_3 X \to \theta_1 X$ is not $(1, 1)$. Let $z \in \theta_1 X \sim X$ be any point such that $g^{-1}(z)$ contains more than one point, and we assume that there is an open neighbourhood G of z in $\theta_1 X$ such that $g^{-1}(G)$ contains two disjoint open subsets H_1, H_2 of $\theta_3 X$ with the following properties

(i) $g^{-1}(G) \cap (\theta_3 X \sim X) \subset H_1 \cup H_2$,

(ii) $g^{-1}(z) \cap H_i \neq \phi$ for $i = 1$ and 2.

Define an equivalence relation $\mathscr{R}$ on $\theta_3 X$ by $(x_1, x_2) \in \mathscr{R}$ if and only if either

$$x_1, x_2 \in H_i \cap g^{-1}(y)$$

for some $y \in \theta_1 X$ and $i = 1$ or 2, or $x_1, x_2 \in g^{-1}(y) \sim (H_1 \cup H_2)$ for some $y \in \theta_1 X$. Thus for each $x \in X$, $\{x\}$ is the equivalence class of x. Let $\theta_2 X$ denote the quotient space $\theta_3 X/\mathscr{R}$. Then

2·5. LEMMA. *$\theta_2 X$ is a compactification of X such that $\theta_3 X \succ \theta_2 X \succ \theta_1 X$, and the natural map $f\colon \theta_2 X \to \theta_1 X$ is not a homeomorphism, but $f^{-1}(y)$ contains at most two elements for each $y \in \theta_1 X \sim X$.*

Proof. Let h be the quotient map $\theta_3 X \to \theta_2 X$. For any $w \in \theta_2 X$, $h^{-1}(w) \subset g^{-1}(y)$ for some $y \in \theta_1 X$. So we have a map $f\colon \theta_2 X \to \theta_1 X$, such that $g = f \circ h$. Thus by (1·2), if we show that $\theta_2 X$ is a compactification of X, it follows that $\theta_2 X \succ \theta_1 X$ with natural map f, which is not a homeomorphism, because $z \in \theta_1 X$ is the image of two points of $\theta_2 X$. Also it is clear from the construction that $f^{-1}(y)$ contains at most two elements for each $y \in \theta_1 X \sim X$.

We need only establish that $\theta_2 X$ is Hausdorff.

Suppose w_1, w_2 are distinct points of $\theta_2 X$. Then $h^{-1}(w_1)$, $h^{-1}(w_2)$ are contained in $g^{-1}(y_1)$, $g^{-1}(y_2)$ respectively for some $y_1, y_2 \in \theta_1 X$.

If $y_1 = y_2$ in $\theta_1 X$, then $g^{-1}(y_1) = h^{-1}(w_1) \cup h^{-1}(w_2)$, and we may assume without loss of generality that $h^{-1}(w_i) \subset H_i$ $(i = 1, 2)$. There are sets L_1, $L_2 \subset \theta_2 X$ such that $h^{-1}(L_i) = H_i$ $(i = 1, 2)$. These subsets L_1, L_2 are thus disjoint open subsets of $\theta_2 X$ containing w_1, w_2 respectively.

If $y_1 \neq y_2$, we can find disjoint G_1, G_2 open in $\theta_1 X$ with $G_i \supset \{y_i\}$ $(i = 1, 2)$. Then $g^{-1}(G_i) = h^{-1}(E_i)$ for some $E_i \subset \theta_2 X$ $(i = 1, 2)$. Thus E_1, E_2 are disjoint open subsets of $\theta_2 X$ containing w_1, w_2 respectively.

2·6. LEMMA. *With the notation of (2·5), if $\theta_1 X$ is a zero-dimensional compactification of X, and G is chosen such that $G \cap (\theta_1 X \sim X)$ is open-and-closed in $\theta_1 X \sim X$, then the compactification $\theta_2 X$ is also zero-dimensional.*

Proof. By (1·5) $\theta_3 X \sim X = h^{-1}(\theta_2 X \sim X) = g^{-1}(\theta_1 X \sim X)$. If $w \in \theta_2 X \sim X$ and L is any open neighbourhood of w in $\theta_2 X$, then $h^{-1}(L)$ is an open neighourhood of $h^{-1}(w)$ in $\theta_3 X$. Also $h^{-1}(w) \subset g^{-1}(y)$ for some $y \in \theta_1 X \sim X$.

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If $h^{-1}(w) = g^{-1}(y)$, then since g is a closed map, there is an open subset M of $\theta_1 X$ containing y such that $g^{-1}(M) \subset h^{-1}(L)$. We can choose M such that $(\theta_1 X \sim X) \cap M$ is open-and-closed in $\theta_1 X \sim X$. Then there is a subset $E \subset \theta_2 X$ such that

$$h^{-1}(E) = g^{-1}(M).$$

So $w \in E \subset L$ and E is open in $\theta_2 X$.

$$\begin{aligned} h^{-1}(E \cap (\theta_2 X \sim X)) &= h^{-1}(E) \cap (\theta_3 X \sim X) \\ &= g^{-1}(M) \cap (\theta_3 X \sim X) \\ &= g^{-1}(M \cap (\theta_1 X \sim X)). \end{aligned}$$

Thus $E \cap (\theta_2 X \sim X)$ is open-and-closed in $\theta_2 X \sim X$.

If $h^{-1}(w)$ is properly contained in $g^{-1}(y)$, then we may assume $h^{-1}(w) = g^{-1}(y) \cap H_1$. Then $h^{-1}(L) \cup H_2$ is an open subset of $\theta_3 X$ containing $g^{-1}(y)$, and so again we can find an open subset M^* of $\theta_1 X$ containing y, such that $M^* \cap (\theta_1 X \sim X)$ is open-and-closed in $\theta_1 X \sim X$, and $g^{-1}(M^*) \subset h^{-1}(L) \cup H_2$. Thus $g^{-1}(M^*) \cap H_1$ is open in $\theta_3 X$. It contains $h^{-1}(w)$, and $g^{-1}(M^*) \cap H_1 \cap (\theta_3 X \sim X)$ is open-and-closed in $\theta_3 X \sim X$. But there is a subset E^* of $\theta_2 X$ with $h^{-1}(E^*) = g^{-1}(M^*) \cap H_1$. Then $L \supset E^* \supset \{w\}$, E^* is open in $\theta_2 X$, and $E^* \cap (\theta_2 X \sim X)$ is open-and-closed in $\theta_2 X \sim X$.

Hence $\theta_2 X$ is a zero-dimensional compactification of X.

2·7. *Note.* A sufficient condition for the required construction of H_1, H_2 in (2·5) would be that $\theta_3 X$ be a zero-dimensional compactification of X.

Then since $g^{-1}(z)$ contains more than one point of $\theta_3 X \sim X$ and is compact, it may be partitioned into disjoint compact subsets F_1, F_2. We can find disjoint open sets G_1, G_2 of $\theta_3 X$ containing F_1, F_2 respectively. Define G to be $\{y | g^{-1}(y) \subset G_1 \cup G_2\}$. The set G is an open neighbourhood of z in $\theta_1 X$, because g is a closed map. Then set $H_i = g^{-1}(G) \cap G_i$. These sets H_1, H_2 satisfy the required conditions for Lemma (2·5).

2·8. THEOREM. *If αX is a maximum countable compactification of X, then it is also a maximum zero-dimensional compactification of X.*

Proof. If not, then there is a zero-dimensional compactification $\epsilon_3 X \succ \alpha X$, the natural map $\epsilon_3 X \to \alpha X$ not being a homeomorphism. Thus by (2·7) we can carry out the construction of (2·5), obtaining a compactification $\epsilon_2 X$ which is strictly greater than αX. However, the construction of $\epsilon_2 X$ is such that $|\epsilon_2 X \sim X| \leqslant 2|\alpha X \sim X|$, which results in the contradiction that $\epsilon_2 X$ is a countable compactification of X. Hence, αX is a maximum zero-dimensional compactification.

3. *Perfect compactifications.* The development of these is attributed by Aleksandrov to Sklyarenko (11). The former on p. 21 of (1) and Isbell on p. 114 of (7) give several equivalent definitions of perfect compactifications, of which we quote three in (3·1).

First, we say that a closed subset F separates a space X into sets G_1, G_2 if

$$X \sim F = G_1 \cup G_2,$$

where G_1 and G_2 are non-empty, disjoint, open subsets of X.

3·1. DEFINITION. We shall call a compactification δX of X a *perfect compactification* if it satisfies any of the following equivalent conditions.

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(*a*) If $y \in \delta X \sim X$, then for any neighbourhood N of y, $N \cap X$ cannot be expressed as the union of two disjoint sets V_1, V_2 each open in $N \cap X$, and having $y \in \mathrm{cl}_\delta V_1 \cap \mathrm{cl}_\delta V_2$.

(*b*) If f is the natural map $\beta X \to \delta X$, then $f^{-1}(z)$ is connected for all $z \in \delta X$.

(*c*) If a closed set F separates X into H_1, H_2, then $\mathrm{cl}_\delta F$ separates δX into G_1, G_2, where $G_i \cap X = H_i$ $(i = 1, 2)$.

In the next theorem we use the following lemma, due to Henriksen and Isbell (6).

3·2. Lemma. *If G is an open subset of βX and $G \cap X$ is partitioned by open sets V_1, V_2 of X, then $\{G \cap \mathrm{cl}_\beta V_1, G \cap \mathrm{cl}_\beta V_2\}$ is a partition of G in βX.*

Our theorem is very similar to Theorem 3 of Sklyarenko (11), the proof of which is given in (13).

3·3. Theorem. *A zero-dimensional compactification γX of X is a maximum zero-dimensional compactification if and only if it is perfect.*

Proof ($\Leftarrow$). Consider any element $y \in \gamma X \sim X$, and let $N_X(y)$ denote the trace on X of the neighbourhood filter of y in γX. Let ϵX be any zero-dimensional compactification of X. If $N_X(y)$, a filter base in ϵX, has only one point in its adherence in ϵX for every $y \in \gamma X \sim X$, then the filter it generates is convergent in the compact space ϵX. Under these conditions we could extend the identity on X to a continuous map from $\gamma X \to \epsilon X$. (See, for example, Bourbaki (2), I, 8·5, Theorem 1.)

We shall show that $\mathrm{adh}_\epsilon N_X(y)$ contains only one point for each $y \in \gamma X \sim X$. Suppose this were not so for some $y \in \gamma X \sim X$. It would follow that there are distinct points $z_1, z_2 \in \mathrm{adh}_\epsilon N_X(y) \subset \epsilon X \sim X$. The set $\mathrm{adh}_\epsilon N_X(y)$ is compact and closed. So we could partition it by disjoint closed subsets K_1, K_2 of ϵX containing z_1, z_2 respectively. Let L_1, L_2 be disjoint open subsets of ϵX containing K_1, K_2 respectively. Set $H_i = X \cap L_i$ and $F = X \sim (H_1 \cup H_2)$. Then F separates X into H_1, H_2 and so, since γX is perfect, $\mathrm{cl}_\gamma F$ separates γX into disjoint open sets G_1, G_2 where $H_i = G_i \cap X$ for $i = 1, 2$. The point $y \notin \mathrm{cl}_\gamma F$, because otherwise F together with $N_X(y)$ would form the base of a filter $\mathscr{F}$ in ϵX, for which

$$\mathrm{adh}_\epsilon \mathscr{F} = \mathrm{cl}_\epsilon F \cap \mathrm{adh}_\epsilon N_X(y) \subset [\epsilon X \sim (L_1 \cup L_2)] \cap (K_1 \cup K_2) = \phi,$$

contradicting the compactness of ϵX. Hence $y \in G_1 \cup G_2$. Suppose $y \in G_1$. Then

$$H_1 \in N_X(y).$$

But since $L_2 \cap H_1 = \phi$, this contradicts the fact that $z_2 \in \mathrm{adh}_\epsilon N_X(y)$. Hence $\mathrm{adh}_\epsilon N_X(y)$ consists of a single point.

($\Rightarrow$). Suppose γX is a zero-dimensional compactification of X, which is not perfect. Then there is a point $z \in \gamma X \sim X$ such that z has an open neighbourhood G with $G \cap X = V_1 \cup V_2$, where $V_1 \cap V_2 = \phi$, $z \in \mathrm{cl}_\gamma V_i$ $(i = 1, 2)$, and V_1, V_2 are open subsets of X. We can also take $G \cap (\gamma X \sim X)$ to be both open-and-closed in $\gamma X \sim X$, since the latter is zero-dimensional.

If $g: \beta X \to \gamma X$ is the natural map, $g^{-1}(G)$ is an open subset of βX and

$$X \cap g^{-1}(G) = V_1 \cup V_2.$$

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By (3·2) if $H_i = \mathrm{cl}_\beta V_i \cap g^{-1}(G)$ for $i = 1, 2$, then $\{H_1, H_2\}$ is a partition of $g^{-1}(G)$ into disjoint open subset of βX. $z \in \mathrm{cl}_\gamma V_i = g(\mathrm{cl}_\beta V_i)$, so that $g^{-1}(z) \cap \mathrm{cl}_\beta V_i \neq \phi$. Hence $g^{-1}(z) \cap H_i \neq \phi$ for $i = 1, 2$.

Thus G, H_1, H_2 satisfy the conditions required in (2·5) and (2·6) for the construction of a zero-dimensional compactification ϵX of X which is strictly greater than γX. Thus γX is not maximum.

We obtain the immediate corollary by (2·8)

3·4. COROLLARY 1. *A countable compactification αX of X is a maximum countable compactification if and only if it is perfect.*

Now suppose that X has a zero-dimensional compactification, and hence a maximum one γX by (2·2). Then γX is perfect, and so if $f: \beta X \to \gamma X$ is the natural map, $f^{-1}(z)$ is connected for each $z \in \gamma X \sim X$. Thus $f^{-1}(z)$ is contained in a component of $\beta X \sim X$.

Conversely if M is a component of $\beta X \sim X$, then $f(M)$ is connected, and so contains only one element. Hence $f^{-1}(z)$ is a component of $\beta X \sim X$ for each $z \in \gamma X \sim X$. Thus

3·5. If γX is a maximum zero-dimensional compactification of X, there is a $(1, 1)$ relation between the components of $\beta X \sim X$ and the elements of $\gamma X \sim X$.

Thus we have

3·6. COROLLARY 2. *If X has at least one zero-dimensional compactification, then X has a maximum countable compactification if and only if $\beta X \sim X$ has only a countable number of components.*

We can in fact slightly extend (3·5), and say that the natural map $f: \beta X \to \gamma X$ induces a $(1, 1)$ relation between the components of any subset S of γX and those of $f^{-1}(S)$.

3·7. COROLLARY 3. *If X has a maximum zero-dimensional compactification γX, this is also the minimum perfect compactification of X.*

Proof. Suppose δX is any perfect compactification of X, and h is the natural map $\beta X \to \delta X$. Let g be the natural map $\beta X \to \gamma X$. Then for any $z \in \delta X \sim X$, $h^{-1}(z)$ is connected in $\beta X \sim X$ by (3·1) (b). The component of $\beta X \sim X$ in which $h^{-1}(z)$ is contained is mapped onto a point $y \in \gamma X \sim X$ by g. Thus we have defined a map $f: \delta X \to \gamma X$, and clearly $g = f \circ h$. So by (1·2) f is continuous, and hence $\delta X \succ \gamma X$.

3·8. DEFINITION. A space is *rim compact* if it has a base consisting of open sets whose frontiers are compact.

3·9. *Example.* The Freudenthal compactification of a rim compact second countable space is its maximum zero-dimensional compactification.

This follows from 7·6 of Freudenthal (4), and our (3·3). The former states that the compactification is both zero-dimensional and perfect.

3·10. *Example.* From the plane R^2 remove any countable subset, leaving a subspace X. Then the one point compactification of R^2 is the maximum countable compactification of X.

This is so because each point of the outgrowth has a base of neighbourhoods, whose trace on X is path connected, and hence (3·1) (a) is satisfied.

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Similarly, we may obtain a maximum zero-dimensional compactification of the plane with any zero-dimensional set along a line removed.

Of course, if we have any compact Hausdorff space Y and remove a zero-dimensional set S leaving a dense subspace X, the restoration of S to X does not necessarily give us the maximum zero-dimensional compactification of X. We illustrate this by an example.

3·11. *Example.* If Q is the space of rationals, then Q is totally disconnected (see Gillman and Jerison (5), p. 91). However, as shown in the same reference, $|\beta Q| = |\beta N|$, and $|\beta N| = 2^c$ (see Pospisil (10)). Hence the two point compactification of the real line is not the maximum zero-dimensional compactification of Q.

Similarly βX is totally disconnected, where X is the set of irrationals on the real line.

The following proposition gives a criterion, that the restoration of a zero-dimensional subset S of a locally connected compact space Y should produce a maximum zero-dimensional compactification of $Y \sim S$.

3·12. Proposition. *If we remove a zero-dimensional subset S from a compact Hausdorff locally connected space Y, leaving a dense subspace $X = Y \sim S$, then the restoration of S to Y gives us Y as the maximum zero-dimensional compactification of X, if and only if every connected open neighbourhood in Y of any point of S remains connected when we remove S.*

Proof. If Y is a maximum zero-dimensional compactification of X, it must be a perfect compactification by (3·3), and hence must satisfy (3·1)(*a*). But if we did have a connected open neighbourhood G of a point of S such that $G \cap X = V_1 \cup V_2$ where the latter are disjoint open subsets of $G \cap X$, then $G \subset \mathrm{cl}_Y(V_1 \cup V_2) = \mathrm{cl}_Y V_1 \cup \mathrm{cl}_Y V_2$. Since G is connected $\mathrm{cl}_Y V_1 \cap \mathrm{cl}_Y V_2 \cap G$ must contain a point y of S. Thus y has a neighbourhood G contradicting condition (3·1)(*a*) for perfect compactifications.

Conversely, suppose that for every connected open neighbourhood G of a point of S, $G \sim S$ is connected. Let y be any point of S and N any neighbourhood such that $X \cap N = V_1 \cup V_2$ where V_1, V_2 are disjoint relatively open subsets of N. The point y has a connected open neighbourhood $G_1 \subset N$. Since $G_1 \cap X$ is connected, we must have $G_1 \cap V_i = \phi$ for $i = 1$ or 2. But, if $G_1 \cap V_1 = \phi$, $y \notin \mathrm{cl}_Y V_1$. Thus (3·1)(*a*) is satisfied. Hence Y is a perfect compactification, and so by (3·3) a maximum zero-dimensional compactification of X.

In particular, if Y is compact and locally connected and if we remove one point y from Y, then Y is the maximum countable compactification of $Y \sim \{y\}$ if and only if y is not a local cut-point of Y.

Another consequence of (3·12) is that if Y is a compact subset of the plane with each point of Y possessing a base of convex non-linear neighbourhoods, and if X is a space obtained by removing a countable subset from Y, then the maximum countable compactification of X is Y.

3·13. Lemma. *If γX is a maximum zero-dimensional compactification of X, then $\mathrm{cl}_\gamma G$ is a maximum zero-dimensional compactification of $\mathrm{cl}_X G$ where G is any open subset of X with compact frontier.*

Proof. The result is trivially true if $\mathrm{cl}_X G = X$. So we suppose $\mathrm{cl}_X G \neq X$. γX is a

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perfect compactification of X, and so by (3·1)(*c*), $\mathrm{cl}_X G \sim G$ separates γX into H_1, H_2, where $H_1 \cap X = G$ and $H_2 \cap X = X \sim \mathrm{cl}_X G$. Then $\mathrm{cl}_\gamma G = \gamma X \sim H_2$, and for any point $y \in (\mathrm{cl}_\gamma G) \sim \mathrm{cl}_X G$ we can take a base of neighbourhoods of y in $\mathrm{cl}_\gamma G$ to be $N \cap H_1$, where N is any neighbourhood of y in γX. This is still a base of neighbourhoods of y in γX, satisfying (3·1)(*a*). Hence $\mathrm{cl}_\gamma G$ is a perfect, and thus a maximum zero-dimensional compactification of $\mathrm{cl}_X G$.

The next definition, due to Magill (8) is closely connected with the study of finite compactifications.

3·14. Definition. $[K, \{G_i | 1 \leqslant i \leqslant N\}]$ is an *N-star* of a space X provided:

(i) $\{G_i | 1 \leqslant i \leqslant N\}$ are mutually disjoint.

(ii) $K = X \sim \bigcup \{G_i | 1 \leqslant i \leqslant N\}$ is compact.

(iii) $K \cup G_i$ is not compact for any i $(1 \leqslant i \leqslant N)$.

The following proposition, useful in determining whether or not a particular zero-dimensional compactification is maximum, is another consequence of (3·3). We here consider all two-stars $[K, \{H_1, H_2\}]$, where K contains some given compact subset K_0 of X. This does not in any way restrict the result, because we could of course take $K_0 = \phi$.

3·15. Proposition. *If K_0 is any compact subset of X, then a zero-dimensional compactification γX of X is maximum if and only if for any two-star $[K, \{H_1, H_2\}]$ of X with $K_0 \subset K$ we have* $\mathrm{cl}_\gamma H_1 \cap \mathrm{cl}_\gamma H_2 \cap (\gamma X \sim X) = \phi$.

Proof ($\Rightarrow$). By (3·3) γX is perfect. Hence for any two-star $[K, \{H_1, H_2\}]$, $\mathrm{cl}_\gamma K = K$ must separate γX into two open subsets G_1, G_2 of γX, whose traces on X are H_1 and H_2 respectively. Clearly $\mathrm{cl}_\gamma H_i \subset G_i \cup K$, and so $\mathrm{cl}_\gamma H_1 \cap \mathrm{cl}_\gamma H_2 \cap (\gamma X \sim X) = \phi$.

($\Leftarrow$). Conversely suppose γX were not maximum. Then by (3·3) and (3·1)(*a*) there is a neighbourhood N in γX of some point $y \in \gamma X \sim X$ such that $N \cap X$ is partitioned by the subsets V_1, V_2, each open in $N \cap X$ such that $y \in \mathrm{cl}_\gamma V_1 \cap \mathrm{cl}_\gamma V_2$.

We can find an open subset G of γX such that $y \in G \subset N$, $G \cap (\gamma X \sim X)$ is open and closed in $\gamma X \sim X$ and $\mathrm{cl}_X G \sim G$ compact. Let $V_i^* = V_i \cap G$ $(i = 1, 2)$. Then $y \in \mathrm{cl}_\gamma V_1^* \cap \mathrm{cl}_\gamma V_2^*$.

Let $H_1 = V_1^* \sim K_0$, $H_2 = [V_2^* \cup (X \sim \mathrm{cl}_X G)] \sim K_0$, and $K = (\mathrm{cl}_X G \sim G) \cup K_0$. Then $[K, \{H_1, H_2\}]$ is a two-star of X with $K_0 \subset K$, and $\mathrm{cl}_\gamma H_1 \cap \mathrm{cl}_\gamma H_2 \cap (\gamma X \sim X) \neq \phi$. This proves sufficiency.

3·16. Corollary. *If γX is a maximum zero-dimensional compactification of X, then for any N-star $[K, \{H_i | 1 \leqslant i \leqslant N\}]$ of X*

$$(\gamma X \sim X) \cap \mathrm{cl}_\gamma H_i \cap \mathrm{cl}_\gamma H_j = \phi \quad \text{for} \quad i \neq j.$$

Proof. $[K, \{H_j, \bigcup \{H_r | r \neq j\}\}]$ is a two-star of X. Hence by (3·15)

$$\begin{aligned} \phi &= (\gamma X \sim X) \cap \mathrm{cl}_\gamma H_j \cap \mathrm{cl}_\gamma(\bigcup \{H_r | r \neq j\}) \\ &\supset (\gamma X \sim X) \cap \mathrm{cl}_\gamma H_j \cap \mathrm{cl}_\gamma H_i, i \neq j. \end{aligned}$$

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