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Dimensionally dependent divergences

BY GREGORY W. HORNDESKI
Department of Applied Mathematics,
University of Waterloo,
Waterloo, Ontario, Canada

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Abstract. It is well known that in an even dimensional Riemannian space there always exists a particular scalar density which is such that its associated Euler–Lagrange tensor vanishes identically. In this note we show that each of these scalar densities can be expressed as the ordinary divergence of a contravariant vector density. Furthermore, there exists an entire class of contravariant vector densities which enjoy this property. These results are discussed with particular reference to the general theory of relativity.

1. *Introduction.* Our considerations will be based upon an everywhere regular region of an n-dimensional Riemannian space with line element†

$$ds^2 = g_{ij}dx^i dx^j,$$

where the x^i's are the local coordinates of the region under consideration and

$$g = \det g_{ij} \neq 0.$$

Since the signature of g_{ij} will not play a role in the forthcoming analysis we shall, without loss of generality, restrict ourselves to the case where g is positive. The Riemann–Christoffel curvature tensor, $R_a{}^b{}_{cd}$, is defined by means of the Ricci identity as follows:

$$A^b{}_{|cd} - A^b{}_{|dc} = A^a R_a{}^b{}_{cd},$$

where a vertical bar denotes covariant differentiation with respect to the Christoffel symbols and A^b is an arbitrary contravariant vector field of class C^2. Lastly, we define the generalized Kronecker delta, $\delta^{i_1\dots i_p}_{j_1\dots j_p}$ $(p = 2, 3, \dots)$ by

$$\delta^{i_1\dots i_p}_{j_1\dots j_p} = \det\begin{vmatrix} \delta^{i_1}_{j_1} \dots \delta^{i_1}_{j_p} \\ \vdots \quad\quad \vdots \\ \delta^{i_p}_{j_1} \dots \delta^{i_p}_{j_p} \end{vmatrix}.$$

It is known (Buchdahl (1), Lovelock (2)) that the Euler–Lagrange tensor obtained from the scalar density‡

$$L = \sqrt{g}\,\delta^{i_1\dots i_{2m}}_{j_1\dots j_{2m}} R_{i_1 i_2}{}^{j_1 j_2} \dots R_{i_{2m-1} i_{2m}}{}^{j_{2m-1} j_{2m}} \tag{1·1}$$

† The summation convention will be used throughout this paper whereby all Latin indices are summed from 1 to n.

‡ In what follows the contravariant and covariant forms of the metric tensor will be used to raise and lower indices; furthermore, it will be assumed that the metric tensor is of class C^3.

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($m = 1, 2, \ldots$) through a variation of the g_{ij}'s is at most of second order in the derivatives of g_{ij}. Furthermore if the dimension of the space under consideration is even and $m = \frac{1}{2}n$ the Euler–Lagrange tensor associated with L vanishes identically. Since it is also known (e.g. Lovelock(3)) that the Euler–Lagrange tensor associated with an ordinary divergence vanishes identically the following question naturally arises: Is the scalar density presented in equation (1·1) an ordinary divergence in a space of even dimension n when $m = \frac{1}{2}n$; and if so can it be obtained from the ordinary divergence of a contravariant vector density?

Buchdahl(1, 4) has shown that the answer to the first part of the above question is in the affirmative when $n = 4$ or 6. However, in both cases the proof depends upon the signature of the metric tensor and in addition the tensorial character (if any) of the differentiated quantities is obscure.

In this note we shall show that in general L can be expressed as the sum of an ordinary divergence (of a contravariant vector density) and additional terms involving $\delta^{i_1 \cdots i_{2m+1}}_{j_1 \cdots j_{2m+1}}$ in a linear fashion. Consequently when we consider (1·1) for a space of dimension $n = 2m$ the terms involving $\delta^{i_1 \cdots i_{2m+1}}_{j_1 \cdots j_{2m+1}}$ will vanish identically, in which case L is a pure divergence. Furthermore, the contravariant vector density, V^a, which appears in this divergence will be of the form†

$$V^a = V^a(g_{ij}; g_{ij,h}; g_{ij,hk}; U^i; U^i{}_{,j}),$$

where U^i is an *arbitrary*, non-null, contravariant vector field of class C^2. Clearly then there exists a class of vector densities V^a (depending on the choice of U^i) which have this property.

In order to illustrate the technique which will be employed to establish the general result mentioned above (and at the same time consider a scalar density which is of particular interest in the general theory of relativity) we shall now show how L may be represented in the simplest case, i.e. when $m = 1$. In this case (1·1) reduces to

$$L = 2\sqrt{g}R, \tag{1·2}$$

where

$$R = g^{ac}R_a{}^b{}_{cb}.$$

To attain this end let us examine the covariant divergence of the vector density

$$V^a = -(4/\rho)\sqrt{g}\,\delta^{ab}_{cd}\,U^c U^d{}_{|b}, \tag{1·3}$$

where the U^p's are the components of a contravariant vector field of class C^2, which is arbitrary except that

$$\rho \equiv U^c U^d g_{cd} \neq 0. \tag{1·4}$$

Using the Ricci identity we find that $V^a{}_{|a}$ is given by

$$V^a{}_{|a} = -\frac{4}{\rho}\sqrt{g}\,\delta^{ab}_{cd}\,U^c{}_{|a}\,U^d{}_{|b} + \frac{8\sqrt{g}}{\rho^2}\delta^{ab}_{cd}\,U^c U_p\,U^p{}_{|a}\,U^d{}_{|b} - \frac{2}{\rho}\sqrt{g}\,\delta^{ab}_{cd}\,U^c U_m R^{md}{}_{ba}. \tag{1·5}$$

Upon expanding the generalized Kronecker deltas appearing in

$$\delta^{abc}_{def}U_a\,U^d U^e{}_{|b}\,U^f{}_{|c}$$

and

$$\delta^{abc}_{def}U_a\,U^d R_{bc}{}^{ef}$$

† A comma denotes partial differentiation.

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about their first row (i.e. the index a) we easily see that the resulting expressions may be used in conjunction with equation (1·5) to yield

$$2\sqrt{g}\,R = V^a{}_{|a} + \frac{\sqrt{g}}{\rho}\delta^{abc}_{def}U_aU^dR_{bc}{}^{ef} + 4\frac{\sqrt{g}}{\rho^2}\delta^{abc}_{def}U_aU^dU^e{}_{|b}U^f{}_{|c}.$$

Thus we have established

THEOREM 1. *In a space of arbitrary dimension n*

$$\sqrt{g}\,R = \left\{-2\frac{\sqrt{g}\,\delta^{ab}_{cd}U^cU^d{}_{|b}}{\rho}\right\}_{|a} + \frac{\sqrt{g}}{2\rho}\delta^{abc}_{def}U_aU^dR_{bc}{}^{ef} + 2\frac{\sqrt{g}}{\rho^2}\delta^{abc}_{def}U_aU^dU^e{}_{|b}U^f{}_{|c},$$

where ρ is given by equation (1·4) *and U^c is any non-null vector field of class C^2. In particular*† *if $n = 2$*

$$\sqrt{g}\,R = \left\{-2\sqrt{g}\frac{\delta^{ab}_{cd}U^cU^d{}_{|b}}{\rho}\right\}_{|a}.$$

Since the covariant divergence of a contravariant vector density is the same as its ordinary divergence we may use Theorem 1 to conclude that $\sqrt{g}\,R$ *is an ordinary divergence in spaces of dimension* 2.

The scalar density $\sqrt{g}\,R$ plays an important role in the general theory of relativity. Due to Theorem 1 we may conclude that since ordinary divergences identically satisfy the Euler–Lagrange equations the following Lagrangian

$$\mathscr{L} = (1/2\rho)\sqrt{g}\,\delta^{abc}_{def}U_aU^dR_{bc}{}^{ef} + (2/\rho^2)\sqrt{g}\,\delta^{abc}_{def}U_aU^dU^e{}_{|b}U^f{}_{|c} \tag{1·6}$$

is equivalent to $\sqrt{g}\,R$ (from the point of view of the calculus of variations) in a space of any dimension. Thus in a space of dimension 4 the Einstein vacuum field equations may be obtained from the Lagrangian (1·6) when U^i is any non-null contravariant vector field of class C^2.

2. *The contravariant vector density V^a.* We shall now proceed to show that (for $m = 2, 3, \ldots$) the scalar density (1·1) is identically equal to the ordinary divergence of a contravariant vector density together with tensorial terms involving $\delta^{i_1\ldots i_{2m+1}}_{j_1\ldots j_{2m+1}}$ in a linear manner.

To see this let us consider the following contravariant vector density

$$V^a = \sqrt{g}\sum_{k=0}^{m-1}\left\{\frac{C_k}{\rho^{k+1}}\delta^{ai_1\ldots i_{2m-1}}_{bj_1\ldots j_{2m-1}}U^bU^{j_1}{}_{|i_1}\ldots U^{j_{2k+1}}{}_{|i_{2k+1}} \times R_{i_{2k+2}i_{2k+3}}{}^{j_{2k+2}j_{2k+3}}\ldots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}\right\}, \tag{2·1}$$

where, for $k = 1, \ldots, m-1$,

$$C_k = -4^{k+1}m\prod_{q=0}^{k-1}\left(\frac{m-q-1}{2q+3}\right) \tag{2·2}$$

with

$$C_0 = -4m. \tag{2·3}$$

† This type of an approach to the derivation of dimensionally dependent identities is due to Lovelock (5, 6).

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Using the Ricci and the Bianchi identities we can easily calculate $V^a{}_{|a}$ viz.

$$V^a{}_{|a} = \sqrt{g}\sum_{k=0}^{m-1}\Big[C_k\Big\{-\frac{2(k+1)}{\rho^{k+2}}U_p\,U^p{}_{|a}\,U^b U^{j_1}{}_{|i_1}\dots U^{j_{2k+1}}{}_{|i_{2k+1}}$$
$$+\frac{1}{\rho^{k+1}}U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2k+1}}{}_{|i_{2k+1}}+\frac{1}{\rho^{k+1}}\frac{2k+1}{2}\,U^b U_m R^{mj_1}{}_{i_1 a}\,U^{j_2}{}_{|i_2}\dots U^{j_{2k+1}}{}_{|i_{2k+1}}\Big\}$$
$$\times\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}R_{i_{2k+2}i_{2k+3}}{}^{j_{2k+2}j_{2k+3}}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}\Big]. \tag{2·4}$$

In order to rewrite $V^a{}_{|a}$ in a more desirable form let us examine the sum of the rth and $r+1$st terms ($r=0,\dots,m-2$) appearing in equation (2·4), viz.

$$\Big[\frac{\sqrt{g}\,C_r}{\rho^{r+1}}\Big\{-\frac{2(r+1)}{\rho}U_p\,U^p{}_{|a}\,U^bU^{j_1}{}_{|i_1}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}+U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}$$
$$+\tfrac{1}{2}(2r+1)\,U^bU_mR^{mj_1}{}_{i_1a}\,U^{j_2}{}_{|i_2}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}\Big\}R_{i_{2r+2}i_{2r+3}}{}^{j_{2r+2}j_{2r+3}}$$
$$+\frac{\sqrt{g}\,C_{r+1}}{\rho^{r+2}}\Big\{-\frac{2(r+2)}{\rho}U_p\,U^p{}_{|a}\,U^bU^{j_1}{}_{|i_1}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}+U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}$$
$$+\tfrac{1}{2}(2r+3)\,U^bU_mR^{mj_1}{}_{i_1a}\,U^{j_2}{}_{|i_2}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}\Big\}\Big]$$
$$\times\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}R_{i_{2r+4}i_{2r+5}}{}^{j_{2r+4}j_{2r+5}}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}.$$

Upon rearrangement we find that the above term may be written as follows:

$$\tfrac{1}{2}\sqrt{g}\,\frac{C_r(2r+1)}{\rho^{r+1}}\,\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}U^bU_mR^{mj_1}{}_{i_1a}\,U^{j_2}{}_{|i_2}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}$$
$$\times R_{i_{2r+2}i_{2r+3}}{}^{j_{2r+2}j_{2r+3}}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}$$
$$+\sqrt{g}\Big[\frac{C_r}{\rho^{r+2}}\{-2(r+1)\,U_p\,U^p{}_{|a}\,U^bU^{j_1}{}_{|i_1}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}+\rho U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}\}$$
$$\times R_{i_{2r+2}i_{2r+3}}{}^{j_{2r+2}j_{2r+3}}+\frac{1}{2}\frac{C_{r+1}}{\rho^{r+2}}(2r+3)\,U^bU_mR^{mj_1}{}_{i_1a}\,U^{j_2}{}_{|i_2}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}\Big]\cdot$$
$$\times\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}R_{i_{2r+4}i_{2r+5}}{}^{j_{2r+4}j_{2r+5}}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}$$
$$+\frac{\sqrt{g}\,C_{r+1}}{\rho^{r+3}}\{-2(r+2)U_p\,U^p{}_{|a}\,U^bU^{j_1}{}_{|i_1}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}+\rho U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}\}$$
$$\times\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}R_{i_{2r+4}i_{2r+5}}{}^{j_{2r+4}j_{2r+5}}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}. \tag{2·5}$$

In the light of expression (2·5) it is now evident that $V^a{}_{|a}$, as given in equation (2·4), may be rewritten in the following form

$$V^a{}_{|a}=\frac{\sqrt{g}\,C_0}{2\rho}\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}U^bU_mR^{mj_1}{}_{i_1a}R_{i_2i_3}{}^{j_2j_3}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}$$
$$+\sqrt{g}\sum_{r=0}^{m-2}\Big(\frac{C_r}{\rho^{r+2}}[\{-2(r+1)\,U_p\,U^p{}_{|a}\,U^bU^{j_1}{}_{|i_1}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}$$
$$+\rho U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2r+1}}{}_{|i_{2r+1}}\}R_{i_{2r+2}i_{2r+3}}{}^{j_{2r+2}j_{2r+3}}+2(m-r-1)\,U^bU_m$$
$$\times R^{mj_1}{}_{i_1a}\,U^{j_2}{}_{|i_2}\dots U^{j_{2r+3}}{}_{|i_{2r+3}}]\,\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}R_{i_{2r+4}i_{2r+5}}{}^{j_{2r+4}j_{2r+5}}\dots R_{i_{2m-2}i_{2m-1}}{}^{j_{2m-2}j_{2m-1}}\Big)$$
$$+\sqrt{g}\,\frac{C_{m-1}}{\rho^{m+1}}\{-2mU_p\,U^p{}_{|a}\,U^bU^{j_1}{}_{|i_1}\dots U^{j_{2m-1}}{}_{|i_{2m-1}}$$
$$+\rho U^b{}_{|a}\,U^{j_1}{}_{|i_1}\dots U^{j_{2m-1}}{}_{|i_{2m-1}}\}\,\delta^{ai_1\dots i_{2m-1}}_{bj_1\dots j_{2m-1}}, \tag{2·6}$$

where we have used equation (2·2) to express C_{r+1} in terms of C_r.

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Let us now consider the following scalar quantities

$$W(s) = \delta^{ai_1\dots i_{2m}}_{bj_1\dots j_{2m}} U_a U^b U^{j_1}{}_{|i_1} \dots U^{j_{2s}}{}_{|i_{2s}} R_{i_{2s+1}i_{2s+2}}{}^{j_{2s+1}j_{2s+2}} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}}, \quad (2{\cdot}7)$$

where $s = 0, 1, \dots, m$. Upon expanding the generalized Kronecker delta appearing in equation (2·7) about its first row we easily find, after some manipulation, that

$$\begin{aligned} W(s) = {} & \rho \delta^{i_1\dots i_{2m}}_{j_1\dots j_{2m}} U^{j_1}{}_{|i_1} \dots U^{j_{2s}}{}_{|i_{2s}} R_{i_{2s+1}i_{2s+2}}{}^{j_{2s+1}j_{2s+2}} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}} \\ & - 2s\, \delta^{i_1\dots i_{2m}}_{bj_2\dots j_{2m}} U^b U_a U^a{}_{|i_1} U^{j_2}{}_{|i_2} \dots U^{j_{2s}}{}_{|i_{2s}} R_{i_{2s+1}i_{2s+2}}{}^{j_{2s+1}j_{2s+2}} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}} \\ & + 2(m-s)\, \delta^{i_{2s+1}i_{2s+2}i_1\dots i_{2s}i_{2s+3}\dots i_{2m}}_{bj_{2s+2}j_1\dots j_{2s}j_{2s+3}\dots j_{2m}} U^b U_a R^{aj_{2s+2}}{}_{i_{2s+2}i_{2s+1}} \\ & \times U^{j_1}{}_{|i_1} \dots U^{j_{2s}}{}_{|i_{2s}} R_{i_{2s+3}i_{2s+4}}{}^{j_{2s+3}j_{2s+4}} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}}. \end{aligned} \quad (2{\cdot}8)$$

An examination of equation (2·8) shows that we may rewrite equation (2·6) as follows:

$$V^a{}_{|a} = \sqrt{g}\, \delta^{i_1\dots 2m}_{j_1\dots 2m} R_{i_1i_2}{}^{j_1j_2} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}} - \frac{\sqrt{g}}{\rho} W(0) + \sqrt{g} \sum_{r=0}^{m-1} \frac{C_r}{\rho^{r+2}} W(r+1), \quad (2{\cdot}9)$$

where we have made use of equation (2·3).

Due to equation (2·9) we have established the following

THEOREM 2. *In a space of arbitrary dimension*

$$\sqrt{g}\, \delta^{i_1\dots i_{2m}}_{j_1\dots j_{2m}} R_{i_1i_2}{}^{j_1j_2} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}} = V^a{}_{|a} + \frac{\sqrt{g}}{\rho} W(0) - \sqrt{g} \sum_{r=0}^{m-1} \frac{C_r}{\rho^{r+2}} W(r+1),$$

where (for $m = 2, 3, \dots$) V^a *and* $W(s)$ *are given by equations* (2·1) *and* (2·7) *respectively.*

Now in a space of dimension less than or equal to $2m$

$$\delta^{i_1\dots i_{2m+1}}_{j_1\dots j_{2m+1}} = 0$$

identically, and consequently so does $W(s)$. Thus we have

THEOREM 3. *In a space of even dimension,* $n = 2m$ $(m = 2, 3, \dots)$,

$$\sqrt{g}\, \delta^{i_1\dots i_{2m}}_{j_1\dots j_{2m}} R_{i_1i_2}{}^{j_1j_2} \dots R_{i_{2m-1}i_{2m}}{}^{j_{2m-1}j_{2m}} = V^a{}_{|a} = V^a{}_{,a},$$

where V^a *is the contravariant vector density defined in equation* (2·1).

Remarks. We emphasize that the results presented in this and the preceeding section are valid for *any* non-null contravariant vector field U^i of class C^2. Thus, for example, we could choose U^i to be any vectorial quantity which is a concomitant of g_{ij} and its derivatives; e.g. we may replace U^i by

$$R^{|i}, \quad (R^{abcd}R_{abcd})^{|i} \quad \text{or} \quad R^{iabc|d}R_{|ab}R_{|cd},$$

provided that these vectors are non-null in the space under consideration.

Lastly we would like to point out that there appears to be no obvious relationship between the vector density V^a given by equation (2·1) (when $n = 4$ and 6, and correspondingly $m = 2$ and 3) and the quantities presented by Buchdahl (1, 4).

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