Econometric Theory, **17**, 2001, 247–256. Printed in the United States of America.

ON THE RANGE OF CORRELATION COEFFICIENTS OF BIVARIATE ORDERED DISCRETE RANDOM VARIABLES

LUNG-FEI LEE
The Ohio State University

The range of correlation coefficient of any bivariate discrete random vector with finite or countably infinite values is derived. We show analytically that the normal-transformed discrete bivariate probability system of Van Ophem (1999, *Econometric Theory* 15, 228–237) has a flexible correlation coefficient. We establish the strictly monotonic relation of its correlation coefficient with the bivariate normal correlation-coefficient parameter in the system.

1. INTRODUCTION

In a recent article in this journal, Van Ophem (1999) considers the formulation of correlated discrete random variables with given marginal probability functions. He applies his proposed technique to the formulation of a bivariate Poisson model. His approach is to transform each of the marginal probability functions through a normal transformation and then describe a bivariate normal probability function. This approach generalizes the distribution-transformation approach of Mardia (1970) to discrete or counting random variables.

Suppose that F_1 and F_2 are two univariate distributions. An interesting problem is to construct a bivariate joint distribution F that contains the given F_1 and F_2 as its marginal distributions but whose correlation coefficient is not restricted in a narrow range. The construction of such bivariate distributions with given marginal distributions was originally considered in Frechet (1951), Plackett (1965), and Mardia (1970). Plackett (1965) introduces the contingency bivariate distribution, and Mardia (1970) suggests distribution-transformation methods. Plackett (1965) and Mardia (1970) consider only continuous random variables. Lee (1983) adopts Mardia's distribution-transformation method for the construction of sample selection models utilizing underlying latent continuous random variables. In a recent publication, Van Ophem (1999) generalizes the Mardia transformation method to the construction of bivariate discrete ran-

Financial support from the RGC of Hong Kong under grant HKUST595/96H for my research is gratefully acknowledged. Address correspondence to: Lung-fei Lee, Department of Economics, Ohio State University, 410 Arps Hall, 1945 N. High Street, Columbus, OH 43210, USA.

© 2001 Cambridge University Press 0266-4666/01 $9.50

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

dom variables with given marginal discrete probability functions. The problem in Van Ophem (1999) differs from the problem considered in Lee (1983) in that there may not be any underlying latent continuous variables that generate the discrete random variables. Van Ophem's idea is to embed a univariate probability function into a univariate normal distribution function and modify Mardia's transformation method to construct a bivariate probability function.

An important issue for such a construction is whether the correlation coefficient of the resulting bivariate random variables is flexible without a restricted range. Furthermore, one has to establish some statistical relations of the correlation coefficient with the underlying parameter of the bivariate system. To illustrate the possible relation of the latter, Van Ophem (1999, pp. 232–233) performs a numerical simulation for a case with given Poisson marginal probability functions. However, he has not been able to provide analytical results. So it remains an open question whether the suggested transformation approach is rich enough that the proposed bivariate probability function has a flexible correlation coefficient. The possible monotonicity of the relation between the correlation coefficient and the underlying parameter also needs to be formally established. In this paper, we provide affirmative analytical results on these issues. Our results are general and have broad implications.

2. BIVARIATE PROBABILITY FUNCTIONS WITH GIVEN MARGINS

Frechet (1951) points out that if F is a bivariate distribution with marginal distributions F_1 and F_2, the following inequalities,

$$\max\{F_1 + F_2 - 1,0\} \leq F \leq \min\{F_1, F_2\}, \tag{2.1}$$

must be satisfied (see also Plackett, 1965; Johnson and Kotz, 1972, pp. 23–24). The upper and lower limits in (2.1) are themselves joint distributions with F_1 and F_2 as their marginal distributions. It has been argued that a rich bivariate distribution should contain the limits in (2.1) and the case of independence as special ones. Plackett (1965) introduces a contingency-type bivariate distribution. Mardia (1970) points out that a simple way of constructing systems that include the limits in (2.1) and the case of independence is by transformations. This is done, for example, by the transformations $u_j = \Phi^{-1}[F_j(\epsilon_j)]$ for $j = 1,2$, where Φ is the standard normal distribution. The ϵ_j is transformed by its distribution function F_j into a uniform random variable on $[0,1]$, and the uniform random variable is then transformed into a standard normal variable u_j. The transformed u_1 and u_2 are then ascribed by a joint bivariate normal distribution. Specifically, let $B(u_1, u_2; \rho)$ be the standard bivariate normal distribution of u_1 and u_2 with a correlation coefficient ρ. The constructed bivariate distribution of ϵ_1 and ϵ_2 based on normal transformations is

$$F(\epsilon_1, \epsilon_2) = B(\Phi^{-1}[F_1(\epsilon_1)], \Phi^{-1}[F_2(\epsilon_2)]; \rho). \tag{2.2}$$

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

Lee (1983) suggests the use of this system for the construction of sample selection models with conditional logit choice probabilities and continuous outcomes. This transformation approach is designed only for continuous random variables as discrete random variables can not be transformed into uniform random variables. Any transformed discrete random variables must be discrete.

For discrete random variables, Van Ophem (1999) argues that the proper objects for transformation are the marginal probability functions. Let Y_1 and Y_2 be two discrete random variables with finite or countable values. Suppose that the cumulative probability function of Y_1 is known to be G_1, that is, $G_1(j) = P(Y_1 \leq j)$, and the cumulative probability function of Y_2 is G_2. Van Ophem's method is to embed the probabilities $G_1(j)$ and $G_2(k)$ of the events $Y_1 \leq j$ and $Y_2 \leq k$ into a bivariate probability function. The terms $G_1(j)$ and $G_2(k)$ can first be transformed into $\Phi^{-1}[G_1(j)]$ and $\Phi^{-1}[G_2(k)]$, and then a bivariate probability function is ascribed:

$$G_{12}(j,k;\rho) = B(\Phi^{-1}[G_1(j)],\Phi^{-1}[G_2(k)];\rho) = \int_{-\infty}^{\Phi^{-1}[G_1(j)]} \int_{-\infty}^{\Phi^{-1}[G_2(k)]} \phi(u,v;\rho)\,du\,dv, \tag{2.3}$$

where $\phi(\cdot,\cdot;\rho)$ is the bivariate standard normal density with correlation coefficient ρ. This cumulative bivariate function G_{12} is a proper probability function with marginal cumulative probability functions G_1 and G_2. The latter is so because when $G_2(m) = 1$ where m is the largest possible value of Y_2, $\Phi^{-1}[G_2(m)] = \infty$ and $G_{12}(j,m;\rho) = B(\Phi^{-1}[G_1(j)],\infty;\rho) = \Phi(\Phi^{-1}[G_1(j)]) = G_1(j)$; similarly, this is so for Y_2. An interesting issue is whether the constructed bivariate cumulative distribution G_{12} in (2.3) is rich enough to have a flexible correlation coefficient. Another issue is the possible relation of ρ with the correlation coefficient of the introduced bivariate probability function.

The ρ in $B(\cdot,\cdot;\rho)$ is the correlation coefficient of two (underlying) standard normal variables. Suppose that the values of Y_1 are $0,1,\ldots,n$ and Y_2 can take values $0,1,\ldots,m$ where n and m may be finite or countable infinity. The correlation coefficient of Y_1 and Y_2 is

$$\rho_{12} = \left\{ \sum_{j=0}^{n} \sum_{k=0}^{m} jkP(Y_1 = j, Y_2 = k) - \left[\sum_{j=0}^{n} jP(Y_1 = j) \right] \left[\sum_{k=0}^{m} kP(Y_2 = k) \right] \right\} \Big/ \sigma_1 \sigma_2, \tag{2.4}$$

where σ_1 and σ_2 are, respectively, the standard deviations of Y_1 and Y_2. Van Ophem (1999) points out that ρ in (2.3) and ρ_{12} from $G_{12}(\cdot,\cdot;\rho)$ should be related. He points out that if $\rho = 0$, $\mathrm{cov}(Y_1,Y_2) = 0$ and, consequently, $\rho_{12} = 0$. The cases $\rho = \pm 1$ would not necessarily translate into $\rho_{12} = \pm 1$, and there might exist some lower bound (> -1) and upper bound ($< +1$) of ρ_{12}. It is expected that ρ_{12} would be an increasing function of ρ. To illustrate the possible relation of ρ and ρ_{12} under (2.3), Van Ophem (1999, pp. 232–233) per-

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

forms a numerical simulation for a case with given Poisson marginal probability functions. However, analytical results are not available in his article.

3. THE RANGE OF CORRELATION COEFFICIENT OF A DISCRETE BIVARIATE RANDOM VECTOR

For discrete random variables Y_1 and Y_2, any joint probability cumulative function G of Y_1 and Y_2 with margins G_1 and G_2 satisfies the inequalities

$$\max\{G_1(j) + G_2(k) - 1, 0\} \leq G(j,k) \leq \min\{G_1(j), G_2(k)\}, \tag{3.1}$$

similarly to (2.1). The second inequality in (3.1) is trivial because $G(j,k) \leq G(j,m) = G_1(j)$ and $G(j,k) \leq G(n,k) = G_2(k)$, where n and m are, respectively, the largest possible values of Y_1 and Y_2. The first inequality in (3.1) is as follows. For any values j of Y_1 and k of Y_2,

$$\begin{aligned} 1 - G(j,k) &= P(Y_1 > j \quad \text{or} \quad Y_2 > k) \leq P(Y_1 > j) + P(Y_2 > k) \\ &= (1 - G_1(j)) + (1 - G_2(k)), \end{aligned}$$

which implies, in turn, $G_1(j) + G_2(k) - 1 \leq G(j,k)$. The latter inequality and the fact that $G(j,k) \geq 0$ imply the first inequality in (3.1). Denote $G_U(j,k) = \min\{G_1(j), G_2(k)\}$ and $G_L(j,k) = \max\{G_1(j) + G_2(k) - 1, 0\}$. Both functions, G_U and G_L, are nonnegative, nondecreasing, and convergent to the unity as j tends to n and k tends to m. Thus the functions G_U and G_L are themselves bivariate cumulative probability functions. It is apparent that the marginal cumulative probability functions G_U and G_L are G_1 and G_2. Thus, any bivariate discrete cumulative probability function G with margins G_1 and G_2 will be dominated by G_U, and G_L will be dominated by G.

Consider first the case in which the values of both Y_1 and Y_2 are finite; that is, n and m are finite positive integers. The correlation coefficient of Y_1 and Y_2 is in (2.4). For any event A of Y_2, one has

$$\begin{aligned} \sum_{j=0}^{n} jP(Y_1 = j, A) &= \sum_{j=1}^{n} P(Y_1 = j, A) + \sum_{j=2}^{n} P(Y_1 = j, A) + \cdots \\ &\quad + \sum_{j=n-1}^{n} P(Y_1 = j, A) + P(Y_1 = n, A) \\ &= [P(A) - P(Y_1 = 0, A)] + [P(A) - P(Y_1 \leq 1, A)] + \cdots \\ &\quad + [P(A) - P(Y_1 \leq n-1, A)] \\ &= nP(A) - \sum_{j=0}^{n-1} P(Y_1 \leq j, A). \end{aligned} \tag{3.2}$$

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

Similarly,

$$\sum_{k=0}^{m} kP(B, Y_2 = k) = mP(B) - \sum_{k=0}^{m-1} P(B, Y_2 \le k) \tag{3.3}$$

for any event B of Y_1. It follows from the identities (3.2) and (3.3) that

$$\sum_{j=0}^{n} jP(Y_1 = j) = n - \sum_{j=0}^{n-1} P(Y_1 \le j), \tag{3.4}$$

$$\sum_{k=0}^{m} kP(Y_2 = k) = m - \sum_{k=0}^{m-1} P(Y_2 \le k), \tag{3.5}$$

and

$$\begin{aligned}\sum_{j=0}^{n}\sum_{k=0}^{m} jkP(Y_1 = j, Y_2 = k) &= \sum_{k=0}^{m} k\left[\sum_{j=0}^{n} jP(Y_1 = j, Y_2 = k)\right] \\ &= n\sum_{k=0}^{m} kP(Y_2 = k) - \sum_{j=0}^{n-1}\sum_{k=0}^{m} kP(Y_1 \le j, Y_2 = k) \\ &= nm - n\sum_{k=0}^{m-1} P(Y_2 \le k) - m\sum_{j=0}^{n-1} P(Y_1 \le j) \\ &\quad + \sum_{j=0}^{n-1}\sum_{k=0}^{m-1} P(Y_1 \le j, Y_2 \le k).\end{aligned} \tag{3.6}$$

It follows from (3.4)–(3.6) that the correlation coefficient of Y_1 and Y_2 in (2.4) can be rewritten as

$$\begin{aligned}\rho_{12} &= \left\{\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} P(Y_1 \le j, Y_2 \le k) - \left[\sum_{j=0}^{n-1} P(Y_1 \le j)\right]\left[\sum_{k=0}^{m-1} P(Y_2 \le k)\right]\right\}\Big/\sigma_1\sigma_2 \\ &= \left\{\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} G(j,k) - \left[\sum_{j=0}^{n-1} G_1(j)\right]\left[\sum_{k=0}^{m-1} G_2(k)\right]\right\}\Big/\sigma_1\sigma_2.\end{aligned} \tag{3.7}$$

Let ρ_U be the correlation coefficient of Y_1 and Y_2 based on G_U, that is,

$$\rho_U = \left\{\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} \min\{G_1(j), G_2(k)\} - \left[\sum_{j=0}^{n-1} G_1(j)\right]\left[\sum_{k=0}^{m-1} G_2(k)\right]\right\}\Big/\sigma_1\sigma_2, \tag{3.8}$$

and ρ_L be that based on G_L:

$$\rho_L = \left\{\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} \max\{G_1(j) + G_2(k) - 1, 0\} - \left[\sum_{j=0}^{n-1} G_1(j)\right]\left[\sum_{k=0}^{m-1} G_2(k)\right]\right\}\Big/\sigma_1\sigma_2. \tag{3.9}$$

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

The inequalities in (3.1) imply that

$$\rho_L \leq \rho_{12} \leq \rho_U. \tag{3.10}$$

As ρ_U and ρ_L in (3.8) and (3.9) are completely determined by the margins G_1 and G_2, we conclude that the range of correlation coefficients of any discrete bivariate probability function of Y_1 and Y_2 given margins G_1 and G_2 is the close interval $[\rho_L, \rho_U]$ contained in $[-1,1]$. If a family of discrete bivariate probability functions of Y_1 and Y_2 with given margins G_1 and G_2 has flexible correlation coefficients, its correlation coefficients should have the full range $[\rho_L, \rho_U]$.

These results need to be generalized to the case where Y_1 or Y_2 are countably infinite variables. It is not meaningful to extend the expression in (3.7) by simply letting n and m go to infinity. This is so because the three series in that expression do not converge but diverge to positive infinity. A possible way to handle the countably infinite case is through an approximation device. Without loss of generality, assume that both Y_1 and Y_2 have countably infinite values. The correlation coefficient of countably infinite discrete variables is well defined only if the second moments of the marginal Y_1 and Y_2 are finite. By Cauchy's inequality, the cross moment of any bivariate probability function of Y_1 and Y_2 with the given margins will exist. The convergence of cross moment implies that

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=0}^{n} \sum_{k=0}^{m} jkP(Y_1 = j, Y_2 = k) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} jkP(Y_1 = j, Y_2 = k). \tag{3.11}$$

As

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} jkP(Y_1 = j, Y_2 = k) = \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} jkP(Y_1 = j, Y_2 = k) + R_{n,m}^{(1)} + R_{n,m}^{(2)} + R_{n,m}^{(3)},$$

where $R_{n,m}^{(1)} = \sum_{j=0}^{n-1} \sum_{k=m}^{\infty} jkP(Y_1 = j, Y_2 = k)$, $R_{n,m}^{(2)} = \sum_{j=n}^{\infty} \sum_{k=0}^{m-1} jkP(Y_1 = j, Y_2 = k)$, and $R_{n,m}^{(3)} = \sum_{j=n}^{\infty} \sum_{k=m}^{\infty} jkP(Y_1 = j, Y_2 = k)$, it follows from (3.11) that $\lim_{n\to\infty} \lim_{m\to\infty} R_{n,m}^{(j)} = 0$ for all $j = 1,2,3$. Define a censored discrete variable $Y_{n,1}^{(c)}$ with finite values $0,1,\ldots,n-1$ such that $Y_{n,1}^{(c)} = Y_1$ if $Y_1 \in \{0,1,\ldots,n-1\}$, and $Y_{n,1}^{(c)} = n$ if $Y_1 \in \{n, n+1, \ldots, \infty\}$. Similarly, define a censored discrete variable $Y_{m,2}^{(c)}$ for Y_2 with values $0,1,\ldots,m$. Under the censoring construction, the bivariate cumulative probability $G^{(c)}$ follows from the cumulative G of Y_1 and Y_2. Let $\rho_{n,m}^{(c)}$ be the correlation coefficient of $Y_{n,1}^{(c)}$ and $Y_{m,2}^{(c)}$ based on $G^{(c)}$. As $Y_{n,1}^{(c)}$ and $Y_{m,2}^{(c)}$ are finite discrete variables, the range of $\rho_{n,m}^{(c)}$ can be established as

$$\rho_L^{(c)} \leq \rho_{n,m}^{(c)} \leq \rho_U^{(c)}, \tag{3.12}$$

where $\rho_L^{(c)}$ is the correlation coefficient based on $G_L^{(c)}$ and $\rho_U^{(c)}$ is based on $G_U^{(c)}$. We argue that the range of ρ_{12} of Y_1 and Y_2 can be established by reducing the degree of censoring.

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

To be rigorous, note that

$$\rho_{n,m}^{(c)} = \left\{ \sum_{j=0}^{n} \sum_{k=0}^{m} jkP(Y_{n,1}^{(c)} = j, Y_{m,2}^{(c)} = k) - \left[\sum_{j=0}^{n} jP(Y_{n,1}^{(c)} = j) \right] \times \left[\sum_{k=0}^{m} kP(Y_{m,2}^{(c)} = k) \right] \right\} \Big/ \sigma_{n,1}^{(c)} \sigma_{m,2}^{(c)}, \tag{3.13}$$

where $\sigma_{n,1}^{(c)}$ and $\sigma_{m,2}^{(c)}$ are, respectively, the standard deviations of $Y_{n,1}^{(c)}$ and $Y_{m,2}^{(c)}$. The cross moment of the censored variables $Y_{n,1}^{(c)}$ and $Y_{m,2}^{(c)}$ is

$$\sum_{j=0}^{n} \sum_{k=0}^{m} jkP(Y_{n,1}^{(c)} = j, Y_{m,2}^{(c)} = k) = \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} jkP(Y_1 = j, Y_2 = k) + S_{n,m}^{(1)} + S_{n,m}^{(2)} + S_{n,m}^{(3)},$$

where $S_{n,m}^{(1)} = m \sum_{j=0}^{n-1} jP(Y_{n,1}^{(c)} = j, Y_{m,2}^{(c)} = m)$, $S_{n,m}^{(2)} = n \sum_{k=0}^{m-1} kP(Y_{n,1}^{(c)} = n, Y_{m,2}^{(c)} = k)$, and $S_{n,m}^{(3)} = nmP(Y_{n,1}^{(c)} = n, Y_{m,2}^{(c)} = m)$. For $j \leq n - 1$, $P(Y_{n,1}^{(c)} = j, Y_{m,2}^{(c)} = m) = P(Y_1 = j, Y_2 \geq m)$, and, hence, $S_{n,m}^{(1)} = m \sum_{j=0}^{n-1} jP(Y_1 = j, Y_2 \geq m) \leq R_{n,m}^{(1)}$. Similarly, $S_{n,m}^{(2)} = n \sum_{k=0}^{m-1} kP(Y_1 \geq n, Y_2 = k) \leq R_{n,m}^{(2)}$, and $S_{n,m}^{(3)} = nmP(Y_1 \geq n, Y_2 \geq m) \leq R_{n,m}^{(3)}$, which go to zero as n and m tend to infinity. These imply that

$$\lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=0}^{n} \sum_{k=0}^{m} jkP(Y_{n,1}^{(c)} = j, Y_{m,2}^{(c)} = k) = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{j=0}^{n-1} \sum_{k=0}^{m-1} jkP(Y_1 = j, Y_2 = k) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} jkP(Y_1 = j, Y_2 = k). \tag{3.14}$$

Similarly,

$$\lim_{n\to\infty} \sum_{j=0}^{n} jP(Y_{n,1}^{(c)} = j) = \sum_{j=0}^{\infty} jP(Y_1 = j),$$

$$\lim_{m\to\infty} \sum_{k=0}^{m} kP(Y_{m,2}^{(c)} = k) = \sum_{k=0}^{\infty} kP(Y_2 = k), \tag{3.15}$$

$\lim_{n\to\infty} \sigma_{n,1}^{(c)} = \sigma_1$, and $\lim_{m\to\infty} \sigma_{m,2}^{(c)} = \sigma_2$. Hence, we can conclude from (3.13)–(3.15) that

$$\lim_{n\to\infty} \lim_{m\to\infty} \rho_{n,m}^{(c)} = \rho_{12}. \tag{3.16}$$

Because the existence of finite second moments of marginal Y_1 and Y_2 implies that the covariance of Y_1 and Y_2 with any bivariate probability function exists and is finite, the preceding analysis is applicable to the upper and lower probability functions of G_U and G_L in (3.1). The corresponding bivariate probability

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

functions generated by G_U and G_L are $P_U(j,k) = G_U(j,k) - G_U(j,k-1) - G_U(j-1,k) + G_U(j-1,k-1)$ and $P_L(j,k) = G_L(j,k) - G_L(j,k-1) - G_L(j-1,k) + G_L(j-1,k-1)$. Therefore,

$$\lim_{n\to\infty}\lim_{m\to\infty}\rho_L^{(c)} = \rho_L \quad \text{and} \quad \lim_{n\to\infty}\lim_{m\to\infty}\rho_U^{(c)} = \rho_U. \tag{3.17}$$

By reducing the degrees of censoring to zero, it follows from (3.12), (3.16), and (3.17) that

$$\rho_L \leq \rho_{12} \leq \rho_U. \tag{3.18}$$

The range of the correlation coefficient of a bivariate countably infinite vector is the interval $[\rho_L, \rho_U]$. For the computation of ρ_L and ρ_U, it is desirable to use the bivariate probability functions P_U and P_L directly in (2.4).

4. THE RELATION OF ρ AND ρ_{12}

The normal transformed bivariate cumulative G_{12} proposed by Van Ophem in (2.3) is a parametric function characterized by parameter ρ, and, hence, the corresponding correlation coefficient ρ_{12} of Y_1 and Y_2 is also a function of ρ. It is of interest to investigate the relation between ρ_{12} and ρ and see whether this class of bivariate probability functions has a flexible correlation coefficient on $[\rho_L, \rho_U]$.

First of all, we show that $G_{12}(j,k;\rho)$ in (2.3) can attain the upper limit $G_U(j,k)$ when ρ tends to one, and the lower limit $G_L(j,k)$ as ρ tends to minus one. It is known that for a bivariate standard normal vector (u,v) with a correlation coefficient ρ, u must equal v almost everywhere when $\rho = 1$. That is, $\lim_{\rho\to 1}\phi(u,v;\rho) = I_a(u,v)\phi(u)$ where $I_a(u,v) = 1$ if $u = v$, and $I_a(u,v) = 0$ whenever $u \neq v$. Therefore,

$$\begin{aligned}
\lim_{\rho\to 1} G_{12}(j,k;\rho) &= \lim_{\rho\to 1}\int_{-\infty}^{\Phi^{-1}[G_1(j)]}\int_{-\infty}^{\Phi^{-1}[G_2(k)]}\phi(u,v;\rho)\,du\,dv \\
&= \int_{-\infty}^{\Phi^{-1}[G_1(j)]}\int_{-\infty}^{\Phi^{-1}[G_2(k)]} I_a(u,v)\phi(u)\,du\,dv \\
&= \int_{-\infty}^{\min\{\Phi^{-1}[G_1(j)],\,\Phi^{-1}[G_2(k)]\}}\phi(u)\,du \\
&= \min\{G_1(j), G_2(k)\}.
\end{aligned} \tag{4.1}$$

On the other hand, when ρ tends to minus one, $u = -v$ almost everywhere. That is, $\lim_{\rho\to -1}\phi(u,v;\rho) = I_b(u,v)\phi(u)$ where $I_b(u,v) = 1$ if $u + v = 0$, and otherwise $I_b(u,v) = 0$. It is apparent that, for any z_1 and z_2 values such that

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

$z_1 + z_2 \leq 0$, $\int_{-\infty}^{z_1}\int_{-\infty}^{z_2} I_b(u,v)\phi(u)du = 0$. It remains to consider the integral with $z_1 + z_2 > 0$. We note that

$$\Phi^{-1}(G_1(j)) + \Phi^{-1}(G_2(k)) > 0 \Leftrightarrow G_1(j) > \Phi[-\Phi^{-1}(G_2(k))]$$
$$\Leftrightarrow G_1(j) + G_2(k) - 1 > 0,$$

where the latter follows because $\Phi(-\Phi^{-1}(G_2(k))) = 1 - G_2(k)$ by the symmetry of a standard normal random variable. Therefore, if $\Phi^{-1}(G_1(j)) + \Phi^{-1}(G_2(k)) > 0$,

$$\begin{aligned}\lim_{\rho\to -1} G_{12}(j,k;\rho) &= \int_{-\infty}^{\Phi^{-1}[G_1(j)]}\int_{-\infty}^{\Phi^{-1}[G_2(k)]} I_b(u,v)\phi(u)dudv \\ &= \int_{-\Phi^{-1}[G_2(k)]}^{\Phi^{-1}[G_1(j)]} I_b(u,v)\phi(u)dudv \\ &= \Phi(\Phi^{-1}(G_1(j))) - \Phi(-\Phi^{-1}(G_2(k))) \\ &= G_1(j) + G_2(k) - 1. \end{aligned} \tag{4.2}$$

The $G_{12}(j,k;\rho)$ attains the lower bound $G_L(j,k)$ as ρ tends to minus one.

Suppose that Y_1 and Y_2 are finite discrete variables with maximum values n and m. The correlation coefficient of Y_1 and Y_2 from (3.7) is

$$\rho_{12}(\rho) = \left\{\sum_{j=0}^{n-1}\sum_{k=0}^{m-1} G_{12}(j,k;\rho) - \left[\sum_{j=0}^{n-1} G_1(j)\right]\left[\sum_{k=0}^{m-1} G_2(k)\right]\right\}\Big/\sigma_1\sigma_2. \tag{4.3}$$

At $\rho = 0$, (2.3) implies that $G_{12}(j,k;0) = G_1(j)G_2(k)$ and, hence, $\rho_{12}(0) = 0$ from (4.3). Thus, zero correlation under G_{12} occurs when $\rho = 0$. For the standard bivariate normal distribution $B(\cdot,\cdot;\rho)$, Plackett (1954) shows that $\partial B(c_1,c_2;\rho)/\partial\rho = \phi(c_1,c_2;\rho)$. Therefore,

$$\begin{aligned}\frac{\partial\rho_{12}(\rho)}{\partial\rho} &= \frac{1}{\sigma_1\sigma_2}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\frac{\partial G_{12}(j,k;\rho)}{\partial\rho} \\ &= \frac{1}{\sigma_1\sigma_2}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\phi(\Phi^{-1}(G_1(j)),\Phi^{-1}(G_2(k));\rho), \end{aligned} \tag{4.4}$$

which is strictly positive. Thus, $\rho_{12}(\rho)$ is a strictly increasing function of ρ, and it has the same trend with ρ. In particular, when $\rho > 0$, Y_1 and Y_2 based on G_{12} are positively correlated; and when $\rho < 0$, they are negatively correlated. An estimate of ρ provides valuable information on the correlation of Y_1 and Y_2 under G_{12}. For discrete variables with countably infinite values, the censoring and limiting arguments can be used. Let G_{12}^c be the censored version of G_{12} and let $\rho_{12}^{(c)}(\rho)$ be the corresponding correlation coefficient of $G_{12}^{(c)}$. The derivative of $\rho_{12}^{(c)}(\rho)$ with respect to ρ will have a similar expression as in (4.4). As censoring becomes less severe and it diminishes to zero,

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press

$$\frac{\partial \rho_{12}(\rho)}{\partial \rho} = \lim_{n\to\infty}\lim_{m\to\infty}\frac{\partial \rho_{12}^{(c)}(\rho)}{\partial \rho}$$

$$= \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{\sigma_1^{(c)}\sigma_2^{(c)}}\sum_{j=0}^{n-1}\sum_{k=0}^{m-1}\phi(\Phi^{-1}(G_1^{(c)}(j)),\Phi^{-1}(G_2^{(c)}(k));\rho)$$

$$= \frac{1}{\sigma_1\sigma_2}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\phi(\Phi^{-1}(G_1(j)),\Phi^{-1}(G_2(k));\rho), \qquad \textbf{(4.5)}$$

which is strictly positive.

5. CONCLUSIONS

This paper has derived the range of correlation coefficients of any bivariate discrete random vector with finite or countably infinite ordered values. It has addressed the related issues of a normal transformed bivariate discrete system with given margins by Van Ophem (1999). Van Ophem (1999) provides numerical illustrations on the correlation coefficient of such a bivariate discrete vector with given marginal Poisson probability functions. Our analysis provides the definite analytical results. This analysis will be useful for other approaches for the construction of bivariate probability functions with given margins.

REFERENCES

Frechet, M. (1951) Sur les tableaux de correlation dont les marges sont donnees. *Annales de l'Université de Lyon Section A, Series 3*, 14, 53–77.

Johnson, N.L. & S. Kotz (1972) *Distributions in Statistics: Continuous Multivariate Distributions*. New York: Wiley.

Lee, L.F. (1983) Generalized econometric models with selectivity. *Econometrica* 51, 507–512.

Mardia, K.V. (1970) A translation family of bivariate distributions and Frechet's bounds. *Sankhya Series A*, 32, 119–121.

Plackett, R.L. (1954) A reduction formula for normal multivariate integrals. *Biometrika* 41, 351–360.

Plackett, R.L. (1965) A class of bivariate distributions. *Journal of the American Statistical Association* 60, 516–522.

Van Ophem, H. (1999) A general method to estimate correlated discrete random variables. *Econometric Theory* 15, 228–237.

https://doi.org/10.1017/S0266466601171082 Published online by Cambridge University Press