CrossMark

Proceedings of the Royal Society of Edinburgh, **152**, 479–517, *2022*
DOI:10.1017/prm.2021.22

Concentration phenomena for fractional magnetic NLS equations

Vincenzo Ambrosio
Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 12, 60131, Ancona, Italy (v.ambrosio@staff.univpm.it)

(Received 28 January 2021; accepted 23 April 2021)

We study the multiplicity and concentration of complex-valued solutions for a fractional magnetic Schrödinger equation involving a scalar continuous electric potential satisfying a local condition and a continuous nonlinearity with subcritical growth. The main results are obtained by applying a penalization technique, generalized Nehari manifold method and Ljusternik–Schnirelman theory. We also prove a Kato's inequality for the fractional magnetic Laplacian which we believe to be useful in the study of other fractional magnetic problems.

Keywords: Fractional magnetic operators; Ljusternik–Schnirelman theory; variational methods

2020 *Mathematics Subject Classification:* 35A15; 35R11; 35S05; 58E05

1. Introduction

In this paper, we deal with the multiplicity of solutions $u : \mathbb{R}^N \to \mathbb{C}$ to the following fractional magnetic nonlinear Schrödinger equation:

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon} u + V(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\varepsilon > 0$ is a small parameter, $N \geqslant 3$, $s \in (0,1)$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ is a Hölder continuous magnetic potential with exponent $\alpha \in (0,1]$. Along the paper, we assume that $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous electric potential satisfying the following del Pino and Felmer [**24**] type assumptions:

(V_1) there exists $V_0 > 0$ such that $V_0 = \inf_{x \in \mathbb{R}^N} V(x)$;

(V_2) there exists a bounded open set $\Lambda \subset \mathbb{R}^N$ such that

$$V_0 < \min_{x \in \partial\Lambda} V(x) \quad \text{and} \quad M = \{x \in \Lambda : V(x) = V_0\} \neq \emptyset. \tag{1.2}$$

Without loss of generality, we may assume that $0 \in M$. Moreover, we suppose that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function fulfilling the following conditions:

(f_1) $f(t) = 0$ for $t \leqslant 0$;

© The Author(s), 2021. Published by Cambridge University Press on behalf of The Royal Society of Edinburgh

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

(f_2) there exists $q \in (2, 2_s^*)$, where $2_s^* = 2N/(N-2s)$ is the fractional critical exponent, such that

$$\lim_{t\to\infty} \frac{f(t)}{t^{(q-2)/2}} = 0;$$

(f_3) there exists $\theta \in (2, q)$ such that $0 < (\theta/2)F(t) \leqslant tf(t)$ for any $t > 0$, where $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$;

(f_4) the function $t \mapsto f(t)$ is increasing in $(0, \infty)$.

Here, $(-\Delta)_A^s$ is the fractional magnetic Schrödinger operator which is defined for any $u \in C_c^\infty(\mathbb{R}^N, \mathbb{C})$ as

$$(-\Delta)_A^s u(x) = C(N,s) \lim_{r\to 0} \int_{B_r^c(x)} \frac{u(x) - \mathrm{e}^{\imath(x-y)\cdot A((x+y)/2)}u(y)}{|x-y|^{N+2s}}\,\mathrm{d}y,$$

$$C(N,s) = s\frac{4^s\Gamma\left((N+2s)/2\right)}{\pi^{N/2}\Gamma(1-s)}.$$

This operator has been recently introduced in [21, 35] and relies essentially on the Lévy–Khintchine formula for the generator of a general Lévy process. We would like to observe that, when $s = \frac{1}{2}$, the above operator takes inspiration from the definition of a quantized operator corresponding to the classical relativistic Hamiltonian symbol for a relativistic particle of mass $m \geqslant 0$, that is

$$\sqrt{(\xi - A(x))^2 + m^2} + V(x), \quad (\xi, x) \in \mathbb{R}^N \times \mathbb{R}^N,$$

which is the sum of the kinetic energy term involving the magnetic vector potential $A(x)$ and the potential energy term given by the electric scalar potential $V(x)$. For the sake of completeness, we emphasize that in the literature other three kinds of quantum relativistic Hamiltonians appear depending on how the kinetic energy term $\sqrt{(\xi - A(x))^2 + m^2}$ is quantized. As explained in [35], these three nonlocal operators are in general different from each other, but coincide when the vector potential A is assumed to be linear. As $s \to 1$ and A sufficiently smooth, $(-\Delta)_A^s$ can be also considered (see [45]) as the fractional analogue of the magnetic Laplacian

$$-\Delta_A u = \left(\frac{1}{\imath}\nabla - A(x)\right)^2 u = -\Delta u - \frac{2}{\imath}A(x)\cdot\nabla u + |A(x)|^2 u - \frac{1}{\imath}u\,\mathrm{div}(A(x)),$$

which plays a fundamental role in quantum mechanics in the description of the dynamics of the particle in a non-relativistic setting. Indeed, in the three-dimensional case, the magnetic field B is exactly the curl of A, while for higher dimensions $N \geqslant 4$, B should be thought of as the 2-form given by $B_{ij} = \partial_j A_k - \partial_k A_j$; see [15, 46] for more physical background. Motivated by this fact, many authors [1, 2, 14, 18, 19, 28, 37] studied the existence and multiplicity of nontrivial solutions for the following nonlinear magnetic Schrödinger equation:

$$-\varepsilon^2 \Delta_A u + V(x)u = f(x, |u|^2)u \quad \text{in } \mathbb{R}^N. \tag{1.3}$$

Equation (1.3) arises when we look for standing wave solutions $\psi(x,t) = u(x)\mathrm{e}^{-\imath(E/\varepsilon)t}$, with $E \in \mathbb{R}$, for the following time-dependent Schrödinger equation

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

in the presence of an external magnetic field:

$$\imath\,\varepsilon\,\frac{\partial\psi}{\partial t} = \Big(\frac{\varepsilon}{\imath}\nabla - A(x)\Big)^2\,\psi + U(x)\psi - f(|\psi|^2)\psi \quad \text{in } (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

where $U(x) = V(x) + E$. An important class of solutions of (1.3) are the so-called semi-classical states which concentrate and develop a spike shape around one, or more, particular points in $\mathbb{R}^N$, while vanishing elsewhere as $\varepsilon \to 0$. This interest is related to the fact that the transition from quantum mechanics to classical mechanics can be formally performed by sending $\varepsilon \to 0$.

In the last few years, much attention has been paid to the following fractional magnetic Schrödinger equation:

$$\varepsilon^{2s}(-\Delta)_A^s u + V(x)u = f(x,|u|^2)u \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

For instance, d'Avenia and Squassina [**21**] considered a class of minimization problems in the spirit of results due to Esteban and Lions in [**28**]. In [**13**] the author and d'Avenia studied the existence and multiplicity of solutions to (1.4) for small $\varepsilon > 0$ when $f \in C^1$ has a subcritical growth and the potential V satisfies the following global condition due to Rabinowitz [**43**]:

$$\liminf_{|x|\to\infty} V(x) > \inf_{x\in\mathbb{R}^N} V(x). \tag{1.5}$$

In [**7**] (see also [**12**]) the author used a penalization argument to prove the existence and concentration of nontrivial solutions to (1.4) under assumptions (V_1)–(V_2) and (f_1)–(f_4). Further interesting results for nonlocal problems involving the operator $(-\Delta)_A^s$ can be found in [**8**–**11**, **33**, **39**, **41**].

In the absence of the magnetic vector potential, i.e. $A \equiv 0$, the operator $(-\Delta)_A^s$ reduces to the celebrated fractional Laplacian operator $(-\Delta)^s$ which arises in a quite natural way in many different physical contexts in which one has to consider long range anomalous diffusions and transport in highly heterogeneous medium; see [**17**, **25**] for more details on this topic. Then (1.4) boils down to the following fractional Schrödinger equation (see [**38**])

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u) \quad \text{in } \mathbb{R}^N, \tag{1.6}$$

for which several existence and multiplicity results under different assumptions on V and f have been established via appropriate variational and topological methods; see [**3**, **5**, **6**, **20**, **26**, **32**] and references therein. In particular, Davila *et al.* [**20**] proved that if $V \in C^{1,\alpha}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ and $\inf_{x\in\mathbb{R}^N} V(x) > 0$, then (1.6) has multi-peak solutions by using the Lyapunov–Schmidt reduction method. Fall *et al.* [**29**] established necessary and sufficient conditions on the smooth potential V in order to produce concentration of solutions of (1.6) as $\varepsilon \to 0$. Alves and Miyagaki [**3**] (see also [**6**]) considered the existence and concentration of positive solutions of (1.6) when V satisfies a local condition and f has subcritical growth at infinity. In [**5**] the author combined the generalized Nehari manifold approach introduced in [**47**] with the Ljusternik–Schnirelman theory to obtain a multiplicity result for (1.6) under assumptions (V_1)–(V_2).

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Particularly motivated by [2, 3, 5, 7, 12, 13], in this paper we deal with the multiplicity and concentration phenomenon as $\varepsilon \to 0$ of nontrivial solutions $u : \mathbb{R}^N \to \mathbb{C}$ to (1.1), under assumptions (V_1)–(V_2) and (f_1)–(f_4). More precisely, our main result can be stated as follows:

THEOREM 1.1. *Assume that (V_1)–(V_2) and (f_1)–(f_4) hold. Then, for any $\delta > 0$ such that*

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leqslant \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ such that, for any $\varepsilon \in (0, \varepsilon_\delta)$, problem (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions. Moreover, if u_ε denotes one of these solutions and x_ε is a global maximum point of $|u_\varepsilon|$, then we have

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0,$$

and there exists a constant $C > 0$ such that

$$|u_\varepsilon(x)| \leqslant \frac{C\,\varepsilon^{N+2s}}{C\,\varepsilon^{N+2s} + |x - x_\varepsilon|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N.$$

The proof of theorem 1.1 will be obtained by combining suitable variational techniques and Ljusternik–Schnirelman theory. As in [7], we adapt the penalization approach in [24] (see also [2]) modifying appropriately the nonlinearity f outside Λ and by considering an auxiliary problem. The main feature of the corresponding modified energy functional J_ε is that it satisfies all the assumptions of the mountain-pass theorem [4]. In order to obtain a multiplicity result for the modified problem, we use a strategy proposed by Benci and Cerami [16] which consists in making precise comparisons between the category of some sublevel sets of J_ε and the category of the set M. Since the nonlinearity f is only continuous, the Nehari manifold associated with J_ε is not differentiable, so we cannot repeat the same arguments used in [2, 13] for C^1-Nehari manifolds [48]. We overcome this obstacle by taking advantage of some abstract critical point theorems from Szulkin and Weth [47]. We recall that a similar approach is also used in [5] where $A \equiv 0$. However, the presence of the magnetic potential creates several difficulties which do not permit to adapt the techniques used in [5] so that a more accurate analysis will be needed in our situation. Indeed, the regularity assumption on A and the use of the fractional diamagnetic inequality [21] will play a crucial role to obtain some refined estimates. Finally, we need to prove that for $\varepsilon > 0$ small enough, the solutions u_ε of the modified problem are also solutions of the original one. To achieve our purpose, we first prove a Kato's inequality [36] for solutions of fractional magnetic problems, which essentially says that if $u : \mathbb{R}^N \to \mathbb{C}$ satisfies $(-\Delta)_A^s u = f \in L^1_{loc}$ in $\mathbb{R}^N$ then $|u| : \mathbb{R}^N \to \mathbb{R}$ satisfies $(-\Delta)^s |u| \leqslant \Re(\text{sign}(\bar{u}) f)$ in the distributional sense. We stress that the proof of this result does not follow the original arguments due to Kato [36] in which some auxiliary regularity lemmas are used and a double passage to the limit was done. Due to the nonlocal character of $(-\Delta)_A^s$ and the presence of the magnetic potential, we choose a suitable test function in the weak formulation of the fractional magnetic problem under consideration and we are able to pass to the limit and to obtain the required inequality. In some sense, we generalize the scheme

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

used in [7]. Unfortunately the approach in [7] was based on the boundedness of the solution and it has only been used so far on specific examples (see [8–10]). Thus it is not always clear to distinguish what is the core of the approach and what belongs to the specific problem under study. An important achievement of our paper is the derivation of a general abstract fractional Kato's inequality. Clearly, with respect to the abovementioned studies, the advantage is the simplicity of the presentation and the 'ready to use' aspect of the result. In light of this result and applying a comparison argument, we show that u_ε's are actually solutions to (1.1) as long as $\varepsilon > 0$ is small enough. We emphasize that theorem 1.1 completes the study started in [7] because we are now considering the question related to the multiplicity of (1.1). Moreover, theorem 1.1 improves and extends in fractional setting theorem 1.1 in [2] in which only C^1-nonlinearities were considered. As far as we know, this is the first time that penalization method jointly with Ljusternik–Schnirelman theory is used to obtain multiple solutions for (1.1) under local conditions on V and the continuity of f. In view of the arguments used along this paper and in [7], it is easy to see that a multiplicity result holds even in the critical and supercritical cases considered in [7], more precisely, when we deal with the following fractional magnetic Schrödinger equation with critical growth:

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u + V(x)u = f(|u|^2)u + |u|^{2^*_s-2}u \quad \text{in } \mathbb{R}^N, \tag{1.7}$$

where f satisfies the following assumptions:

- (h_1) $f(t) = 0$ for $t \leqslant 0$;
- (h_2) there exist $C_0 > 0$ and $q, \sigma \in (2, 2^*_s)$ such that

$$f(t) \geqslant C_0 t^{(q-2)/2} \text{ for all } t \geqslant 0 \text{ and } \lim_{t\to\infty} \frac{f(t)}{t^{(\sigma-2)/2}} = 0;$$

  and $C_0 > 0$ if either $N \geqslant 4s$, or $2s < N < 4s$ and $2^*_s - 2 < q < 2^*_s$, $C_0 > 0$ is sufficiently large if $2s < N < 4s$ and $2 < q \leqslant 2^*_s - 2$.
- (h_3) there exists $\theta \in (2, \sigma)$ such that $0 < (\theta/2)F(t) \leqslant tf(t)$ for any $t > 0$, where $F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau$;
- (h_4) the function $t \mapsto f(t)$ is increasing in $(0, \infty)$;

and when we study the following fractional magnetic Schrödinger equation with supercritical growth:

$$\varepsilon^{2s}(-\Delta)^s_{A/\varepsilon}u + V(x)u = |u|^{q-2}u + \lambda|u|^{r-2}u \quad \text{in } \mathbb{R}^N, \tag{1.8}$$

where $2 < q < 2^*_s < r$. The proofs are only a simple adaptation of the techniques used in this paper with minor modifications. For completeness, we state without proofs the following theorems:

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

THEOREM 1.2. *Assume that* (V_1)–(V_2) *and* (h_1)–(h_4) *hold. Then, for any* $\delta > 0$ *such that*

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leqslant \delta\} \subset \Lambda,$$

there exists $\varepsilon_\delta > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_\delta)$, *problem* (1.7) *has at least* $cat_{M_\delta}(M)$ *nontrivial solutions. Moreover, if* u_ε *denotes one of these solutions and* x_ε *is a global maximum point of* $|u_\varepsilon|$, *then we have*

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0.$$

THEOREM 1.3. *Assume that* (V_1)–(V_2) *hold. Then there exists* $\lambda_0 > 0$ *with the following property: for any* $\lambda \in (0, \lambda_0)$ *and for any* $\delta > 0$ *such that*

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leqslant \delta\} \subset \Lambda,$$

there exists $\varepsilon_{\lambda,\delta} > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_{\lambda,\delta})$, *problem* (1.8) *has at least* $cat_{M_\delta}(M)$ *nontrivial solutions. Moreover, if* u_ε *denotes one of these solutions and* x_ε *is a global maximum point of* $|u_\varepsilon|$, *then we have*

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = V_0.$$

The paper is structured as follows. In § 2 we introduce the notations and we collect some preliminary results for fractional Sobolev spaces. In § 3 we study the modified functional. In § 4 we consider the scalar limiting problem. In § 5 we provide a multiplicity result for the modified problem. Finally, in § 6, we give the proof of theorem 1.1.

2. Preliminaries

Fix $s \in (0, 1)$ and we denote by $\mathcal{D}^{s,2}(\mathbb{R}^N, \mathbb{R})$ the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{R})$ with respect to the Gagliardo seminorm

$$[u] = [u]_s = \sqrt{\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy}.$$

When $N > 2s$, we also know (see theorem 2.2 in [**23**]) that

$$\mathcal{D}^{s,2}(\mathbb{R}^N, \mathbb{R}) = \{u \in L^{2_s^*}(\mathbb{R}^N, \mathbb{R}) : [u] < \infty\}.$$

We denote by $H^s(\mathbb{R}^N, \mathbb{R})$ the fractional Sobolev space

$$H^s(\mathbb{R}^N, \mathbb{R}) = \{u \in L^2(\mathbb{R}^N, \mathbb{R}) : [u] < \infty\}.$$

It is well-known that the embedding $H^s(\mathbb{R}^N, \mathbb{R}) \subset L^q(\mathbb{R}^N, \mathbb{R})$ is continuous for all $q \in [2, 2_s^*)$ and locally compact for all $q \in [1, 2_s^*)$; see [**25**].

Let $L^2(\mathbb{R}^N, \mathbb{C})$ be the space of complex-valued functions such that $\|u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |u|^2 \, dx < \infty$ endowed with the inner product $\langle u, v \rangle_{L^2} = \Re \int_{\mathbb{R}^N} u\bar{v} \, dx$, where

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

$\Re(z)$ denotes the real part of $z \in \mathbb{C}$ and $\bar{z}$ is its conjugate. Let us denote the magnetic Gagliardo seminorm by

$$[u]_A = [u]_{s,A} = \sqrt{\iint_{\mathbb{R}^{2N}} \frac{|u(x) - \mathrm{e}^{\imath(x-y)\cdot A(\frac{x+y}{2})}u(y)|^2}{|x-y|^{3+2s}}\,\mathrm{d}x\,\mathrm{d}y},$$

and consider

$$\mathcal{D}_A^{s,2}(\mathbb{R}^N, \mathbb{C}) = \left\{u \in L^{2_s^*}(\mathbb{R}^N, \mathbb{C}) : [u]_A^2 < \infty\right\}.$$

Set $A_\varepsilon(x) = A(\varepsilon x)$ and $V_\varepsilon(x) = V(\varepsilon x)$. Then we define the Hilbert space

$$\mathcal{H}_\varepsilon = \left\{u \in \mathcal{D}_{A_\varepsilon}^{s,2}(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2\,\mathrm{d}x < \infty\right\}$$

endowed with the scalar product

$$\begin{aligned}\langle u, v\rangle_\varepsilon = \Re \iint_{\mathbb{R}^{2N}} \frac{(u(x) - \mathrm{e}^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}u(y))\overline{(v(x) - \mathrm{e}^{\imath(x-y)\cdot A_\varepsilon(\frac{x+y}{2})}v(y))}}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\ + \Re \int_{\mathbb{R}^N} V_\varepsilon(x)u\bar{v}\,\mathrm{d}x\end{aligned}$$

for all $u, v \in \mathcal{H}_\varepsilon$, and let

$$\|u\|_\varepsilon = \sqrt{\langle u, u\rangle_\varepsilon}.$$

In what follows we list some useful technical lemmas which will be frequently used along the paper; see [**13**, **21**] for more details.

THEOREM 2.1 [**13**, **21**]. *The space $\mathcal{H}_\varepsilon$ is complete and $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ is dense in $\mathcal{H}_\varepsilon$.*

LEMMA 2.2 [**21**]. *If $u \in \mathcal{D}_A^{s,2}(\mathbb{R}^N, \mathbb{C})$ then $|u| \in \mathcal{D}^{s,2}(\mathbb{R}^N, \mathbb{R})$ and we have*

$$[|u|] \leqslant [u]_A.$$

THEOREM 2.3 [**21**]. *The space $\mathcal{H}_\varepsilon$ is continuously embedded in $L^r(\mathbb{R}^N, \mathbb{C})$ for all $r \in [2, 2_s^*]$, and compactly embedded in $L^r(B_R, \mathbb{C})$ for all $R > 0$ and any $r \in [1, 2_s^*)$.*

LEMMA 2.4 [**13**]. *If $u \in H^s(\mathbb{R}^N, \mathbb{R})$ and u has compact support, then $w = e^{\imath A(0)\cdot x}u \in \mathcal{H}_\varepsilon$.*

We also recall a fractional version of Lions lemma.

LEMMA 2.5 [**30**]. *Let $s \in (0, 1)$ and $R > 0$. If (u_n) is a bounded sequence in $H^s(\mathbb{R}^N, \mathbb{R})$ and if*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 dx = 0,$$

then $u_n \to 0$ in $L^r(\mathbb{R}^N, \mathbb{R})$ for all $r \in (2, 2_s^*)$.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

3. Variational setting and the modified problem

Using the change of variable $u(x) \mapsto u(\varepsilon x)$, we see that (1.1) is equivalent to

$$(-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x)u = f(|u|^2)u \quad \text{in } \mathbb{R}^N. \tag{3.1}$$

Fix $k > 1$ and $a > 0$ such that $f(a) = V_0/k$, and we define the function

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \leqslant a, \\ \dfrac{V_0}{k} & \text{if } t > a. \end{cases}$$

We introduce the penalized nonlinearity $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ given by

$$g(x,t) = \chi_\Lambda(x) f(t) + (1 - \chi_\Lambda(x))\tilde{f}(t),$$

where χ_Λ is the characteristic function of Λ, and we set $G(x,t) = \int_0^t g(x,\tau)\,\mathrm{d}\tau$.

By (f_1)–(f_4), it follows that g is a Carathéodory function satisfying the following properties:

(g_1) $\lim_{t\to 0} g(x,t) = 0$ uniformly in $x \in \mathbb{R}^N$;

(g_2) $g(x,t) \leqslant f(t)$ for any $x \in \mathbb{R}^N$ and $t > 0$;

(g_3) (i) $0 < (\theta/2)G(x,t) \leqslant g(x,t)t$ for any $x \in \Lambda$ and $t > 0$,
(ii) $0 \leqslant G(x,t) \leqslant g(x,t)t \leqslant (V_0/k)t$ for any $x \in \Lambda^c$ and $t > 0$;

(g_4) for any $x \in \Lambda$, the function $t \mapsto g(x,t)$ is increasing in $(0,\infty)$, and for any $x \in \Lambda^c$, the function $t \mapsto g(x,t)$ is increasing in $(0,a)$.

We used the notation $A^c = \mathbb{R}^N \setminus A$ for $A \subset \mathbb{R}^N$. Set $g_\varepsilon(x,t) = g(\varepsilon x, t)$ and we consider the following modified problem

$$\begin{cases} (-\Delta)^s_{A_\varepsilon} u + V_\varepsilon(x)u = g_\varepsilon(x,|u|^2)u & \text{in } \mathbb{R}^N, \\ u \in \mathcal{H}_\varepsilon. \end{cases} \tag{3.2}$$

Let us note that if u is a solution of (3.2) such that

$$|u(x)| \leqslant \sqrt{a} \quad \text{for all } x \in \Lambda^c_\varepsilon, \tag{3.3}$$

where $\Lambda_\varepsilon = \{x \in \mathbb{R}^N : \varepsilon x \in \Lambda\}$, then u is also a solution of (3.1). In order to study weak solutions of (3.2), we look for critical points of the functional $J_\varepsilon : \mathcal{H}_\varepsilon \to \mathbb{R}$ defined as

$$J_\varepsilon(u) = \frac{1}{2}\|u\|^2_\varepsilon - \frac{1}{2}\int_{\mathbb{R}^N} G_\varepsilon(x,|u|^2)\,\mathrm{d}x.$$

It is easy to check that $J_\varepsilon \in C^1(\mathcal{H}_\varepsilon, \mathbb{R})$ and that its differential is given by

$$\langle J'_\varepsilon(u), v\rangle = \langle u, v\rangle_\varepsilon - \Re\int_{\mathbb{R}^N} g_\varepsilon(x,|u|^2)u\bar{v}\,\mathrm{d}x \quad \text{for all } u, v \in \mathcal{H}_\varepsilon.$$

Therefore, weak solutions to (3.2) can be found as critical points of J_ε. Since we are looking for multiple critical points of the functional J_ε, we shall consider it

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

constrained to an appropriated subset of $\mathcal{H}_\varepsilon$. More precisely, we introduce the Nehari manifold associated with J_ε, namely

$$\mathcal{N}_\varepsilon = \{u \in \mathcal{H}_\varepsilon \setminus \{0\} : \langle J'_\varepsilon(u), u\rangle = 0\}.$$

From the growth conditions of g and theorem 2.3, we see that for any fixed $u \in \mathcal{N}_\varepsilon$ and $\delta > 0$ small enough

$$\begin{aligned} 0 = \langle J'_\varepsilon(u), u\rangle &= \|u\|^2_\varepsilon - \int_{\mathbb{R}^N} g_\varepsilon(x, |u|^2)|u|^2 \,\mathrm{d}x \\ &\geqslant \|u\|^2_\varepsilon - \delta C_1 \|u\|^2_\varepsilon - C_\delta \|u\|^{2^*_s}_\varepsilon \\ &\geqslant C_2\|u\|^2_\varepsilon - C_3\|u\|^{2^*_s}_\varepsilon, \end{aligned}$$

so there exists $r > 0$ independent of u such that

$$\|u\|_\varepsilon \geqslant r \quad \text{for all } u \in \mathcal{N}_\varepsilon. \tag{3.4}$$

Let us consider

$$\mathcal{H}^+_\varepsilon = \{u \in \mathcal{H}_\varepsilon : |\mathrm{supp}(|u|) \cap \Lambda_\varepsilon| > 0\} \subset \mathcal{H}_\varepsilon.$$

Let $\mathbb{S}_\varepsilon$ be the unit sphere of $\mathcal{H}_\varepsilon$ and we denote by $\mathbb{S}^+_\varepsilon = \mathbb{S}_\varepsilon \cap \mathcal{H}^+_\varepsilon$. We observe that $\mathcal{H}^+_\varepsilon$ is open in $\mathcal{H}_\varepsilon$. By the definition of $\mathbb{S}^+_\varepsilon$ and the fact that $\mathcal{H}^+_\varepsilon$ is open in $\mathcal{H}_\varepsilon$, it follows that $\mathbb{S}^+_\varepsilon$ is an incomplete $C^{1,1}$-manifold of codimension 1, modelled on $\mathcal{H}_\varepsilon$ and contained in the open $\mathcal{H}^+_\varepsilon$; see [**31**, **47**]. Then, $\mathcal{H}_\varepsilon = T_u\mathbb{S}^+_\varepsilon \oplus \mathbb{R}u$ for each $u \in \mathbb{S}^+_\varepsilon$, where

$$T_u\mathbb{S}^+_\varepsilon = \{v \in \mathcal{H}_\varepsilon : \langle u, v\rangle_\varepsilon = 0\}.$$

Next we prove that J_ε possesses a mountain pass geometry [**4**].

LEMMA 3.1. *The functional J_ε satisfies the following conditions:*

(i) $J_\varepsilon(0) = 0$;

(ii) there exist $\alpha, \rho > 0$ *such that* $J_\varepsilon(u) \geqslant \alpha$ *for any* $u \in \mathcal{H}_\varepsilon$ *such that* $\|u\|_\varepsilon = \rho$;

(iii) there exists $e \in \mathcal{H}_\varepsilon$ *such that* $\|e\|_\varepsilon > \rho$ *and* $J_\varepsilon(e) < 0$.

Proof. Clearly, $J_\varepsilon(0) = 0$. By (g_1) and (g_2), for all $\delta > 0$ there exists $C_\delta > 0$ such that

$$|G_\varepsilon(x, t^2)| \leqslant \delta|t|^2 + C_\delta|t|^{2^*_s} \quad \text{for every } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

This fact combined with theorem 2.3 implies that for all $u \in \mathcal{H}_\varepsilon$

$$J_\varepsilon(u) \geqslant \left(\frac{1}{2} - \delta C\right)\|u\|^2_\varepsilon - C'_\delta\|u\|^{2^*_s}_\varepsilon$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

which implies that (ii) holds true. Concerning (iii), for each $u \in \mathcal{H}_\varepsilon^+$ and $t > 0$, we get

$$\begin{aligned} J_\varepsilon(tu) &\leqslant \frac{t^2}{2}\|u\|_\varepsilon^2 - \frac{1}{2}\int_{\Lambda_\varepsilon} F(t^2|u|^2)\,\mathrm{d}x \\ &\leqslant \frac{t^2}{2}\|u\|_\varepsilon^2 - Ct^\theta \int_{\Lambda_\varepsilon} |u|^\theta\,\mathrm{d}x + C'|\mathrm{supp}(|u|)\cap\Lambda_\varepsilon| \end{aligned}$$

where we used (g_2) and (f_3). Since $\theta > 2$ we see that $J_\varepsilon(tu) \to -\infty$ as $t \to \infty$. □

Since f is merely continuous, the next results will be fundamental to overcome the non-differentiability of $\mathcal{N}_\varepsilon$ and the incompleteness of $\mathbb{S}_\varepsilon^+$.

LEMMA 3.2. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *hold. Then,*

(i) For each $u \in \mathcal{H}_\varepsilon^+$, *let* $h : \mathbb{R}_+ \to \mathbb{R}$ *be defined by* $h_u(t) = J_\varepsilon(tu)$. *Then, there is a unique* $t_u > 0$ *such that*

$$\begin{aligned} h_u'(t) &> 0 \text{ in } (0, t_u), \\ h_u'(t) &< 0 \text{ in } (t_u, \infty). \end{aligned}$$

(ii) There exists $\tau > 0$ *independent of* u *such that* $t_u \geqslant \tau$ *for any* $u \in \mathbb{S}_\varepsilon^+$. *Moreover, for each compact set* $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ *there is a positive constant* $C_\mathbb{K}$ *such that* $t_u \leqslant C_\mathbb{K}$ *for any* $u \in \mathbb{K}$.

(iii) The map $\hat{m}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathcal{N}_\varepsilon$ *given by* $\hat{m}_\varepsilon(u) = t_u u$ *is continuous and* $m_\varepsilon = \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ *is a homeomorphism between* $\mathbb{S}_\varepsilon^+$ *and* $\mathcal{N}_\varepsilon$. *Moreover,* $m_\varepsilon^{-1}(u) = u/\|u\|_\varepsilon$.

(iv) If there is a sequence $(u_n) \subset \mathbb{S}_\varepsilon^+$ *such that* $dist(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$, *then* $\|m_\varepsilon(u_n)\|_\varepsilon \to \infty$ *and* $J_\varepsilon(m_\varepsilon(u_n)) \to \infty$.

Proof. (i) We note that $h_u \in C^1(\mathbb{R}_+, \mathbb{R})$, and arguing as in the proof of lemma 3.1 it is easy to verify that $h_u(0) = 0$, $h_u(t) > 0$ for $t > 0$ small enough and $h_u(t) < 0$ for $t > 0$ sufficiently large. Therefore, $\max_{t\geqslant 0} h_u(t)$ is achieved at some $t_u > 0$ verifying $h_u'(t_u) = 0$ and $t_u u \in \mathcal{N}_\varepsilon$. Next we claim the uniqueness of such a t_u. Let $t_1 > t_2 > 0$ be such that $h_u'(t_i) = 0$ for $i = 1, 2$, that is $\|u\|_\varepsilon^2 = \int_{\mathbb{R}^N} g_\varepsilon(x, |t_i u|^2)|u|^2\,\mathrm{d}x$ for $i = 1, 2$. By using the definition of g, (f_4), (g_4) and $u \in \mathcal{H}_\varepsilon^+$, we see that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \left[g_\varepsilon(x, |t_1 u|^2)|u|^2 - g_\varepsilon(x, |t_2 u|^2)|u|^2\right]\mathrm{d}x \\ &= \int_{\Lambda_\varepsilon^c} \left[g_\varepsilon(x, |t_1 u|^2)|u|^2 - g_\varepsilon(x, |t_2 u|^2)|u|^2\right]\mathrm{d}x \\ &\quad + \int_{\Lambda_\varepsilon} \left[f(|t_1 u|^2)|u|^2 - f(|t_2 u|^2)|u|^2\right]\mathrm{d}x \end{aligned}$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

$$> \int_{\Lambda_\varepsilon^c \cap \{|t_2 u|^2 \leqslant a < |t_1 u|^2\}} \left[\frac{V_0}{k}|u|^2 - f(|t_2 u|^2)|u|^2\right] \mathrm{d}x + \int_{\Lambda_\varepsilon^c \cap \{|t_1 u|^2 \leqslant a\}} \left[f(|t_1 u|^2)|u|^2 - f(|t_2 u|^2)|u|^2\right] \mathrm{d}x \geqslant 0$$

which gives a contradiction.

(ii) Let $u \in \mathbb{S}_\varepsilon^+$. By (i) there exists $t_u > 0$ such that $h_u'(t_u) = 0$, or equivalently

$$t_u = \int_{\mathbb{R}^N} g_\varepsilon(x, |t_u u|^2) t_u |u|^2 \, \mathrm{d}x.$$

By assumptions (g_1) and (g_2), given $\xi > 0$ there exists a positive constant C_ξ such that

$$|g_\varepsilon(x,t)| \leqslant \xi + C_\xi |t|^{(2_s^* - 2)/2}, \quad \text{for every } (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.5}$$

Thus (3.5) and theorem 2.3 yield

$$t_u \leqslant \xi t_u C_1 + C_\xi C_2 t_u^{2_s^* - 1}.$$

Taking $\xi > 0$ sufficiently small, we obtain that there exists $\tau > 0$, independent of u, such that $t_u \geqslant \tau$. Now, let $\mathbb{K} \subset \mathbb{S}_\varepsilon^+$ be a compact set and we show that t_u can be estimated from above by a constant depending on $\mathbb{K}$. Assume by contradiction that there exists a sequence $(u_n) \subset \mathbb{K}$ such that $t_n = t_{u_n} \to \infty$. Therefore, there exists $u \in \mathbb{K}$ such that $u_n \to u$ in $\mathcal{H}_\varepsilon$. From (iii) in lemma 3.1 we get

$$J_\varepsilon(t_n u_n) \to -\infty. \tag{3.6}$$

Fix $v \in \mathcal{N}_\varepsilon$. Then, using the fact that $\langle J_\varepsilon'(v), v\rangle = 0$, and assumption (g_3), we can infer

$$\begin{aligned} J_\varepsilon(v) &= J_\varepsilon(v) - \frac{1}{\theta}\langle J_\varepsilon'(v), v\rangle \\ &= \left(\frac{1}{2} - \frac{1}{\theta}\right)\|v\|_\varepsilon^2 + \int_{\mathbb{R}^N} \frac{1}{\theta} g_\varepsilon(x, |v|^2)|v|^2 - \frac{1}{2} G_\varepsilon(x, |v|^2)] \, \mathrm{d}x \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\theta}\right)\|v\|_\varepsilon^2 + \frac{1}{\theta}\int_{\Lambda_\varepsilon^c} [g_\varepsilon(x, |v|^2)|v|^2 - \frac{\theta}{2} G_\varepsilon(x, |v|^2)] \, \mathrm{d}x \\ &\geqslant \left(\frac{\theta - 2}{2\theta}\right)\|v\|_\varepsilon^2 - \left(\frac{\theta - 2}{2\theta}\right)\frac{1}{k}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|v|^2 \, \mathrm{d}x \\ &\geqslant \left(\frac{\theta - 2}{2\theta}\right)\left(1 - \frac{1}{k}\right)\|v\|_\varepsilon^2. \end{aligned} \tag{3.7}$$

Taking into account that $(t_{u_n} u_n) \subset \mathcal{N}_\varepsilon$ and $\|t_{u_n} u_n\|_\varepsilon = t_{u_n} \to \infty$, from (3.7) we deduce that (3.6) does not hold.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

(iii) First, we note that $\hat{m}_\varepsilon$, m_ε and m_ε^{-1} are well defined. Indeed, by (i), for each $u \in \mathcal{H}_\varepsilon^+$ there exists a unique $m_\varepsilon(u) \in \mathcal{N}_\varepsilon$. On the other hand, if $u \in \mathcal{N}_\varepsilon$ then $u \in \mathcal{H}_\varepsilon^+$. Otherwise, if $u \notin \mathcal{H}_\varepsilon^+$, we have

$$|\mathrm{supp}(|u|) \cap \Lambda_\varepsilon| = 0,$$

which together with (g_3)-(ii) gives

$$\begin{aligned} 0 < \|u\|_\varepsilon^2 &= \int_{\mathbb{R}^N} g_\varepsilon(x, |u|^2)|u|^2 \,\mathrm{d}x \\ &= \int_{\Lambda_\varepsilon} g_\varepsilon(x, |u|^2)|u|^2 \,\mathrm{d}x + \int_{\Lambda_\varepsilon^c} g_\varepsilon(x, |u|^2)|u|^2 \,\mathrm{d}x \\ &= \int_{\Lambda_\varepsilon^c} g_\varepsilon(x, |u|^2)|u|^2 \,\mathrm{d}x \\ &\leqslant \frac{1}{k}\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u|^2 \,\mathrm{d}x \leqslant \frac{1}{k}\|u\|_\varepsilon^2. \end{aligned} \tag{3.8}$$

Using (3.8) we get

$$0 < \|u\|_\varepsilon^2 \leqslant \frac{1}{k}\|u\|_\varepsilon^2$$

and this leads to a contradiction because $k > 1$. Consequently, $m_\varepsilon^{-1}(u) = u/\|u\|_\varepsilon \in \mathbb{S}_\varepsilon^+$, m_ε^{-1} is well defined and continuous.
Let $u \in \mathbb{S}_\varepsilon^+$. Then,

$$m_\varepsilon^{-1}(m_\varepsilon(u)) = m_\varepsilon^{-1}(t_u u) = \frac{t_u u}{\|t_u u\|_\varepsilon} = \frac{u}{\|u\|_\varepsilon} = u$$

from which m_ε is a bijection. Now, our aim is to prove that $\hat{m}_\varepsilon$ is a continuous function. Let $(u_n) \subset \mathcal{H}_\varepsilon^+$ and $u \in \mathcal{H}_\varepsilon^+$ such that $u_n \to u$ in $\mathcal{H}_\varepsilon^+$. Hence,

$$\frac{u_n}{\|u_n\|_\varepsilon} \to \frac{u}{\|u\|_\varepsilon} \quad \text{in } \mathcal{H}_\varepsilon.$$

Set $v_n = u_n/\|u_n\|_\varepsilon$ and $t_n = t_{v_n}$. By (ii) there exists $t_0 > 0$ such that $t_n \to t_0$. Since $t_n v_n \in \mathcal{N}_\varepsilon$ and $\|v_n\|_\varepsilon = 1$, we have

$$t_n^2 = \int_{\mathbb{R}^N} g_\varepsilon(x, |t_n v_n|^2)|t_n v_n|^2 \,\mathrm{d}x.$$

Passing to the limit as $n \to \infty$ we obtain

$$t_0^2 = \int_{\mathbb{R}^N} g_\varepsilon(x, |t_0 v|^2)|t_0 v|^2 \,\mathrm{d}x,$$

where $v = u/\|u\|_\varepsilon$, which implies that $t_0 v \in \mathcal{N}_\varepsilon$. By (i) we deduce that $t_v = t_0$, and this shows that

$$\hat{m}_\varepsilon(u_n) = \hat{m}_\varepsilon\left(\frac{u_n}{\|u_n\|_\varepsilon}\right) \to \hat{m}_\varepsilon\left(\frac{u}{\|u\|_\varepsilon}\right) = \hat{m}_\varepsilon(u) \quad \text{in } \mathcal{H}_\varepsilon.$$

Therefore, $\hat{m}_\varepsilon$ and m_ε are continuous functions.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

(iv) Let $(u_n) \subset \mathbb{S}_\varepsilon^+$ be such that $\operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+) \to 0$. Observe that, by the Sobolev inequality and (V_1)–(V_2), for each $p \in [2, 2_s^*]$ there exists $\kappa_p > 0$ such that it holds

$$\begin{aligned}\|u_n\|_{L^p(\Lambda_\varepsilon)}^p &\leqslant \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_{L^p(\Lambda_\varepsilon)}^p \\ &\leqslant \kappa_p \inf_{v\in\partial\mathbb{S}_\varepsilon^+} \|u_n - v\|_\varepsilon^p \\ &\leqslant \kappa_p \operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^p, \qquad \text{for all } n \in \mathbb{N}.\end{aligned}$$

Then, by the above inequality, (g_1), (g_2) and (g_3)-(ii), we can infer that, for all $t > 0$,

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N} G_\varepsilon(x, |tu_n|^2)\,\mathrm{d}x \\ &\quad= \frac{1}{2}\int_{\Lambda_\varepsilon^c} G_\varepsilon(x, |tu_n|^2)\,\mathrm{d}x + \frac{1}{2}\int_{\Lambda_\varepsilon} G_\varepsilon(x, |tu_n|^2)\,\mathrm{d}x \\ &\quad\leqslant \frac{t^2}{2k}\int_{\Lambda_\varepsilon^c} V(\varepsilon x)|u_n|^2\,\mathrm{d}x + \int_{\Lambda_\varepsilon}\frac{1}{2}F(|tu_n|^2)\,\mathrm{d}x \\ &\quad\leqslant \frac{t^2}{2k}\|u_n\|_\varepsilon^2 + C_1 t^2\int_{\Lambda_\varepsilon}|u_n|^2\,\mathrm{d}x + C_2 t^{2_s^*}\int_{\Lambda_\varepsilon}|u_n|^{2_s^*}\,\mathrm{d}x \\ &\quad\leqslant \frac{t^2}{2k} + C_1\kappa_2 t^2 \operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^2 + C_2\kappa_{2_s^*} t^{2_s^*}\operatorname{dist}(u_n, \partial\mathbb{S}_\varepsilon^+)^{2_s^*}\end{aligned}$$

from which

$$\frac{1}{2}\limsup_{n\to\infty}\int_{\mathbb{R}^N} G_\varepsilon(x, |tu_n|^2)\,\mathrm{d}x \leqslant \frac{t^2}{2k} \qquad \text{for all } t > 0. \tag{3.9}$$

Bearing in mind the definition of $m_\varepsilon(u_n)$, and by using (3.9), we have

$$\begin{aligned}\liminf_{n\to\infty} J_\varepsilon(m_\varepsilon(u_n)) &\geqslant \liminf_{n\to\infty} J_\varepsilon(tu_n) \\ &= \liminf_{n\to\infty}\left[\frac{1}{2}\|tu_n\|_\varepsilon^2 - \frac{1}{2}\int_{\mathbb{R}^N} G_\varepsilon(x, |tu_n|^2)\,\mathrm{d}x\right] \\ &\geqslant \left(\frac{1}{2} - \frac{1}{2k}\right)t^2 \qquad \text{for all } t > 0.\end{aligned}$$

Recalling that $k > 1$ we get

$$\lim_{n\to\infty} J_\varepsilon(m_\varepsilon(u_n)) = \infty.$$

Moreover, by the definition of J_ε, we see that

$$\liminf_{n\to\infty}\frac{1}{2}\|m_\varepsilon(u_n)\|_\varepsilon^2 \geqslant \liminf_{n\to\infty} J_\varepsilon(m_\varepsilon(u_n)) = \infty.$$

This completes the proof of the lemma. □

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Let us consider the maps

$$\hat{\psi}_\varepsilon : \mathcal{H}_\varepsilon^+ \to \mathbb{R} \quad \text{and} \quad \psi_\varepsilon : \mathbb{S}_\varepsilon^+ \to \mathbb{R},$$

by setting $\hat{\psi}_\varepsilon(u) = J_\varepsilon(\hat{m}_\varepsilon(u))$ and $\psi_\varepsilon = \hat{\psi}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$.

The next result is a direct consequence of lemma 3.2 and corollary 10 in [47].

PROPOSITION 3.3. *Assume that* (V_1)–(V_2) *and* (f_1)–(f_4) *hold. Then,*

(a) $\hat{\psi}_\varepsilon \in C^1(\mathcal{H}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}'_\varepsilon(u), v\rangle = \frac{\|\hat{m}_\varepsilon(u)\|_\varepsilon}{\|u\|_\varepsilon} \langle J'_\varepsilon(\hat{m}_\varepsilon(u)), v\rangle,$$

for every $u \in \mathcal{H}_\varepsilon^+$ *and* $v \in \mathcal{H}_\varepsilon$.

(b) $\psi_\varepsilon \in C^1(\mathbb{S}_\varepsilon^+, \mathbb{R})$ *and*

$$\langle \psi'_\varepsilon(u), v\rangle = \|m_\varepsilon(u)\|_\varepsilon \langle J'_\varepsilon(m_\varepsilon(u)), v\rangle,$$

for every $v \in T_u\mathbb{S}_\varepsilon^+$.

(c) If (u_n) *is a Palais–Smale sequence for* ψ_ε, *then* $(m_\varepsilon(u_n))$ *is a Palais–Smale sequence for* J_ε. *If* $(u_n) \subset \mathcal{N}_\varepsilon$ *is a bounded Palais–Smale sequence for* J_ε, *then* $(m_\varepsilon^{-1}(u_n))$ *is a Palais–Smale sequence for* ψ_ε.

(d) u *is a critical point of* ψ_ε *if and only if* $m_\varepsilon(u)$ *is a nontrivial critical point for* J_ε. *Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_\varepsilon^+} \psi_\varepsilon(u) = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u).$$

REMARK 3.4. As in [47], we have the following variational characterization of the infimum of J_ε over $\mathcal{N}_\varepsilon$:

$$0 < c_\varepsilon = \inf_{u\in\mathcal{N}_\varepsilon} J_\varepsilon(u) = \inf_{u\in\mathcal{H}_\varepsilon^+} \max_{t>0} J_\varepsilon(tu) = \inf_{u\in\mathbb{S}_\varepsilon^+} \max_{t>0} J_\varepsilon(tu).$$

Moreover, if

$$c'_\varepsilon = \inf_{\gamma\in\Gamma_\varepsilon} \max_{t\in[0,1]} J_\varepsilon(\gamma(t)), \quad \text{where}$$
$$\Gamma_\varepsilon = \{\gamma \in C([0,1], \mathcal{H}_\varepsilon) : \gamma(0) = 0 \text{ and } J_\varepsilon(\gamma(1)) < 0\},$$

then we can argue as in [24, 43, 48] to verify that $c_\varepsilon = c'_\varepsilon$.

The main feature of J_ε is that it satisfies the following compactness property:

LEMMA 3.5. *The functional* J_ε *satisfies the* $(PS)_c$ *condition at any level* $c \in \mathbb{R}$.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Proof. Let $(u_n) \subset \mathcal{H}_\varepsilon$ be a Palais–Smale sequence for J_ε at the level c, that is, as $n \to \infty$

$$J_\varepsilon(u_n) \to c \text{ and } J'_\varepsilon(u_n) \to 0.$$

First, we show that (u_n) is bounded in $\mathcal{H}_\varepsilon$. Indeed, using (g_3), we get

$$\begin{aligned} C(1+\|u_n\|_\varepsilon) &\geqslant J_\varepsilon(u_n) - \frac{1}{\theta}\langle J'_\varepsilon(u_n), u_n\rangle \\ &= \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_\varepsilon^2 + \frac{1}{\theta}\int_{\mathbb{R}^N}\left[g_\varepsilon(x,|u_n|^2)|u_n|^2 - \frac{\theta}{2}G_\varepsilon(x,|u_n|^2)\right]\mathrm{d}x \\ &\geqslant \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_\varepsilon^2 - \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\Lambda_\varepsilon^c} G_\varepsilon(x,|u_n|^2)\,\mathrm{d}x \\ &\geqslant \left(\frac{1}{2}-\frac{1}{\theta}\right)\|u_n\|_\varepsilon^2 - \left(\frac{1}{2}-\frac{1}{\theta}\right)\frac{1}{k}\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x \\ &\geqslant \left(\frac{\theta-2}{2\theta}\right)\left(1-\frac{1}{k}\right)\|u_n\|_\varepsilon^2, \end{aligned}$$

and recalling that $k>1$ and $\theta>2$, we deduce that (u_n) is bounded in $\mathcal{H}_\varepsilon$. Since $\mathcal{H}_\varepsilon$ is a reflexive space, we can find a subsequence, still denoted by (u_n), and $u \in \mathcal{H}_\varepsilon$ such that

$$\begin{array}{ll} u_n \rightharpoonup u & \text{in } \mathcal{H}_\varepsilon \text{ as } n\to\infty, \\ u_n \to u & \text{in } L^q_{loc}(\mathbb{R}^N,\mathbb{C}) \text{ for all } q\in[1,2^*_s) \text{ as } n\to\infty, \\ |u_n| \to |u| & \text{a.e. in } \mathbb{R}^N \text{ as } n\to\infty. \end{array} \tag{3.10}$$

Using (g_1) and (g_2), we see that

$$\lim_{n\to\infty} \Re\int_{\mathbb{R}^N} g_\varepsilon(x,|u_n|^2)u_n\overline{\phi}\,\mathrm{d}x = \Re\int_{\mathbb{R}^N} g_\varepsilon(x,|u|^2)u\overline{\phi}\,\mathrm{d}x \quad \text{for all } \phi\in C_c^\infty(\mathbb{R}^N,\mathbb{C}). \tag{3.11}$$

Taking into account (3.10), (3.11) and theorem 2.1, we deduce that

$$\langle J'_\varepsilon(u),\phi\rangle = 0 \quad \text{for all } \phi\in\mathcal{H}_\varepsilon,$$

that is u is a critical point for J_ε. In particular, $\langle J'_\varepsilon(u),u\rangle = 0$, or equivalently

$$[u]^2_{A_\varepsilon} + \int_{\Lambda_\varepsilon} V_\varepsilon(x)|u|^2\,\mathrm{d}x + \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x,|u|^2)\,\mathrm{d}x = \int_{\Lambda_\varepsilon} f(|u|^2)|u|^2\,\mathrm{d}x, \tag{3.12}$$

where $\mathcal{C}(x,t) = V(x)t - g(x,t)t$. Note that, by (g_3)-(ii), it holds

$$V(x)t \geqslant \mathcal{C}(x,t) \geqslant \left(1-\frac{1}{k}\right)V(x)t \geqslant 0 \quad \text{for all } x\in\Lambda^c,\ t\geqslant 0. \tag{3.13}$$

Recalling that $\langle J'_\varepsilon(u_n),u_n\rangle = o_n(1)$, we also know that

$$[u_n]^2_{A_\varepsilon} + \int_{\Lambda_\varepsilon} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x + \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x,|u_n|^2)\,\mathrm{d}x = \int_{\Lambda_\varepsilon} f(|u_n|^2)|u_n|^2\,\mathrm{d}x + o_n(1). \tag{3.14}$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Since Λ_ε is bounded, it follows from the local compact embeddings in theorem 2.3 that

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon} f(|u_n|^2)|u_n|^2\,\mathrm{d}x = \int_{\Lambda_\varepsilon} f(|u|^2)|u|^2\,\mathrm{d}x, \tag{3.15}$$

and

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x = \int_{\Lambda_\varepsilon} V_\varepsilon(x)|u|^2\,\mathrm{d}x. \tag{3.16}$$

Putting together (3.12), (3.14), (3.15) and (3.16), we deduce that

$$\limsup_{n\to\infty}\left([u_n]^2_{A_\varepsilon} + \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u_n|^2)\,\mathrm{d}x\right) = [u]^2_{A_\varepsilon} + \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u|^2)\,\mathrm{d}x.$$

Now, by (3.13) and Fatou's lemma, we get

$$\liminf_{n\to\infty}\left([u_n]^2_{A_\varepsilon} + \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u_n|^2)\,\mathrm{d}x\right) \geqslant [u]^2_{A_\varepsilon} + \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u|^2)\,\mathrm{d}x.$$

Hence,

$$\lim_{n\to\infty}[u_n]^2_{A_\varepsilon} = [u]^2_{A_\varepsilon}, \tag{3.17}$$

and

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u_n|^2)\,\mathrm{d}x = \int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u|^2)\,\mathrm{d}x.$$

The last limit, Fatou's lemma and (3.13) lead to

$$\begin{aligned}
\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u|^2\,\mathrm{d}x &\leqslant \liminf_{n\to\infty}\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x\\
&\leqslant \limsup_{n\to\infty}\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x\\
&\leqslant \left(\frac{k}{k-1}\right)\limsup_{n\to\infty}\int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u_n|^2)\,\mathrm{d}x\\
&= \left(\frac{k}{k-1}\right)\int_{\Lambda_\varepsilon^c}\mathcal{C}(\varepsilon x, |u|^2)\,\mathrm{d}x\\
&\leqslant \left(\frac{k}{k-1}\right)\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u|^2\,\mathrm{d}x \quad \text{for all } k>1,
\end{aligned}$$

and by sending $k\to\infty$ we find

$$\lim_{n\to\infty}\int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x = \int_{\Lambda_\varepsilon^c} V_\varepsilon(x)|u|^2\,\mathrm{d}x$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

which combined with (3.16) gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} V_\varepsilon(x)|u_n|^2\,\mathrm{d}x = \int_{\mathbb{R}^N} V_\varepsilon(x)|u|^2\,\mathrm{d}x. \tag{3.18}$$

Putting together (3.17) and (3.18), we obtain that

$$\lim_{n\to\infty}\|u_n\|_\varepsilon^2 = \|u\|_\varepsilon^2.$$

Since $\mathcal{H}_\varepsilon$ is a Hilbert space and $u_n \rightharpoonup u$ in $\mathcal{H}_\varepsilon$ as $n\to\infty$, we conclude that $u_n \to u$ in $\mathcal{H}_\varepsilon$ as $n\to\infty$. □

COROLLARY 3.6. *The functional ψ_ε satisfies the $(PS)_c$ condition on $\mathbb{S}_\varepsilon^+$ at any level $c\in\mathbb{R}$.*

Proof. Let $(u_n)\subset\mathbb{S}_\varepsilon^+$ be a Palais–Smale sequence for ψ_ε at the level c. Then,

$$\psi_\varepsilon(u_n)\to c \quad \text{and} \quad \|\psi'_\varepsilon(u_n)\|_*\to 0,$$

where $\|\cdot\|_*$ denotes the norm in the dual space $(T_{u_n}\mathbb{S}_\varepsilon^+)^*$. It follows from proposition 3.3-(c) that $(m_\varepsilon(u_n))\subset\mathcal{H}_\varepsilon$ is a Palais–Smale sequence for J_ε at the level c. Then, using lemma 3.5, there exists $u\in\mathbb{S}_\varepsilon^+$ such that, up to a subsequence,

$$m_\varepsilon(u_n)\to m_\varepsilon(u) \quad \text{in } \mathcal{H}_\varepsilon.$$

Applying lemma 3.2-(iii) we can infer that $u_n\to u$ in $\mathbb{S}_\varepsilon^+$. □

We end this section by showing the following existence result for (3.2).

THEOREM 3.7. *Assume that (V_1)–(V_2) and (f_1)–(f_4) hold. Then, for all $\varepsilon>0$, there exists a ground state solution to (3.2).*

Proof. In view of lemmas **3.1** and 3.5, we can apply the mountain pass theorem [**4**] to see that for all $\varepsilon>0$ there exists a nontrivial critical point $u_\varepsilon\in\mathcal{H}_\varepsilon$ of J_ε. By remark 3.4, we deduce the thesis. □

4. The limiting scalar problem

In this section, we focus our attention on the limiting scalar problem associated to (3.2), namely

$$\begin{cases}(-\Delta)^s u + V_0 u = f(u^2)u \text{ in } \mathbb{R}^N,\\ u\in H^s(\mathbb{R}^N,\mathbb{R}).\end{cases} \tag{4.1}$$

The corresponding energy functional $I_0:\mathcal{H}_0\to\mathbb{R}$ is given by

$$I_0(u)=\frac{1}{2}\|u\|_0^2-\int_{\mathbb{R}^N}\frac{1}{2}F(u^2)\,\mathrm{d}x,$$

where $\mathcal{H}_0$ stands for the fractional Sobolev space $H^s(\mathbb{R}^N,\mathbb{R})$ endowed with the norm

$$\|u\|_0=\sqrt{[u]^2+V_0\|u\|^2_{L^2(\mathbb{R}^N)}}.$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

For any $u, v \in \mathcal{H}_0$, we set

$$\langle u, v\rangle_0 = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y + V_0 \int_{\mathbb{R}^N} uv\,\mathrm{d}x.$$

Let

$$\mathcal{M}_0 = \{u \in \mathcal{H}_0 \setminus \{0\} : \langle I'_\mu(u), u\rangle = 0\}$$

be the Nehari manifold associated with I_0. Let us consider

$$\mathcal{H}_0^+ = \{u \in \mathcal{H}_0 : |\mathrm{supp}(|u|)| > 0\}.$$

Let $\mathbb{S}_0$ be the unit sphere of $\mathcal{H}_0$ and we denote by $\mathbb{S}_0^+ = \mathbb{S}_\varepsilon \cap \mathcal{H}_0^+$. We observe that $\mathbb{S}_0^+$ is an incomplete $C^{1,1}$-manifold of codimension 1, modelled on $\mathcal{H}_0$ and contained in the open $\mathcal{H}_0^+$. Then, $\mathcal{H}_0 = T_u\mathbb{S}_0^+ \oplus \mathbb{R}u$ for each $u \in \mathbb{S}_0^+$, where

$$T_u\mathbb{S}_0^+ = \{v \in \mathcal{H}_0 : \langle u, v\rangle_0 = 0\}.$$

Arguing as in the proof of lemma 3.2 (see also [**5**]) we have the following result.

LEMMA 4.1. *Assume that* (f_1)–(f_4) *hold. Then,*

(i) For each $u \in \mathcal{H}_0^+$, *let* $h : \mathbb{R}_+ \to \mathbb{R}$ *be defined by* $h_u(t) = I_0(tu)$. *Then, there is a unique* $t_u > 0$ *such that*

$$h'_u(t) > 0 \text{ in } (0, t_u),$$
$$h'_u(t) < 0 \text{ in } (t_u, \infty).$$

(ii) There exists $\tau > 0$ *independent of* u *such that* $t_u \geqslant \tau$ *for any* $u \in \mathbb{S}_0^+$. *Moreover, for each compact set* $\mathbb{K} \subset \mathbb{S}_0^+$ *there is a positive constant* $C_\mathbb{K}$ *such that* $t_u \leqslant C_\mathbb{K}$ *for any* $u \in \mathbb{K}$.

(iii) The map $\hat{m}_0 : \mathcal{H}_0^+ \to \mathcal{M}_0$ *given by* $\hat{m}_0(u) = t_u u$ *is continuous and* $m_0 = \hat{m}_0|_{\mathbb{S}_0^+}$ *is a homeomorphism between* $\mathbb{S}_0^+$ *and* $\mathcal{M}_0$. *Moreover,* $m_0^{-1}(u) = u/\|u\|_0$.

(iv) If there is a sequence $(u_n) \subset \mathbb{S}_0^+$ *such that* $dist(u_n, \partial\mathbb{S}_0^+) \to 0$ *then* $\|m_0(u_n)\|_0 \to \infty$ *and* $I_0(m_0(u_n)) \to \infty$.

Let us define the maps

$$\hat{\psi}_0 : \mathcal{H}_0^+ \to \mathbb{R} \quad \text{and} \quad \psi_0 : \mathbb{S}_0^+ \to \mathbb{R},$$

by $\hat{\psi}_0(u) = I_0(\hat{m}_0(u))$ and $\psi_0 = \hat{\psi}_0|_{\mathbb{S}_0^+}$.

The next result is a direct consequence of lemma 4.1 and corollary 10 in [**47**].

PROPOSITION 4.2. *Assume that* (f_1)–(f_4) *hold. Then,*

1. $\hat{\psi}_0 \in C^1(\mathcal{H}_0^+, \mathbb{R})$ *and*

$$\langle \hat{\psi}'_0(u), v\rangle = \frac{\|\hat{m}_0(u)\|_0}{\|u\|_0} \langle I'_0(\hat{m}_0(u)), v\rangle,$$

for every $u \in \mathcal{H}_0^+$ *and* $v \in \mathcal{H}_0$.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

2. $\psi_0 \in C^1(\mathbb{S}_0^+, \mathbb{R})$ *and*

$$\langle \psi_0'(u), v\rangle = \|m_0(u)\|_0 \langle I_0'(m_0(u)), v\rangle,$$

for every $v \in T_u\mathbb{S}_0^+$.

3. *If* (u_n) *is a Palais–Smale sequence for* ψ_0*, then* $(m_0(u_n))$ *is a Palais–Smale sequence for* I_0*. If* $(u_n) \subset \mathcal{M}_0$ *is a bounded Palais–Smale sequence for* I_0*, then* $(m_0^{-1}(u_n))$ *is a Palais–Smale sequence for* ψ_0.

4. u *is a critical point of* ψ_0 *if and only if* $m_0(u)$ *is a nontrivial critical point for* I_0*. Moreover, the corresponding critical values coincide and*

$$\inf_{u\in\mathbb{S}_0^+} \psi_0(u) = \inf_{u\in\mathcal{M}_0} I_0(u).$$

REMARK 4.3. As in [47], we have the following variational characterization of the infimum of I_0 over $\mathcal{M}_0$:

$$0 < d_0 = \inf_{u\in\mathcal{M}_0} I_0(u) = \inf_{u\in\mathcal{H}_0^+} \max_{t>0} I_0(tu) = \inf_{u\in\mathbb{S}_0^+} \max_{t>0} I_0(tu).$$

The next lemma allows us to assume that the weak limit of a Palais–Smale sequence is nontrivial.

LEMMA 4.4. *Let* $(u_n) \subset \mathcal{H}_0$ *be a Palais–Smale sequence for* I_0 *at the level* d_0*. Then, one and only one of the following alternatives holds:*

(i) $u_n \to 0$ *in* $\mathcal{H}_0$,

(ii) there exist a sequence $(y_n) \subset \mathbb{R}^N$ *and constants* $R, \beta > 0$ *such that*

$$\liminf_{n\to\infty} \int_{B_R(y_n)} |u_n|^2 dx \geqslant \beta > 0.$$

Proof. Assume that (ii) does not occur. Arguing as in the proof of lemma 3.5 we see that (u_n) is bounded in $\mathcal{H}_0$. Then we use lemma 2.5 to deduce that $u_n \to 0$ in $L^r(\mathbb{R}^N, \mathbb{R})$ for all $r \in (2, 2_s^*)$. In view of (f_1)–(f_2), we get $\int_{\mathbb{R}^N} f(u_n^2)u_n^2 \mathrm{d}x = o_n(1)$. This fact combined with $\langle I_0'(u_n), u_n\rangle = o_n(1)$ yields $\|u_n\|_0^2 = o_n(1)$. □

REMARK 4.5. As it has been mentioned, if u is the weak limit of a Palais–Smale sequence $(u_n) \subset \mathcal{H}_0$ of I_0 at the level d_0, then we can assume that $u \neq 0$. Otherwise, we would have $u_n \rightharpoonup 0$ in $\mathcal{H}_0$ and, if $u_n \nrightarrow 0$ in $\mathcal{H}_0$, we deduce from lemma 4.4 that there are $(y_n) \subset \mathbb{R}^N$ and $R, \beta > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(y_n)} u_n^2 \, \mathrm{d}x \geqslant \beta > 0.$$

Set $v_n(x) = u_n(x + y_n)$. Then we see that (v_n) is a Palais–Smale sequence for I_0 at the level d_0, (v_n) is bounded in $\mathcal{H}_0$ and there exists $v \in \mathcal{H}_0$ such that $v_n \rightharpoonup v$ in $\mathcal{H}_0$ with $v \neq 0$.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Next we prove an existence result for the scalar autonomous problem (4.1).

THEOREM 4.6. *Let $(u_n) \subset \mathcal{H}_0$ be a Palais–Smale sequence of I_0 at the level d_0. Then there exists $u \in \mathcal{H}_0 \setminus \{0\}$, with $u \geqslant 0$, such that, up to a subsequence, we have $u_n \to u$ in $\mathcal{H}_0$. Moreover, u is a positive ground state for (4.1).*

Proof. Arguing as in the proof of lemma 3.5, we can see that (u_n) is bounded in $\mathcal{H}_0$. Therefore, up to going a subsequence, we may assume that $u_n \rightharpoonup u$ in $\mathcal{H}_0$. Standard arguments show that u is a critical point of I_0. In light of remark 4.5, we may assume that u is not trivial. Hence, $u \in \mathcal{M}_0$. Now we prove that $I_0(u) = d_0$. Indeed, by $u \in \mathcal{M}_0$, (f_3) and Fatou's lemma, we have

$$\begin{aligned} d_0 &\leqslant I_0(u) - \frac{1}{\theta}\langle I_0'(u), u\rangle \\ &\leqslant \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{\theta}\right)\|u_n\|_0^2 + \int_{\mathbb{R}^N} \frac{1}{\theta} f(u_n^2)u_n^2 - \frac{1}{2}F(u_n^2)\,\mathrm{d}x\right] \\ &= \liminf_{n\to\infty} \left[I_0(u_n) - \frac{1}{\theta}\langle I_0'(u_n), u_n\rangle\right] \\ &= d_0. \end{aligned}$$

Since $\langle I_0'(u), u^-\rangle = 0$, where $u^- = \min\{u, 0\}$, and using (f_1), we can see that $u \geqslant 0$ in $\mathbb{R}^N$. Performing a standard Moser iteration argument (see proposition 5.1.1 in [**26**]) we see that $u \in L^\infty(\mathbb{R}^N, \mathbb{R})$ and applying proposition 2.9 in [**44**] we infer that $u \in C^{0,\alpha}(\mathbb{R}^N, \mathbb{R})$ for some $\alpha \in (0, 1)$. By the strong maximum principle (see theorem 1.4 in [**22**]), we deduce that $u > 0$ in $\mathbb{R}^N$. □

The next lemma is a compactness result for the autonomous problem which will be used later.

LEMMA 4.7. *Let $(u_n) \subset \mathcal{M}_0$ be a sequence such that $I_0(u_n) \to d_0$. Then (u_n) has a convergent subsequence in $\mathcal{H}_0$.*

Proof. Since $(u_n) \subset \mathcal{M}_0$ and $I_0(u_n) \to d_0$, we can apply lemma 4.1-(iii), proposition 4.2-(d) and the definition of d_0 to infer that

$$\nu_n = m_0^{-1}(u_n) = \frac{u_n}{\|u_n\|_0} \in \mathbb{S}_0^+$$

and

$$\psi_0(\nu_n) = I_0(u_n) \to d_0 = \inf_{v\in\mathbb{S}_0^+} \psi_0(v).$$

Let us introduce the following map $\mathcal{F} : \overline{\mathbb{S}}_0^+ \to \mathbb{R} \cup \{\infty\}$ defined by setting

$$\mathcal{F}(u) = \begin{cases} \psi_0(u) & \text{if } u \in \mathbb{S}_0^+, \\ \infty & \text{if } u \in \partial\mathbb{S}_0^+. \end{cases}$$

We note that

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

- $(\overline{\mathbb{S}}_0^+, \delta_0)$, where $\delta_0(u,v) = \|u - v\|_0$, is a complete metric space;
- $\mathcal{F} \in C(\overline{\mathbb{S}}_0^+, \mathbb{R} \cup \{\infty\})$, by lemma 4.1-(iv);
- $\mathcal{F}$ is bounded below, by proposition 4.2-(d).

By applying Ekeland's variational principle [**27**], we can find a Palais–Smale sequence $(\hat{v}_n) \subset \mathbb{S}_0^+$ for ψ_0 at the level d_0 and such that $\|\hat{v}_n - \nu_n\|_0 = o_n(1)$. Now the remainder of the proof follows from proposition 4.2, theorem 4.6 and arguing as in the proof of corollary 3.6. □

Finally, we prove the following interesting relation between c_ε and d_0.

LEMMA 4.8. *The numbers c_ε and d_0 satisfy the following inequality:*

$$\limsup_{\varepsilon \to 0} c_\varepsilon \leqslant d_0.$$

Proof. Let $w \in H^s(\mathbb{R}^N, \mathbb{R})$ be a positive ground state to (4.1), so that $I_0'(w) = 0$ and $I_0(w) = d_0$, and let $\eta \in C_c^\infty(\mathbb{R}^N, [0,1])$ be a cut-off function such that $\eta = 1$ in $B_{\delta/2}$ and $\operatorname{supp}(\eta) \subset B_\delta \subset \Lambda$ for some $\delta > 0$. We recall that the existence of w is guaranteed by theorem 4.6. Moreover, from theorem 1.5 in [**30**], we know that w satisfies the following power-type decay estimate:

$$0 < w(x) \leqslant \frac{C}{|x|^{N+2s}} \quad \text{for all } |x| > 1. \tag{4.2}$$

Let us define $w_\varepsilon(x) = \eta_\varepsilon(x) w(x) \mathrm{e}^{\imath A(0)\cdot x}$, with $\eta_\varepsilon(x) = \eta(\varepsilon x)$ for $\varepsilon > 0$, and we observe that $|w_\varepsilon| = \eta_\varepsilon w$ and $w_\varepsilon \in \mathcal{H}_\varepsilon$ in light of lemma 2.4. Next we claim that

$$\lim_{\varepsilon \to 0} \|w_\varepsilon\|_\varepsilon^2 = \|w\|_0^2 \in (0, \infty). \tag{4.3}$$

Clearly, $\int_{\mathbb{R}^N} V_\varepsilon(x)|w_\varepsilon|^2 \mathrm{d}x \to \int_{\mathbb{R}^N} V_0 |w|^2 \mathrm{d}x$. Then, it remains to show that

$$\lim_{\varepsilon \to 0} [w_\varepsilon]_{A_\varepsilon}^2 = [w]^2. \tag{4.4}$$

Using lemma 5 in [**42**], we know that

$$[\eta_\varepsilon w] \to [w] \text{ as } \varepsilon \to 0. \tag{4.5}$$

On the other hand

$$\begin{aligned}
[w_\varepsilon]_{A_\varepsilon}^2 &= \iint_{\mathbb{R}^{2N}} \frac{|\mathrm{e}^{\imath A(0)\cdot x}\eta_\varepsilon(x)w(x) - \mathrm{e}^{\imath A_\varepsilon(\frac{x+y}{2})\cdot(x-y)}\mathrm{e}^{\imath A(0)\cdot y}\eta_\varepsilon(y)w(y)|^2}{|x-y|^{N+2s}} \mathrm{d}x\,\mathrm{d}y \\
&= [\eta_\varepsilon w]^2 + \iint_{\mathbb{R}^{2N}} \frac{\eta_\varepsilon^2(y) w^2(y) |\mathrm{e}^{\imath[A_\varepsilon(\frac{x+y}{2}) - A(0)]\cdot(x-y)} - 1|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y + 2\Re \iint_{\mathbb{R}^{2N}} \\
&\quad \times \frac{(\eta_\varepsilon(x)w(x) - \eta_\varepsilon(y)w(y))\eta_\varepsilon(y)w(y)(1 - \mathrm{e}^{-\imath[A_\varepsilon(\frac{x+y}{2}) - A(0)]\cdot(x-y)})}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&= [\eta_\varepsilon w]^2 + X_\varepsilon + 2Y_\varepsilon.
\end{aligned}$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Then, in view of $|Y_\varepsilon| \leqslant [\eta_\varepsilon w]\sqrt{X_\varepsilon}$ and (4.5), it is suffices to prove that $X_\varepsilon \to 0$ as $\varepsilon \to 0$ to deduce that (4.4) is satisfied. Let us note that for $0 < \beta < \alpha/(1+\alpha-s)$,

$$\begin{aligned} X_\varepsilon &\leqslant \int_{\mathbb{R}^N} w^2(y)\mathrm{d}y \int_{|x-y|\geqslant \varepsilon^{-\beta}} \frac{|\mathrm{e}^{\imath[A_\varepsilon(\frac{x+y}{2})-A(0)]\cdot(x-y)}-1|^2}{|x-y|^{N+2s}}\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} w^2(y)\mathrm{d}y \int_{|x-y|<\varepsilon^{-\beta}} \frac{|\mathrm{e}^{\imath[A_\varepsilon(\frac{x+y}{2})-A(0)]\cdot(x-y)}-1|^2}{|x-y|^{N+2s}}\mathrm{d}x \\ &= X_\varepsilon^1 + X_\varepsilon^2. \end{aligned} \tag{4.6}$$

Using $|\mathrm{e}^{\imath t}-1|^2 \leqslant 4$ and $w \in H^s(\mathbb{R}^N,\mathbb{R})$, we get

$$X_\varepsilon^1 \leqslant C\int_{\mathbb{R}^N} w^2(y)\mathrm{d}y \int_{\varepsilon^{-\beta}}^{\infty} \rho^{-1-2s}\mathrm{d}\rho \leqslant C\,\varepsilon^{2\beta s} \to 0. \tag{4.7}$$

Since $|\mathrm{e}^{\imath t}-1|^2 \leqslant t^2$ for all $t \in \mathbb{R}$, $A \in C^{0,\alpha}(\mathbb{R}^N,\mathbb{R}^N)$ with $\alpha \in (0,1]$, and $|x+y|^2 \leqslant 2(|x-y|^2+4|y|^2)$, we have

$$\begin{aligned} X_\varepsilon^2 &\leqslant \int_{\mathbb{R}^N} w^2(y)\,\mathrm{d}y \int_{|x-y|<\varepsilon^{-\beta}} \frac{|A_\varepsilon\left(\frac{x+y}{2}\right)-A(0)|^2}{|x-y|^{N+2s-2}}\mathrm{d}x \\ &\leqslant C\,\varepsilon^{2\alpha}\int_{\mathbb{R}^N} w^2(y)\,\mathrm{d}y \int_{|x-y|<\varepsilon^{-\beta}} \frac{|x+y|^{2\alpha}}{|x-y|^{N+2s-2}}\mathrm{d}x \\ &\leqslant C\,\varepsilon^{2\alpha}\left(\int_{\mathbb{R}^N} w^2(y)\,\mathrm{d}y \int_{|x-y|<\varepsilon^{-\beta}} \frac{1}{|x-y|^{N+2s-2-2\alpha}}\mathrm{d}x \right. \\ &\qquad \left. + \int_{\mathbb{R}^N} |y|^{2\alpha}w^2(y)\,\mathrm{d}y \int_{|x-y|<\varepsilon^{-\beta}} \frac{1}{|x-y|^{N+2s-2}}\mathrm{d}x\right) \\ &= C\,\varepsilon^{2\alpha}(X_\varepsilon^{2,1}+X_\varepsilon^{2,2}). \end{aligned} \tag{4.8}$$

Then

$$X_\varepsilon^{2,1} = C\int_{\mathbb{R}^N} w^2(y)\,\mathrm{d}y \int_0^{\varepsilon^{-\beta}} \rho^{1+2\alpha-2s}\,\mathrm{d}\rho \leqslant C\,\varepsilon^{-2\beta(1+\alpha-s)}. \tag{4.9}$$

On the other hand, using (4.2), we infer that

$$\begin{aligned} X_\varepsilon^{2,2} &\leqslant C\int_{\mathbb{R}^N} |y|^{2\alpha}w^2(y)\,\mathrm{d}y \int_0^{\varepsilon^{-\beta}} \rho^{1-2s}\,\mathrm{d}\rho \\ &\leqslant C\,\varepsilon^{-2\beta(1-s)}\left[\int_{B_1} w^2(y)\,\mathrm{d}y + \int_{\mathbb{R}^N\setminus B_1} \frac{1}{|y|^{2(N+2s)-2\alpha}}\,\mathrm{d}y\right] \\ &\leqslant C\,\varepsilon^{-2\beta(1-s)}. \end{aligned} \tag{4.10}$$

Taking into account (4.6), (4.7), (4.8), (4.9) and (4.10) we can conclude that $X_\varepsilon \to 0$. Therefore (4.3) holds. Now, let $t_\varepsilon > 0$ be the unique number such that

$$J_\varepsilon(t_\varepsilon w_\varepsilon) = \max_{t\geqslant 0} J_\varepsilon(t w_\varepsilon).$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Then t_ε satisfies

$$\|w_\varepsilon\|_\varepsilon^2 = \int_{\mathbb{R}^N} g_\varepsilon(x, t_\varepsilon^2|w_\varepsilon|^2)|w_\varepsilon|^2 \,\mathrm{d}x = \int_{\mathbb{R}^N} f(t_\varepsilon^2|w_\varepsilon|^2)|w_\varepsilon|^2 \,\mathrm{d}x \tag{4.11}$$

where we used $\mathrm{supp}(\eta) \subset \Lambda$ and $g = f$ on $\Lambda \times \mathbb{R}$. Let us prove that $t_\varepsilon \to 1$ as $\varepsilon \to 0$. Using $\eta = 1$ in $B_{\delta/2}$ and (f_4) we find

$$\|w_\varepsilon\|_\varepsilon^2 \geqslant f(t_\varepsilon^2 \alpha_0^2) \int_{B_{\delta/2}} |w|^2 \,\mathrm{d}x,$$

where $\alpha_0 = \min_{\overline{B}_{\delta/2}} w > 0$ (remark that w is a continuous positive function). So, if $t_\varepsilon \to \infty$ as $\varepsilon \to 0$, then we can use (f_3) to deduce that $\|w\|_0^2 = \infty$, which gives a contradiction by (4.3). On the other hand, if $t_\varepsilon \to 0$ as $\varepsilon \to 0$, we can exploit the growth assumptions on f and (4.11) to infer that $\|w\|_0^2 = 0$ which is in contrast with (4.3). In conclusion, $t_\varepsilon \to t_0 \in (0, \infty)$ as $\varepsilon \to 0$. Now, taking the limit as $\varepsilon \to 0$ in (4.11) and using (4.3), we can see that

$$\|w\|_0^2 = \int_{\mathbb{R}^N} f(t_0^2 w^2) w^2 \,\mathrm{d}x. \tag{4.12}$$

In view of $w \in \mathcal{M}_0$ and (f_4), we have that $t_0 = 1$. Then, by (4.3), $t_\varepsilon \to 1$ and applying the dominated convergence theorem, we obtain that $\lim_{\varepsilon \to 0} J_\varepsilon(t_\varepsilon w_\varepsilon) = I_0(w) = d_0$. Since $c_\varepsilon \leqslant \max_{t \geqslant 0} J_\varepsilon(t w_\varepsilon) = J_\varepsilon(t_\varepsilon w_\varepsilon)$, we can conclude that $\limsup_{\varepsilon \to 0} c_\varepsilon \leqslant d_0$. □

5. Multiplicity result for the modified problem

In this section, we make use of the Ljusternik–Schnirelman category theory to obtain multiple solutions to (3.2). In particular, we relate the number of positive solutions of (3.2) to the topology of the set M. To do this, we introduce some useful tools needed to implement the barycenter machinery below. Let $\delta > 0$ be such that

$$M_\delta = \{x \in \mathbb{R}^N : \mathrm{dist}(x, M) \leqslant \delta\} \subset \Lambda,$$

and consider a smooth nonincreasing function $\eta : [0, \infty) \to \mathbb{R}$ such that $\eta(t) = 1$ if $0 \leqslant t \leqslant \delta/2$, $\eta(t) = 0$ if $t \geqslant \delta$, $0 \leqslant \eta \leqslant 1$ and $|\eta'(t)| \leqslant c$ for some $c > 0$. For any $y \in M$, we introduce

$$\Psi_{\varepsilon,y}(x) = \eta(|\,\varepsilon\, x - y|) w\left(\frac{\varepsilon\, x - y}{\varepsilon}\right) \mathrm{e}^{\imath \tau_y((\varepsilon\, x - y)/\varepsilon)},$$

where $\tau_y(x) = \sum_{j=1}^N A_j(y) x_j$ and $w \in \mathcal{H}_0$ is a positive ground state solution to the autonomous problem (4.1) whose existence is guaranteed by theorem **4.6**. Let $t_\varepsilon > 0$ be the unique number such that

$$J_\varepsilon(t_\varepsilon \Psi_{\varepsilon,y}) = \max_{t \geqslant 0} J_\varepsilon(t \Psi_{\varepsilon,y}).$$

Finally, we consider $\Phi_\varepsilon : M \to \mathcal{N}_\varepsilon$ defined by setting

$$\Phi_\varepsilon(y) = t_\varepsilon \Psi_{\varepsilon,y}.$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

LEMMA 5.1. *The functional Φ_ε satisfies the following limit*

$$\lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = d_0 \quad \text{uniformly in } y \in M.$$

Proof. Assume by contradiction that there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|J_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - d_0| \geqslant \delta_0. \tag{5.1}$$

Applying lemma 4.1 in [13] and the dominated convergence theorem, we see that

$$\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} \to \|w\|^2_0 \in (0,\infty). \tag{5.2}$$

On the other hand, since $\langle J'_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)), \Phi_{\varepsilon_n}(y_n)\rangle = 0$ and using the change of variable $z = (\varepsilon_n x - y_n)/\varepsilon_n$, it follows that

$$t^2_{\varepsilon_n}\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} = \int_{\mathbb{R}^N} g(\varepsilon_n z + y_n, |t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)|^2)|t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)|^2 \mathrm{d}z.$$

If $z \in B_{\delta/\varepsilon_n}$, then $\varepsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Lambda$. Since $g(x,t) = f(t)$ for $(x,t) \in \Lambda \times [0,\infty)$, we have

$$t^2_{\varepsilon_n}\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} = \int_{\mathbb{R}^N} f(|t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z)|^2)(t_{\varepsilon_n}\eta(|\varepsilon_n z|)w(z))^2 \,\mathrm{d}z. \tag{5.3}$$

In view of $\eta(|x|) = 1$ for $x \in B_{\delta/2}$ and that $B_{\delta/2} \subset B_{\delta/2\,\varepsilon_n}$ for all n large enough, it follows from (5.3) and (f_4) that

$$\begin{aligned}\|\Psi_{\varepsilon_n,y_n}\|^2_{\varepsilon_n} &\geqslant \int_{B_{\delta/2}} \frac{f(|t_{\varepsilon_n}w(z)|^2)(t_{\varepsilon_n}w(z))^2}{t^2_{\varepsilon_n}}\mathrm{d}z \\ &\geqslant f(|t_{\varepsilon_n}w(\hat{z})|^2)\int_{B_{\delta/2}} w^2(z)\,\mathrm{d}z,\end{aligned} \tag{5.4}$$

where

$$w(\hat{z}) = \min_{z\in\overline{B}_{\delta/2}} w(z) > 0.$$

Now, assume by contradiction that $t_{\varepsilon_n} \to \infty$. This fact, (5.2) and (5.4) yield

$$\|w\|^2_0 = \infty,$$

that is a contradiction. Hence, (t_{ε_n}) is bounded and, up to subsequence, we may assume that $t_{\varepsilon_n} \to t_0$ for some $t_0 \geqslant 0$. In particular, $t_0 > 0$. In fact, if $t_0 = 0$, we

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

see that (3.4) and (5.3) imply that

$$r \leqslant \int_{\mathbb{R}^3} f(|t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z)|^2)(t_{\varepsilon_n}\eta(|\,\varepsilon_n\, z|)w(z))^2\,\mathrm{d}z.$$

Using (f_1), (f_2), (5.2) and the above inequality, we deduce that $t_0 > 0$. Thus, letting $n \to \infty$ in (5.3), we have that

$$\|w\|_0^2 = \int_{\mathbb{R}^N} f((t_0 w)^2)w^2\,\mathrm{d}x.$$

Bearing in mind that $w \in \mathcal{M}_0$ and using (f_4), we infer that $t_0 = 1$. Passing to the limit as $n \to \infty$ and using $t_{\varepsilon_n} \to 1$ we conclude that

$$\lim_{n\to\infty} J_{\varepsilon_n}(\Phi_{\varepsilon_n,y_n}) = I_0(w) = d_0,$$

which is in contrast with (5.1). □

Let us fix $\rho = \rho(\delta) > 0$ satisfying $M_\delta \subset B_\rho$, and we consider $\Upsilon : \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\Upsilon(x) = \begin{cases} x & \text{if } |x| < \rho, \\ \dfrac{\rho x}{|x|} & \text{if } |x| \geqslant \rho. \end{cases}$$

Then we define the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \to \mathbb{R}^N$ as follows:

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \Upsilon(\varepsilon\, x)|u(x)|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N} |u(x)|^2\,\mathrm{d}x}.$$

Lemma 5.2. *The following limit holds true:*

$$\lim_{\varepsilon\to 0} \beta_\varepsilon(\Phi_\varepsilon(y)) = y \quad \textit{uniformly in } y \in M.$$

Proof. Suppose by contradiction that there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\varepsilon_n \to 0$ such that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| \geqslant \delta_0. \tag{5.5}$$

Using the definitions of $\Phi_{\varepsilon_n}(y_n)$, β_{ε_n}, ψ and the change of variable $z = (\varepsilon_n(x - y)_n)/\varepsilon_n$, we can see that

$$\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varepsilon_n\, z + y_n) - y_n]|\eta(|\,\varepsilon_n\, z|)w(z)|^2\,\mathrm{d}z}{\int_{\mathbb{R}^N} |\eta(|\,\varepsilon_n\, z|)w(z)|^2\,\mathrm{d}z}.$$

Since $(y_n) \subset M \subset B_\rho$, it follows from the dominated convergence theorem that

$$|\beta_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) - y_n| = o_n(1)$$

which is in contrast with (5.5). □

The next compactness result will play a fundamental role to prove that the solutions of (3.2) are also solution to (3.1).

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

LEMMA 5.3. *Let $\varepsilon_n \to 0$ and $(u_n) = (u_{\varepsilon_n}) \subset \mathcal{N}_{\varepsilon_n}$ be such that $J_{\varepsilon_n}(u_n) \to d_0$. Then there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) = |u_n|(x + \tilde{y}_n)$ has a convergent subsequence in $\mathcal{H}_0$. Moreover, up to a subsequence, $y_n = \varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$.*

Proof. Taking into account $\langle J'_{\varepsilon_n}(u_n), u_n\rangle = 0$, $J_{\varepsilon_n}(u_n) \to d_0$, it is easy to see that (u_n) is bounded in $\mathcal{H}_{\varepsilon_n}$. Then, there exists $C > 0$ (independent of n) such that $\|u_n\|_{\varepsilon_n} \leqslant C$ for all $n \in \mathbb{N}$. Moreover, from lemma 2.2, we also know that $(|u_n|)$ is bounded in $H^s(\mathbb{R}^N, \mathbb{R})$.

Now we prove that there exist a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants $R, \gamma > 0$ such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} |u_n|^2 \,\mathrm{d}x \geqslant \gamma > 0. \tag{5.6}$$

If by contradiction (5.6) does not hold, then for all $R > 0$ we get

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \,\mathrm{d}x = 0.$$

From the boundedness of $(|u_n|)$ in $H^s(\mathbb{R}^N, \mathbb{R})$ and lemma 2.5, we can see that $|u_n| \to 0$ in $L^q(\mathbb{R}^N, \mathbb{R})$ for any $q \in (2, 2^*_s)$. This fact together with (3.5) and the boundedness of $(|u_n|)$ in $L^2(\mathbb{R}^N, \mathbb{R})$ yields that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \,\mathrm{d}x = 0 = \lim_{n\to\infty} \int_{\mathbb{R}^N} G_{\varepsilon_n}(x, |u_n|^2) \,\mathrm{d}x. \tag{5.7}$$

Taking into account $\langle J'_{\varepsilon_n}(u_n), u_n\rangle = 0$ and (5.7), we can infer that $\|u_n\|_{\varepsilon_n} \to 0$ as $n \to \infty$ and this implies that $J_{\varepsilon_n}(u_n) \to 0$ which is in contrast with $d_0 > 0$. Now, we set $v_n(x) = |u_n|(x + \tilde{y}_n)$. Then (v_n) is bounded in $H^s(\mathbb{R}^N, \mathbb{R})$, and we may assume that $v_n \rightharpoonup v \not\equiv 0$ in $H^s(\mathbb{R}^N, \mathbb{R})$ as $n \to \infty$. Fix $t_n > 0$ such that $\tilde{v}_n = t_n v_n \in \mathcal{M}_0$. By lemma 2.2, $(u_n) \subset \mathcal{N}_{\varepsilon_n}$ and $J_{\varepsilon_n}(u_n) \to d_0$, we can see that

$$d_0 \leqslant I_0(\tilde{v}_n) \leqslant \max_{t\geqslant 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) = d_0 + o_n(1)$$

which implies that $I_0(\tilde{v}_n) \to d_0$. In particular, $(\tilde{v}_n)$ is bounded in $H^s(\mathbb{R}^N, \mathbb{R})$ and $\tilde{v}_n \rightharpoonup \tilde{v}$ in $H^s(\mathbb{R}^N, \mathbb{R})$. Since $v_n \not\to 0$ in $H^s(\mathbb{R}^N, \mathbb{R})$ and $(\tilde{v}_n)$ is bounded in $H^s(\mathbb{R}^N, \mathbb{R})$, we deduce that (t_n) is bounded in $\mathbb{R}$ and, up to a subsequence, we can assume that $t_n \to t^* \geqslant 0$. Indeed $t^* > 0$ due to the boundedness of (v_n) in $H^s(\mathbb{R}^N, \mathbb{R})$, and the fact that $\tilde{v}_n \not\to 0$ in $H^s(\mathbb{R}^N, \mathbb{R})$ (otherwise, by the continuity of I_0, $I_0(\tilde{v}_n) \to 0$ which is impossible because $d_0 > 0$). Then, from the uniqueness of the weak limit, we have that $\tilde{v}_n \rightharpoonup \tilde{v} = t^* v \not\equiv 0$ in $H^s(\mathbb{R}^N, \mathbb{R})$. This fact combined with lemma 4.7 yields

$$\tilde{v}_n \to \tilde{v} \quad \text{ in } H^s(\mathbb{R}^N, \mathbb{R}). \tag{5.8}$$

Consequently, $v_n \to v$ in $H^s(\mathbb{R}^N, \mathbb{R})$ as $n \to \infty$. Moreover,

$$I_0(\tilde{v}) = 0 \text{ and } \langle I'_0(\tilde{v}), \tilde{v}\rangle = 0. \tag{5.9}$$

Now, we put $y_n = \varepsilon_n \tilde{y}_n$ and we claim that (y_n) admits a subsequence, still denoted by (y_n), such that $y_n \to y_0$ for some $y_0 \in M$. First, we prove that (y_n) is bounded

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

in $\mathbb{R}^N$. Assume by contradiction that, up to a subsequence, $|y_n| \to \infty$ as $n \to \infty$. Take $R > 0$ such that $\Lambda \subset B_R$. Since we may suppose that $|y_n| > 2R$ for n large, we have that for any $z \in B_{R/\varepsilon_n}$

$$|\varepsilon_n z + y_n| \geqslant |y_n| - |\varepsilon_n z| > R.$$

Hence, using $(u_n) \subset \mathcal{N}_{\varepsilon_n}$, (V_1), lemma 2.2 and the change of variable $x \mapsto z + \tilde{y}_n$, we obtain that

$$\begin{aligned}[v_n]^2 + \int_{\mathbb{R}^N} V_0 v_n^2 \,\mathrm{d}x &\leqslant \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, |v_n|^2)|v_n|^2 \,\mathrm{d}x \\ &\leqslant \int_{B_{R/\varepsilon_n}} \tilde{f}(|v_n|^2)|v_n|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N \setminus B_{R/\varepsilon_n}} f(|v_n|^2)|v_n|^2 \,\mathrm{d}x. \end{aligned} \tag{5.10}$$

Recalling that $v_n \to v$ in $H^s(\mathbb{R}^N, \mathbb{R})$ as $n \to \infty$ and that $\tilde{f}(t) \leqslant V_0/k$, we can see that (5.10) and the dominated convergence theorem yield

$$\min\left\{1, V_0\left(1 - \frac{1}{k}\right)\right\}\left([v_n]^2 + \int_{\mathbb{R}^N} |v_n|^2 \,\mathrm{d}x\right) = o_n(1),$$

that is $v_n \to 0$ in $H^s(\mathbb{R}^N, \mathbb{R})$, which gives a contradiction. Therefore, (y_n) is bounded and we may assume that $y_n \to y_0 \in \mathbb{R}^N$. If $y_0 \notin \overline{\Lambda}$, then we can argue as before to infer that $v_n \to 0$ in $H^s(\mathbb{R}^N, \mathbb{R})$, which is impossible. Hence $y_0 \in \overline{\Lambda}$. Let us note that if $V(y_0) = V_0$, then we can infer that $y_0 \notin \partial\Lambda$ in view of (V_2). Therefore, it is enough to verify that $V(y_0) = V_0$ to deduce that $y_0 \in M$. Suppose by contradiction that $V(y_0) > V_0$. Then, using (5.8), Fatou's lemma, the invariance of $\mathbb{R}^N$ by translations and lemma 2.2, we get

$$\begin{aligned} d_0 = I_0(\tilde{v}) &< \frac{1}{2}[\tilde{v}]^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(y_0)\tilde{v}^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} F(|\tilde{v}|^2)\,\mathrm{d}x \\ &\leqslant \liminf_{n\to\infty}\left[\frac{1}{2}[\tilde{v}_n]^2 + \frac{1}{2}\int_{\mathbb{R}^N} V(\varepsilon_n x + y_n)|\tilde{v}_n|^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} F(|\tilde{v}_n|^2)\,\mathrm{d}x\right] \\ &\leqslant \liminf_{n\to\infty}\left[\frac{t_n^2}{2}[|u_n|]^2 + \frac{t_n^2}{2}\int_{\mathbb{R}^N} V(\varepsilon_n z)|u_n|^2 \,\mathrm{d}z - \frac{1}{2}\int_{\mathbb{R}^N} F(|t_n u_n|^2)\,\mathrm{d}z\right] \\ &\leqslant \liminf_{n\to\infty} J_{\varepsilon_n}(t_n u_n) \leqslant \liminf_{n\to\infty} J_{\varepsilon_n}(u_n) \leqslant d_0 \end{aligned}$$

which is impossible. This ends the proof of the lemma. □

Now, we consider the following subset of $\mathcal{N}_\varepsilon$

$$\widetilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : J_\varepsilon(u) \leqslant d_0 + h(\varepsilon)\},$$

where $h(\varepsilon) = \sup_{y\in M} |J_\varepsilon(\Phi_\varepsilon(y)) - d_0| \to 0$ as $\varepsilon \to 0$ as a consequence of lemma 5.1. By the definition of $h(\varepsilon)$, we know that, for all $y \in M$ and $\varepsilon > 0$, $\Phi_\varepsilon(y) \in \widetilde{\mathcal{N}}_\varepsilon$ and $\widetilde{\mathcal{N}}_\varepsilon \neq \emptyset$. We present below an interesting relation between $\widetilde{\mathcal{N}}_\varepsilon$ and the barycenter map β_ε.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

LEMMA 5.4. *For any $\delta > 0$, there holds that*

$$\lim_{\varepsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\varepsilon} dist(\beta_\varepsilon(u), M_\delta) = 0.$$

Proof. Let $(\varepsilon_n) \subset (0,\infty)$ such that $\varepsilon_n \to 0$. Then there exists $(u_n) \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that

$$\sup_{u\in\widetilde{\mathcal{N}}_{\varepsilon_n}} \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u) - y| = \inf_{y\in M_\delta} |\beta_{\varepsilon_n}(u_n) - y| + o_n(1).$$

Hence, it is enough to find a sequence $(y_n) \subset M_\delta$ such that

$$\lim_{n\to\infty} |\beta_{\varepsilon_n}(u_n) - y_n| = 0. \tag{5.11}$$

By lemma 2.2 we know that $I_0(t|u_n|) \leqslant J(tu_n)$ for all $t \geqslant 0$. Therefore, recalling that $(u_n) \subset \widetilde{\mathcal{N}}_{\varepsilon_n} \subset \mathcal{N}_{\varepsilon_n}$, we deduce that

$$d_0 \leqslant \max_{t\geqslant 0} I_0(t|u_n|) \leqslant \max_{t\geqslant 0} J_{\varepsilon_n}(tu_n) = J_{\varepsilon_n}(u_n) \leqslant d_0 + h(\varepsilon_n)$$

which leads to $J_{\varepsilon_n}(u_n) \to d_0$. By invoking lemma 5.3, we can find $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $y_n = \varepsilon_n \tilde{y}_n \in M_\delta$ for n sufficiently large. Consequently,

$$\beta_{\varepsilon_n}(u_n) = y_n + \frac{\int_{\mathbb{R}^N} [\Upsilon(\varepsilon_n z + y_n) - y_n]|u_n(z+\tilde{y}_n)|^2\,\mathrm{d}z}{\int_{\mathbb{R}^N} |u_n(z+\tilde{y}_n)|^2\,\mathrm{d}z}.$$

Taking into account that $|u_n|(\cdot + \tilde{y}_n)$ strongly converges in $\mathcal{H}_0$ and that $\varepsilon_n z + y_n \to y_0 \in M_\delta$, we find $\beta_{\varepsilon_n}(u_n) = y_n + o_n(1)$, that is (5.11) is satisfied. □

We end this section by proving a multiplicity result for (3.2). Since $\mathbb{S}^+_\varepsilon$ is not a completed metric space, we cannot use directly an abstract result as in [**8**, **10**, **13**]. Instead, we invoke the abstract category result in [**47**].

THEOREM 5.5. *For any $\delta > 0$ such that $M_\delta \subset \Lambda$, there exists $\tilde{\varepsilon}_\delta > 0$ such that, for any $\varepsilon \in (0, \tilde{\varepsilon}_\delta)$, problem (3.2) has at least $cat_{M_\delta}(M)$ nontrivial solutions.*

Proof. For any $\varepsilon > 0$, we consider the map $\alpha_\varepsilon : M \to \mathbb{S}^+_\varepsilon$ defined as $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$.

Using lemma 5.1, we see that

$$\lim_{\varepsilon\to 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon\to 0} J_\varepsilon(\Phi_\varepsilon(y)) = d_0 \quad \text{uniformly in } y \in M. \tag{5.12}$$

Set

$$\widetilde{\mathcal{S}}^+_\varepsilon = \{w \in \mathbb{S}^+_\varepsilon : \psi_\varepsilon(w) \leqslant d_0 + h(\varepsilon)\},$$

where $h(\varepsilon) = \sup_{y\in\Lambda} |\psi_\varepsilon(\alpha_\varepsilon(y)) - d_0|$. It follows from (5.12) that $h(\varepsilon) \to 0$ as $\varepsilon \to 0$. Moreover, $\alpha_\varepsilon(y) \in \widetilde{\mathcal{S}}^+_\varepsilon$ for all $y \in M$ and this shows that $\widetilde{\mathcal{S}}^+_\varepsilon \neq \emptyset$ for all $\varepsilon > 0$.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

In light of lemmas **5.1**, **3.2**-(iii), **5.2** and **5.4**, we can find $\bar{\varepsilon} = \bar{\varepsilon}_\delta > 0$ such that the following diagram

$$M \overset{\Phi_\varepsilon}{\to} \Phi_\varepsilon(M) \overset{m_\varepsilon^{-1}}{\to} \alpha_\varepsilon(M) \overset{m_\varepsilon}{\to} \Phi_\varepsilon(M) \overset{\beta_\varepsilon}{\to} M_\delta$$

is well defined for any $\varepsilon \in (0, \bar{\varepsilon})$. Thanks to lemma 5.2, and decreasing $\bar{\varepsilon}$ if necessary, we see that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \theta(\varepsilon, y)$ for all $y \in M$, for some function $\theta(\varepsilon, y)$ satisfying $|\theta(\varepsilon, y)| < \delta/2$ uniformly in $y \in M$ and for all $\varepsilon \in (0, \bar{\varepsilon})$. Define $H(t, y) = y + (1 - t)\theta(\varepsilon, y)$. Then $H : [0, 1] \times M \to M_\delta$ is continuous. Clearly, $H(0, y) = \beta_\varepsilon(\Phi_\varepsilon(y))$ and $H(1, y) = y$ for all $y \in M$. Consequently, $H(t, y)$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ (m_\varepsilon^{-1} \circ \Phi_\varepsilon)$ and the inclusion map $id : M \to M_\delta$. This fact yields

$$cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M) \geqslant cat_{M_\delta}(M). \tag{5.13}$$

Applying corollary 3.6, lemma 4.8 and theorem 27 in [**47**] with $c = c_\varepsilon \leqslant d_0 + h(\varepsilon) = d$ and $K = \alpha_\varepsilon(M)$, we obtain that ψ_ε has at least $cat_{\alpha_\varepsilon(M)}\alpha_\varepsilon(M)$ critical points on $\widetilde{\mathcal{S}}_\varepsilon^+$. Taking into account proposition 3.3-(d) and (5.13), we infer that J_ε admits at least $cat_{M_\delta}(M)$ critical points in $\widetilde{\mathcal{N}}_\varepsilon$. □

6. Proof of theorem 1.1

This last section is devoted to the proof of the main result of this paper. In order to show that the solutions of (3.2) are indeed solutions to (3.1) for $\varepsilon > 0$ small, we need to verify that $|u_\varepsilon| \leqslant \sqrt{a}$ in Λ_ε^c holds true provided that $\varepsilon > 0$ is sufficiently small. We start by proving a fractional Kato's inequality in the spirit of [**36**] for the solutions of fractional magnetic problems.

THEOREM 6.1. *Let $u \in \mathcal{D}_A^{s,2}(\mathbb{R}^N, \mathbb{C})$ and $f \in L^1_{loc}(\mathbb{R}^N, \mathbb{C})$ be such that*

$$\Re\left(\iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{N+2s}} \overline{(\psi(x) - \psi(y)e^{\imath A(\frac{x+y}{2})\cdot(x-y)})}\, dx\, dy\right)$$
$$= \Re\left(\int_{\mathbb{R}^N} f\bar{\psi}\, dx\right) \tag{6.1}$$

for all $\psi : \mathbb{R}^N \to \mathbb{C}$ measurable with compact support and such that $[\psi]_A < \infty$.

Then it holds $(-\Delta)^s|u| \leqslant \Re(sign(\bar{u})f)$ in the distributional sense, that is

$$\iint_{\mathbb{R}^{2N}} \frac{(|u(x)| - |u(y)|)(\varphi(x) - \varphi(y))}{|x-y|^{N+2s}}\, dx\, dy \leqslant \Re\left(\int_{\mathbb{R}^N} sign(\bar{u})f\varphi\, dx\right) \tag{6.2}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\varphi \geqslant 0$, where

$$sign(\bar{u})(x) = \begin{cases} \dfrac{\overline{u(x)}}{|u(x)|} & \text{if } u(x) \neq 0, \\ 0 & \text{if } u(x) = 0. \end{cases}$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Proof. Take $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\varphi \geqslant 0$, and for all $\delta > 0$ we consider

$$\omega_\delta(x) = \frac{u(x)}{\sqrt{|u(x)|^2 + \delta^2}}\varphi(x) = \frac{u(x)}{u_\delta(x)}\varphi(x)$$

as test function in (6.1). First, we show that ω_δ is admissible. It is clear that ω_δ has compact support. On the other hand, we can observe

$$\begin{aligned}
\omega_\delta(x) - \omega_\delta(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)} &= \left(\frac{u(x)}{u_\delta(x)}\right)\varphi(x) - \left(\frac{u(y)}{u_\delta(y)}\right)\varphi(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)} \\
&= \left[u(x) - u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}\right]\frac{\varphi(x)}{u_\delta(x)} \\
&\quad + \left[\frac{\varphi(x)}{u_\delta(x)} - \frac{\varphi(y)}{u_\delta(y)}\right]u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)} \\
&= \left[u(x) - u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}\right]\frac{\varphi(x)}{u_\delta(x)} \\
&\quad + \left[\frac{1}{u_\delta(x)} - \frac{1}{u_\delta(y)}\right]\varphi(x)u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)} \\
&\quad + [\varphi(x) - \varphi(y)]\frac{u(y)}{u_\delta(y)}\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}
\end{aligned}$$

which implies that

$$\begin{aligned}
&|\omega_\delta(x) - \omega_\delta(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}|^2 \\
&\leqslant \frac{4}{\delta^2}|u(x) - u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}|^2\|\varphi\|^2_{L^\infty(\mathbb{R}^N)} \\
&\quad + 4\left|\frac{u(y)}{u_\delta(y)}\right|^2\frac{1}{|u_\delta(x)|^2}\|\varphi\|^2_{L^\infty(\mathbb{R}^N)}|u_\delta(y) - u_\delta(x)|^2 + 4|\varphi(x) - \varphi(y)|^2 \\
&\leqslant \frac{4}{\delta^2}|u(x) - u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}|^2\|\varphi\|^2_{L^\infty(\mathbb{R}^N)} + \frac{4}{\delta^2}||u(x)| - |u(y)||^2\|\varphi\|^2_{L^\infty(\mathbb{R}^N)} \\
&\quad + 4|\varphi(x) - \varphi(y)|^2,
\end{aligned}$$

where we used the following elementary inequalities

$$\begin{aligned}
&|z + w + k|^2 \leqslant 4(|z|^2 + |w|^2 + |k|^2) \quad \text{for all } z, w, k \in \mathbb{C}, \\
&|\sqrt{|z|^2 + \delta^2} - \sqrt{|w|^2 + \delta^2}| \leqslant ||z| - |w|| \quad \text{for all } z, w \in \mathbb{C},
\end{aligned}$$

and that $|\mathrm{e}^{\imath t}| = 1$ for all $t \in \mathbb{R}$, $u_\delta \geqslant \delta$, $|u/u_\delta| \leqslant 1$. Since $u \in \mathcal{D}^{s,2}_A(\mathbb{R}^N, \mathbb{C})$, $|u| \in \mathcal{D}^{s,2}(\mathbb{R}^N, \mathbb{R})$ (by lemma 2.2) and $\varphi \in C_c^\infty(\mathbb{R}^N, \mathbb{R})$, we deduce that $[\omega_\delta]_A < \infty$. Then

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

we have

$$\begin{aligned}
&\Re\left[\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{N+2s}}\right.\\
&\qquad\left.\times\left(\overline{\frac{u(x)}{u_\delta(x)}}\varphi(x)-\overline{\frac{u(y)}{u_\delta(y)}}\varphi(y)\mathrm{e}^{-\imath A(\frac{x+y}{2})\cdot(x-y)}\right)\mathrm{d}x\,\mathrm{d}y\right]\\
&\qquad=\Re\left(\int_{\mathbb{R}^N} f\frac{\overline{u}}{u_\delta}\varphi\,\mathrm{d}x\right). \qquad (6.3)
\end{aligned}$$

Now, using $\Re(z)\leqslant|z|$ for all $z\in\mathbb{C}$ and $|\mathrm{e}^{\imath t}|=1$ for all $t\in\mathbb{R}$, we see that

$$\begin{aligned}
&\Re\left[(u(x)-u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)})\left(\overline{\frac{u(x)}{u_\delta(x)}}\varphi(x)-\overline{\frac{u(y)}{u_\delta(y)}}\varphi(y)\mathrm{e}^{-\imath A(\frac{x+y}{2})\cdot(x-y)}\right)\right]\\
&=\Re\left[\frac{|u(x)|^2}{u_\delta(x)}\varphi(x)+\frac{|u(y)|^2}{u_\delta(y)}\varphi(y)-\frac{u(x)\overline{u(y)}}{u_\delta(y)}\varphi(y)\mathrm{e}^{-\imath A(\frac{x+y}{2})\cdot(x-y)}\right.\\
&\quad\left.-\frac{u(y)\overline{u(x)}}{u_\delta(x)}\varphi(x)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)}\right]\\
&\geqslant\left[\frac{|u(x)|^2}{u_\delta(x)}\varphi(x)+\frac{|u(y)|^2}{u_\delta(y)}\varphi(y)-|u(x)|\frac{|u(y)|}{u_\delta(y)}\varphi(y)-|u(y)|\frac{|u(x)|}{u_\delta(x)}\varphi(x)\right]. \qquad (6.4)
\end{aligned}$$

Let us note that

$$\begin{aligned}
&\frac{|u(x)|^2}{u_\delta(x)}\varphi(x)+\frac{|u(y)|^2}{u_\delta(y)}\varphi(y)-|u(x)|\frac{|u(y)|}{u_\delta(y)}\varphi(y)-|u(y)|\frac{|u(x)|}{u_\delta(x)}\varphi(x)\\
&\quad=\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)\varphi(x)-\frac{|u(y)|}{u_\delta(y)}(|u(x)|-|u(y)|)\varphi(y)\\
&\quad=\left[\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)\varphi(x)-\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)\varphi(y)\right]\\
&\qquad+\left(\frac{|u(x)|}{u_\delta(x)}-\frac{|u(y)|}{u_\delta(y)}\right)(|u(x)|-|u(y)|)\varphi(y)\\
&\quad=\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))\\
&\qquad+\left(\frac{|u(x)|}{u_\delta(x)}-\frac{|u(y)|}{u_\delta(y)}\right)(|u(x)|-|u(y)|)\varphi(y)\\
&\quad\geqslant\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y)), \qquad (6.5)
\end{aligned}$$

where in the last inequality we used the fact that

$$\left(\frac{|u(x)|}{u_\delta(x)}-\frac{|u(y)|}{u_\delta(y)}\right)(|u(x)|-|u(y)|)\varphi(y)\geqslant 0$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

because

$$h(t) = \frac{t}{\sqrt{t^2+\delta^2}} \text{ is increasing for } t \geqslant 0 \quad \text{and} \quad \varphi \geqslant 0 \text{ in } \mathbb{R}^N.$$

Since

$$\frac{\left|\frac{|u(x)|}{u_\delta(x)}(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))\right|}{|x-y|^{N+2s}} \leqslant \frac{||u(x)|-|u(y)||}{|x-y|^{(N+2s)/2}} \frac{|\varphi(x)-\varphi(y)|}{|x-y|^{(N+2s)/2}} \in L^1(\mathbb{R}^{2N},\mathbb{R}),$$

and $|u(x)|/u_\delta(x) \to 1$ a.e. in $\mathbb{R}^N$ as $\delta \to 0$, we can use (6.4), (6.5) and the dominated convergence theorem to deduce that

$$\begin{aligned}
&\liminf_{\delta\to 0} \Re\left[\iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y)\mathrm{e}^{\imath A(\frac{x+y}{2})\cdot(x-y)})}{|x-y|^{N+2s}}\right.\\
&\qquad\left.\times\left(\frac{\overline{u(x)}}{u_\delta(x)}\varphi(x)-\frac{\overline{u_n(y)}}{u_\delta(y)}\varphi(y)\mathrm{e}^{-\imath A(\frac{x+y}{2})\cdot(x-y)}\right)\mathrm{d}x\mathrm{d}y\right]\\
&\geqslant \liminf_{\delta\to 0}\iint_{\mathbb{R}^{2N}}\frac{|u(x)|}{u_\delta(x)}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\mathrm{d}y\\
&= \iint_{\mathbb{R}^{2N}}\frac{(|u(x)|-|u(y)|)(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \qquad (6.6)
\end{aligned}$$

On the other hand, observing that $|f(\overline{u}/u_\delta)\varphi| \leqslant |f\varphi| \in L^1(\mathbb{R}^N,\mathbb{R})$ and $f(\overline{u}/u_\delta)\varphi \to f\,\mathrm{sign}(\overline{u})\varphi$ a.e. in $\mathbb{R}^N$ as $\delta \to 0$, we can invoke the dominated convergence theorem to infer that as $\delta \to 0$

$$\Re\left(\int_{\mathbb{R}^N} f\frac{\overline{u}}{u_\delta}\varphi\,\mathrm{d}x\right) \to \Re\left(\int_{\mathbb{R}^N} f\,\mathrm{sign}(\overline{u})\varphi\,\mathrm{d}x\right). \qquad (6.7)$$

Putting together (6.3), (6.6) and (6.7), we see that (6.2) holds true. □

Remark 6.2. A pointwise Kato's inequality for $(-\Delta)_A^s$ is proved in [**7**]. In [**34**] the authors established a Kato's inequality for the fractional magnetic operator $((-\imath\nabla - A(x))^2 + m^2)^{\alpha/2}$ with $\alpha \in (0,1]$ and $m > 0$, or $\alpha = 1$ and $m = 0$, borrowing some arguments used in [**36**]. As observed in [**21**], when $\alpha = 1$ and $m = 0$, this operator coincides with $(-\Delta)_A^{1/2}$.

Now we prove the following crucial result.

Lemma 6.3. *Let $\varepsilon_n \to 0$ and $(u_n) \subset \widetilde{\mathcal{N}}_{\varepsilon_n}$ be a sequence of solutions to (3.2). Then, $v_n = |u_n|(\cdot + \tilde{y}_n)$ satisfies $v_n \in L^\infty(\mathbb{R}^N,\mathbb{R})$ and there exists $C > 0$ such that*

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C \quad \text{for all } n \in \mathbb{N},$$

where $(\tilde{y}_n) \subset \mathbb{R}^N$ is given by lemma 5.3. Moreover,

$$\lim_{|x|\to\infty} v_n(x) = 0 \quad \text{uniformly in } n \in \mathbb{N}.$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Proof. Since $J_{\varepsilon_n}(u_n) \leqslant d_0 + h(\varepsilon_n)$ with $h(\varepsilon_n) \to 0$ as $n \to \infty$, we can argue as at the beginning of the proof of lemma 5.3 to deduce that $J_{\varepsilon_n}(u_n) \to d_0$. Thus we may invoke lemma 5.3 to obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $\varepsilon_n \tilde{y}_n \to y_0 \in M$ and $v_n = |u_n|(\cdot + \tilde{y}_n)$ strongly converges in $\mathcal{H}_0$. Let $\tilde{u}_n(x) = u_n(\cdot + \tilde{y}_n)$ and note that it solves

$$(-\Delta)^s_{\tilde{A}_n} \tilde{u}_n + \tilde{V}_n(x)\tilde{u}_n = \tilde{g}_n(x, v_n^2)\tilde{u}_n \quad \text{in } \mathbb{R}^N, \tag{6.8}$$

where

$$\tilde{A}_n(x) = A_{\varepsilon_n}(x + \tilde{y}_n),$$

$$\tilde{V}_n(x) = V_{\varepsilon_n}(x + \tilde{y}_n),$$

and

$$\tilde{g}_n(x, v_n^2) = g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n^2(x)).$$

Using theorem 6.1, we deduce that v_n satisfies (in the distributional sense)

$$(-\Delta)^s v_n + \tilde{V}_n(x)v_n \leqslant \tilde{g}_n(x, v_n^2)v_n = h_n \quad \text{in } \mathbb{R}^N. \tag{6.9}$$

By performing a Moser iteration argument [**40**] as in lemma 5.1 in [**6**], we obtain that

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \leqslant K \quad \text{for all } n \in \mathbb{N}. \tag{6.10}$$

Moreover, by interpolation, $v_n \to v$ strongly converges in $L^r(\mathbb{R}^N, \mathbb{R})$ for all $r \in [2, \infty)$. In view of the growth assumptions on g, we can also see that $h_n \to h = f(v^2)v$ in $L^r(\mathbb{R}^N, \mathbb{R})$ for all $r \in [2, \infty)$, and $\|h_n\|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$. Note that, by (V_1), we have

$$(-\Delta)^s v_n + V_0 v_n \leqslant h_n \quad \text{in } \mathbb{R}^N. \tag{6.11}$$

Let us denote by $z_n \in H^s(\mathbb{R}^N, \mathbb{R})$ the unique solution to

$$(-\Delta)^s z_n + V_0 z_n = h_n \quad \text{in } \mathbb{R}^N. \tag{6.12}$$

Since v_n satisfies (6.11) and z_n solves (6.12), by comparison we see that $0 \leqslant v_n \leqslant z_n$ in $\mathbb{R}^N$ for all $n \in \mathbb{N}$. Next we show that

$$z_n(x) \to 0 \text{ as } |x| \to \infty \text{ uniformly in } n \in \mathbb{N}. \tag{6.13}$$

Note that $z_n(x) = (\mathcal{K} * h_n)(x)$, where the kernel $\mathcal{K}(x) = \mathcal{F}^{-1}((|k|^{2s} + V_0)^{-1})$ satisfies the following properties (see [**30**]):

- (b_1) $\mathcal{K}$ is positive, radially symmetric and smooth in $\mathbb{R}^N \setminus \{0\}$;
- (b_2) there exists a positive constant $C > 0$ such that $\mathcal{K}(x) \leqslant K_1/|x|^{N+2s}$ for any $x \in \mathbb{R}^N \setminus \{0\}$;
- (b_3) $\mathcal{K} \in L^\nu(\mathbb{R}^N)$ for any $\nu \in [1, (N/(N-2s))$.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Now we borrow some arguments used in [3] to prove that (6.13) holds true. Fix $\delta > 0$ and we observe that

$$0 \leqslant z_n(x) = (\mathcal{K} * h_n)(x) = \int_{B^c_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y + \int_{B_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y. \tag{6.14}$$

From (b_2) we deduce that

$$\begin{aligned}\int_{B^c_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y &\leqslant K_1\|h_n\|_{L^\infty(\mathbb{R}^N)} \int_{B^c_{1/\delta}(x)} \frac{\mathrm{d}y}{|x-y|^{N+2s}} \\ &\leqslant c_1\delta^{2s}K_1 \int_{|\xi|\geqslant 1} \frac{\mathrm{d}\xi}{|\xi|^{N+2s}} = C_1\delta^{2s}. \end{aligned} \tag{6.15}$$

On the other hand,

$$\begin{aligned}\left|\int_{B_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y\right| &\leqslant \int_{B_{1/\delta}(x)} \mathcal{K}(x-y)|h_n(y)-h(y)|\,\mathrm{d}y \\ &\quad + \int_{B_{1/\delta}(x)} \mathcal{K}(x-y)|h(y)|\,\mathrm{d}y.\end{aligned}$$

Fix

$$\nu \in \left(1, \min\left\{\frac{N}{N-2s}, 2\right\}\right).$$

Note that, if $N \geqslant 4s$ then $\nu \in (1, (N/(N-2s))$ and we have $\nu' = (\nu/(\nu-1)) > (N/2s) \geqslant 2$, and that when $2s < N < 4s$ then $2 < (N/(N-2s))$, $\nu \in (1,2)$ and $\nu' \in (2,\infty)$. In any case, $\nu \in (1, (N/(N-2s))$ and $\nu' \in (2,\infty)$. Then, by (b_3), we deduce that $\mathcal{K} \in L^\nu(\mathbb{R}^N)$. Using Hölder's inequality we have

$$\begin{aligned}\left|\int_{B_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y\right| &\leqslant \|\mathcal{K}\|_{L^\nu(\mathbb{R}^N)}\|h_n - h\|_{L^{\nu'}(\mathbb{R}^N)} \\ &\quad + \|\mathcal{K}\|_{L^\nu(\mathbb{R}^N)}\|h\|_{L^{\nu'}(B_{1/\delta}(x))}.\end{aligned}$$

Since $\|h_n - h\|_{L^{\nu'}(\mathbb{R}^N)} \to 0$ as $n \to \infty$ and $\|h\|_{L^{\nu'}(B_{1/\delta}(x))} \to 0$ as $|x| \to \infty$, we see that there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$\int_{B_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y \leqslant C_2\delta \quad \text{for all } |x| \geqslant R,\ n \geqslant n_0. \tag{6.16}$$

Putting together (6.14), (6.15) and (6.16), we obtain that

$$z_n(x) = \int_{\mathbb{R}^N} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y \leqslant C_1\delta^{2s} + C_2\delta \quad \text{for all } |x| \geqslant R,\ n \geqslant n_0. \tag{6.17}$$

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Now, for each $n \in \{1, \dots, n_0 - 1\}$, there is $R_n > 0$ such that $\|h_n\|_{L^{\nu'}(B_{1/\delta}(x))} < \delta$ for $|x| \geqslant R_n$. Then, for $|x| \geqslant R_n$, we have

$$\begin{aligned}\int_{\mathbb{R}^N} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y &\leqslant C_1\delta^{2s} + \int_{B_{1/\delta}(x)} \mathcal{K}(x-y)h_n(y)\,\mathrm{d}y \\ &\leqslant C_1\delta^{2s} + \|\mathcal{K}\|_{L^{\nu}(\mathbb{R}^N)}\|h_n\|_{L^{\nu'}(B_{1/\delta}(x))} \\ &\leqslant C_1\delta^{2s} + C_3\delta. \end{aligned} \tag{6.18}$$

Set $\bar{R} = \max\{R_1, \dots, R_{n_0-1}, R\}$. By (6.17) and (6.18) we find

$$z_n(x) \leqslant C_4(\delta^{2s} + \delta) \quad \text{for all } |x| \geqslant \bar{R},\ n \in \mathbb{N}.$$

From the arbitrariness of $\delta > 0$, we deduce that (6.13) holds true. Consequently, combining (6.13) with $0 \leqslant v_n \leqslant z_n$ in $\mathbb{R}^N$, we obtain that $v_n(x) \to 0$ as $|x| \to \infty$ uniformly with respect to $n \in \mathbb{N}$. □

Now, we are ready to give the proof of theorem 1.1.

Proof of theorem 1.1. Let $\delta > 0$ be such that $M_\delta \subset \Lambda$. First, we claim that there exists $\hat{\varepsilon}_\delta > 0$ such that for any $\varepsilon \in (0, \hat{\varepsilon}_\delta)$ and any solution $u \in \widetilde{\mathcal{N}}_\varepsilon$ of (3.2), it holds

$$\|u\|_{L^\infty(\Lambda_\varepsilon^c)} < \sqrt{a}. \tag{6.19}$$

We argue by contradiction and assume that for some subsequence $\varepsilon_n \to 0$ we can obtain $u_n = u_{\varepsilon_n} \in \widetilde{\mathcal{N}}_{\varepsilon_n}$ such that $J'_\varepsilon(u_n) = 0$ and

$$\|u_n\|_{L^\infty(\Lambda_\varepsilon^c)} \geqslant \sqrt{a}. \tag{6.20}$$

Since $J_{\varepsilon_n}(u_n) \leqslant d_0 + h(\varepsilon_n)$, we can argue as in the first part of lemma 5.3 to deduce that $J_{\varepsilon_n}(u_n) \to d_0$. In view of lemma 5.3, there exists $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $\varepsilon_n \tilde{y}_n \to y_0$ for some $y_0 \in M$ and $v_n = |u_n|(\cdot + \tilde{y}_n)$ strongly converges in $H^s(\mathbb{R}^N, \mathbb{R})$.

Take $r > 0$ such that, for some subsequence still denoted by itself, it holds $B_r(\varepsilon_n \tilde{y}_n) \subset \Lambda$ for all $n \in \mathbb{N}$. Hence $B_{r/\varepsilon_n}(\tilde{y}_n) \subset \Lambda_{\varepsilon_n}$ for all $n \in \mathbb{N}$, and consequently $\Lambda_{\varepsilon_n}^c \subset B_{r/\varepsilon_n}^c(\tilde{y}_n)$ for all $n \in \mathbb{N}$. By lemma 6.3, we can find $R > 0$ such that

$$v_n(x) < \sqrt{a} \quad \text{for all } |x| \geqslant R,\ n \in \mathbb{N},$$

from which we deduce that $|u_n(x)| < \sqrt{a}$ for any $x \in B_R^c(\tilde{y}_n)$ and $n \in \mathbb{N}$. On the other hand, there exists $\nu \in \mathbb{N}$ such that for any $n \geqslant \nu$ it holds

$$\Lambda_{\varepsilon_n}^c \subset B_{r/\varepsilon_n}^c(\tilde{y}_n) \subset B_R^c(\tilde{y}_n).$$

Therefore, $|u_n(x)| < \sqrt{a}$ for any $x \in \Lambda_{\varepsilon_n}^c$ and $n \geqslant \nu$, and this contradicts (6.20).

Let $\tilde{\varepsilon}_\delta > 0$ be given by theorem 5.5 and we set $\varepsilon_\delta = \min\{\tilde{\varepsilon}_\delta, \hat{\varepsilon}_\delta\}$. Let us fix $\varepsilon \in (0, \varepsilon_\delta)$. Applying theorem 5.5 we obtain at least $cat_{M_\delta}(M)$ nontrivial solutions to (3.2). If $u_\varepsilon \in \mathcal{H}_\varepsilon$ is one of these solutions, then $u_\varepsilon \in \widetilde{\mathcal{N}}_\varepsilon$, and in view of (6.19) and the definition of g, we infer that u_ε is also a solution to (3.1). Since $\hat{u}_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a solution to (1.1), we deduce that (1.1) has at least $cat_{M_\delta}(M)$ nontrivial solutions.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

Finally, we investigate the behaviour of the maximum points of $|\hat{u}_\varepsilon|$. Take $\varepsilon_n \to 0$ and let $(u_n) \subset \mathcal{H}_{\varepsilon_n}$ be a sequence of solutions to (3.2) as above. From (g_1), there exists $\gamma \in (0, \sqrt{a})$ such that

$$g_\varepsilon(x, t^2)t^2 \leqslant \frac{V_0}{2}t^2 \quad \text{for all } x \in \mathbb{R}^N, |t| \leqslant \gamma. \tag{6.21}$$

Arguing as above, we can find $R > 0$ such that

$$\|u_n\|_{L^\infty(B_R^c(\tilde{y}_n))} < \gamma. \tag{6.22}$$

Up to a subsequence, we may also assume that

$$\|u_n\|_{L^\infty(B_R(\tilde{y}_n))} \geqslant \gamma. \tag{6.23}$$

Indeed, if (6.23) does not hold, we get $\|u_n\|_{L^\infty(\mathbb{R}^N)} < \gamma$, and using $J'_{\varepsilon_n}(u_n) = 0$, (6.21) and lemma 2.2, we deduce that

$$[|u_n|]^2 + \int_{\mathbb{R}^N} V_0|u_n|^2 \mathrm{d}x \leqslant \|u_n\|^2_{\varepsilon_n} = \int_{\mathbb{R}^N} g_{\varepsilon_n}(x, |u_n|^2)|u_n|^2 \,\mathrm{d}x \leqslant \frac{V_0}{2}\int_{\mathbb{R}^N} |u_n|^2 \,\mathrm{d}x.$$

This fact yields $|||u_n|||_0 = 0$, which is impossible. Hence, (6.23) is satisfied.

In light of (6.22) and (6.23), we can see that if p_n is a global maximum point of $|u_n|$, then p_n belongs to $B_R(\tilde{y}_n)$, that is $p_n = \tilde{y}_n + q_n$ for some $q_n \in B_R$. Since $\hat{u}_n(x) = u_n(x/\varepsilon_n)$ is solution of (1.1), we infer that $\eta_n = \varepsilon_n \tilde{y}_n + \varepsilon_n q_n$ is a global maximum point of $|\hat{u}_n|$. Using $(q_n) \subset B_R$, $\varepsilon_n \tilde{y}_n \to y_0 \in M$, and the continuity of V, we deduce that

$$\lim_{n\to\infty} V(\eta_n) = V(y_0) = V_0.$$

Finally, we provide a decay estimate for $|\hat{u}_n|$. By using lemma 4.3 in [**30**] there exists a continuous function w such that

$$0 < w(x) \leqslant \frac{C}{1 + |x|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N, \tag{6.24}$$

and satisfying in the classical sense

$$(-\Delta)^s w + \frac{V_0}{2}w = 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}_{R_1}, \tag{6.25}$$

for some suitable $R_1 > 0$. By lemma 6.3, we can find $R_2 > 0$ such that

$$h_n = g(\varepsilon_n \cdot + \varepsilon_n \tilde{y}_n, v_n^2)v_n \leqslant \frac{V_0}{2}v_n \quad \text{in } \mathbb{R}^N \setminus \overline{B}_{R_2}. \tag{6.26}$$

Let us denote by w_n the unique solution to

$$(-\Delta)^s w_n + V_0 w_n = h_n \quad \text{in } \mathbb{R}^N.$$

Then, arguing as in the proof of (6.13), we see that $w_n(x) \to 0$ as $|x| \to \infty$ uniformly in $n \in \mathbb{N}$. Thus, by comparison, we obtain that $0 \leqslant v_n \leqslant w_n$ in $\mathbb{R}^N$. Moreover, in

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

light of (6.26), it holds

$$(-\Delta)^s w_n + \frac{V_0}{2} w_n = h_n - \frac{V_0}{2} w_n \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}_{R_2}.$$

Take $R_3 = \max\{R_1, R_2\}$ and we set

$$c = \min_{\overline{B}_{R_3}} w > 0 \quad \text{and} \quad \tilde{w}_n = (b+1)w - cw_n, \tag{6.27}$$

where $b = \sup_{n\in\mathbb{N}} \|w_n\|_{L^\infty(\mathbb{R}^N)} < \infty$. Our aim is to prove that

$$\tilde{w}_n \geqslant 0 \quad \text{in } \mathbb{R}^N. \tag{6.28}$$

First, we observe that

$$\tilde{w}_n \geqslant bc + w - bc > 0 \quad \text{in } \overline{B}_{R_3},$$
$$(-\Delta)^s \tilde{w}_n + \frac{V_0}{2} \tilde{w}_n \geqslant 0 \quad \text{in } \mathbb{R}^N \setminus \overline{B}_{R_3}.$$

Then we can apply a comparison principle (see theorem 7.1 in [**23**]) to deduce that (6.28) holds true. From (6.24), (6.28) and $0 \leqslant v_n \leqslant w_n$ in $\mathbb{R}^N$, we get

$$0 \leqslant v_n(x) \leqslant w_n(x) \leqslant \frac{(b+1)}{c} w(x) \leqslant \frac{\tilde{C}}{1+|x|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N, n \in \mathbb{N},$$

for some constant $\tilde{C} > 0$. Therefore, recalling the definition of v_n, we infer that

$$\begin{aligned} |\hat{u}_n|(x) = |u_n|\left(\frac{x}{\varepsilon_n}\right) = v_n\left(\frac{x}{\varepsilon_n} - \tilde{y}_n\right) \\ \leqslant \frac{\tilde{C}}{1+|\frac{x}{\varepsilon_n} - \tilde{y}_n|^{N+2s}} \\ = \frac{\tilde{C}\,\varepsilon_n^{N+2s}}{\varepsilon_n^{N+2s} + |x - \varepsilon_n\,\tilde{y}_n|^{N+2s}} \\ \leqslant \frac{\tilde{C}\,\varepsilon_n^{N+2s}}{\varepsilon_n^{N+2s} + |x - \eta_n|^{N+2s}} \quad \text{for all } x \in \mathbb{R}^N. \end{aligned}$$

This ends the proof of theorem 1.1. □

Remark 6.4. The approach used in this paper can be easily extended to deal with fractional Kirchhoff problems [**10**], fractional Choquard equations [**9**] and fractional Schrödinger–Poisson systems [**11**].

References

1 C. O. Alves and G. M. Figueiredo. Multiple solutions for a semilinear elliptic equation with critical growth and magnetic field. *Milan J. Math.* **82** (2014), 389–405.

2 C. O. Alves, G. M. Figueiredo and M. F. Furtado. Multiple solutions for a nonlinear Schrödinger equation with magnetic fields. *Commun. Partial Differ. Equ.* **36** (2011), 1565–1586.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

3 C. O. Alves and O. H. Miyagaki. Existence and concentration of solution for a class of fractional elliptic equation in $\mathbb{R}^N$ via penalization method. *Calc. Var. Partial Differ. Equ.* **55** (2016), 19pp. art. 47.

4 A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *J. Funct. Anal.* **14** (1973), 349–381.

5 V. Ambrosio. Multiplicity of positive solutions for a class of fractional Schrödinger equations via penalization method. *Ann. Mat. Pura Appl. (4)* **196** (2017), 2043–2062.

6 V. Ambrosio. Concentrating solutions for a class of nonlinear fractional Schrödinger equations in $\mathbb{R}^N$. *Rev. Mat. Iberoam.* **35** (2019), 1367–1414.

7 V. Ambrosio. Existence and concentration results for some fractional Schrödinger equations in $\mathbb{R}^N$ with magnetic fields. *Commun. Partial Differ. Equ.* **44** (2019), 637–680.

8 V. Ambrosio. On a fractional magnetic Schrödinger equation in $\mathbb{R}$ with exponential critical growth. *Nonlinear Anal.* **183** (2019), 117–148.

9 V. Ambrosio. Concentration phenomena for a fractional Choquard equation with magnetic field. *Dyn. Partial Differ. Equ.* **16** (2019), 125–149.

10 V. Ambrosio. Multiple concentrating solutions for a fractional Kirchhoff equation with magnetic fields. *Dis. Contin. Dyn. Syst.* **40** (2020), 781–815.

11 V. Ambrosio. Existence and concentration of nontrivial solutions for a fractional magnetic Schrödinger–Poisson type equation. *Ann. Sci. Norm. Super. Pisa Cl. Sci. (5)* **XXI** (2020), 1023–1061.

12 V. Ambrosio. A local mountain pass approach for a class of fractional NLS equations with magnetic fields. *Nonlinear Anal.* **190** (2020), 14 pp. 111622.

13 V. Ambrosio and P. d'Avenia. Nonlinear fractional magnetic Schrödinger equation: existence and multiplicity. *J. Diff. Equ.* **264** (2018), 3336–3368.

14 G. Arioli and A. Szulkin. A semilinear Schrödinger equation in the presence of a magnetic field. *Arch. Ration. Mech. Anal.* **170** (2003), 277–295.

15 J. Avron, I. Herbst and B. Simon. Schrödinger operators with magnetic fields I. General interactions. *Duke Math. J.* **45** (1978), 847–883.

16 V. Benci and G. Cerami. Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. *Calc. Var. Partial Differ. Equ.* **2** (1994), 29–48.

17 C. Bucur and E. Valdinoci. *Nonlocal diffusion and applications*. Lecture Notes of the Unione Matematica Italiana, vol. 20 (Bologna: Springer, [Cham]; Unione Matematica Italiana, 2016, xii+155 pp.).

18 J. Chabrowski and A. Szulkin. On the Schrödinger equation involving a critical Sobolev exponent and magnetic field. *Topol. Methods Nonlinear Anal.* **25** (2005), 3–21.

19 S. Cingolani and S. Secchi. Semiclassical limit for nonlinear Schrödinger equations with electromagnetic fields. *J. Math. Anal. Appl.* **275** (2002), 108–130.

20 J. Dávila, M. del Pino and J. Wei. Concentrating standing waves for the fractional nonlinear Schrödinger equation. *J. Diff. Equ.* **256** (2014), 858–892.

21 P. d'Avenia and M. Squassina. Ground states for fractional magnetic operators. *ESAIM Control Optim. Calc. Var.* **24** (2018), 1–24.

22 L. M. Del Pezzo and A. Quaas. A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian. *J. Diff. Equ.* **263** (2017), 765–778.

23 L. M. Del Pezzo and A. Quaas. Spectrum of the fractional p-Laplacian in $\mathbb{R}^N$ and decay estimate for positive solutions of a Schrödinger equation. *Nonlinear Anal.* **193** (2020), 111479.

24 M. del Pino and P. L. Felmer. Local mountain passes for semilinear elliptic problems in unbounded domains. *Calc. Var. Partial Differ. Equ.* **4** (1996), 121–137.

25 E. Di Nezza, G. Palatucci and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.* **136** (2012), 521–573.

26 S. Dipierro, M. Medina and E. Valdinoci. *Fractional elliptic problems with critical growth in the whole of* $\mathbb{R}^n$, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)] vol. 15, viii+152 pp. (Pisa: Edizioni della Normale, 2017).

27 I. Ekeland. On the variational principle. *J. Math. Anal. Appl.* **47** (1974), 324–353.

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press

28 M. Esteban and P. L. Lions. Stationary solutions of nonlinear Schrödinger equations with an external magnetic field. In F. Colombini, A. Marino, L. Modica and S. Spagnolo (eds), *Partial differential equations and the calculus of variations,* Vol. I, Progr. Nonlinear Differential Equations Appl., vol. 1, pp. 401–449 (Boston, MA: Birkhäuser Boston, 1989).

29 M. M. Fall, F. Mahmoudi and E. Valdinoci. Ground states and concentration phenomena for the fractional Schrödinger equation. *Nonlinearity* **28** (2015), 1937–1961.

30 P. Felmer, A. Quaas and J. Tan. Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian. *Proc. R. Soc. Edinburgh Sect. A* **142** (2012), 1237–1262.

31 G. M. Figueiredo and J. R. Santos. Multiplicity and concentration behavior of positive solutions for a Schrödinger–Kirchhoff type problem via penalization method. *ESAIM Control Optim. Calc. Var.* **20** (2014), 389–415.

32 G. M. Figueiredo and G. Siciliano. A multiplicity result via Ljusternick–Schnirelman category and Morse theory for a fractional Schrödinger equation in $\mathbb{R}^N$. *NoDEA Nonlinear Diff. Equ. Appl.* **23** (2016), 22 pp., Art. 12.

33 A. Fiscella, A. Pinamonti and E. Vecchi. Multiplicity results for magnetic fractional problems. *J. Diff. Equ.* **263** (2017), 4617–4633.

34 F. Hiroshima, T. Ichinose and J. Lörinczi. Kato's inequality for magnetic relativistic Schrödinger operators. *Publ. Res. Inst. Math. Sci.* **53** (2017), 79–117.

35 T. Ichinose. Magnetic relativistic Schrödinger operators and imaginary-time path integrals. In M. Demuth and W. Kirsch (eds), *Mathematical physics, spectral theory and stochastic analysis*. Oper. Theory Adv. Appl., vol. 232, pp. 247–297 (Basel: Birkhäuser/Springer Basel AG, 2013).

36 T. Kato. Schrödinger operators with singular potentials, Proceedings of the International Symposium on Partial Differential Equations and the Geometry of Normed Linear Spaces (Jerusalem, 1972). *Israel J. Math.* **13** (1973), 135–148.

37 K. Kurata. Existence and semi-classical limit of the least energy solution to a nonlinear Schrödinger equation with electromagnetic fields. *Nonlinear Anal.* **41** (2000), 763–778.

38 N. Laskin. Fractional quantum mechanics and Lévy path integrals. *Phys. Lett. A* **268** (2000), 298–305.

39 X. Mingqi, V. D. Rădulescu and B. Zhang. A critical fractional Choquard–Kirchhoff problem with magnetic field. *Commun. Contemp. Math.* **21** (2019), 36 pp., 1850004.

40 J. Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Commun. Pure Appl. Math.* **13** (1960), 457–468.

41 H. M. Nguyen, A. Pinamonti, M. Squassina and E. Vecchi. New characterizations of magnetic Sobolev spaces. *Adv. Nonlinear Anal.* **7** (2018), 227–245.

42 G. Palatucci and A. Pisante. Improved Sobolev embeddings, profile decomposition, and concentration-compactness for fractional Sobolev spaces. *Calc. Var. Partial Differ. Equ.* **50** (2014), 799–829.

43 P. H. Rabinowitz. On a class of nonlinear Schrödinger equations. *Z. Angew. Math. Phys.* **43** (1992), 270–291.

44 L. Silvestre. Regularity of the obstacle problem for a fractional power of the Laplace operator. *Commun. Pure Appl. Math.*, **60** (2007), 67–112.

45 M. Squassina and B. Volzone. Bourgain–Brezis–Mironescu formula for magnetic operators. *C. R. Math.* **354** (2016), 825–831.

46 C. Sulem and P.-L. Sulem. *The nonlinear Schrödinger equation. Self-focusing and wave collapse,* Applied Mathematical Sciences, vol. 139, xvi+350 pp. (New York: Springer-Verlag, 1999).

47 A. Szulkin and T. Weth. The method of Nehari manifold. In David Yang Gao and D. Motreanu (eds), *Handbook of Nonconvex Analysis and Applications*, pp. 597–632 (Somerville, MA: Int. Press, 2010).

48 M. Willem. *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, vol. 24 (Boston, MA: Birkhäuser Boston, Inc., 1996).

https://doi.org/10.1017/prm.2021.22 Published online by Cambridge University Press