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Representations of complex imprimitive reflection groups

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Introduction

In his attempt to give the irreducible characters of the Weyl groups of type A, B and D, Mayer[3] found a unified theory using their common structure as reflection groups. In this paper, we give the characters of the complex imprimitive reflection groups. We are able to give an algorithm which allows us to calculate the irreducible components of the principal character of a 'Weyl' subgroup induced up to the whole group. This is a generalization of the algorithm given by Mayer.

Section 1

Let V be a complex vector space of dimension n.

A group G of unitary automorphisms of V is called *imprimitive* if V is a direct sum $V = V_1 \oplus V_2 \oplus \ldots \oplus V_t$ of non-trivial proper linear subspaces V_i $(1 \leqslant i \leqslant t)$ of V such that $\{V_i | i = 1, \ldots, t\}$ is invariant under G. In this situation the family $\{V_i\}$ $(1 \leqslant i \leqslant t)$ is called a *system of imprimitivity* for G. If such a direct splitting of V does not exist, then G is called *primitive*.

Let $\mathscr{S}_n$ be the group of all $n \times n$ permutation matrices and let $A(m, p, n)$, where $m, p, n \in \mathbb{Z}^+$, and $p|m$, be the set of all diagonal $n \times n$ matrices with ξ^{ρ_i}, $\rho_i \in \mathbb{Z}$ in the (i, i) position, where ξ is a primitive mth root of unity and $\Sigma_{i=1}^{n} \rho_i \equiv 0 \pmod{p}$. Then $\mathscr{S}_n$ normalizes $A(m, p, n)$ and we let $G(m, p, n) = A(m, p, n).\ \mathscr{S}_n$ (semi-direct product).

$G(m, p, n)$ is a group generated by unitary reflections, and Cohen [1] proves that all irreducible imprimitive reflection groups are of the form $G(m, p, n)$.

The aim of this paper is to present some results about the characters of $G(m, p, n)$.

THEOREM 1. *If* $(p, q) = (q, n) = 1$, *then* $G(m, p, n) = G(m, m, n) \times Z_q$, *where* $q = m/p$, *and* Z_q *denotes the cyclic group of order* q.

Proof. The element

$$\begin{pmatrix} \xi^{\rho} & & & \\ & \xi^{\rho} & & \\ & & \ddots & \\ & & & \xi^{\rho} \end{pmatrix} \quad \text{for} \quad \rho = 1, \ldots, q-1,$$

where ξ is a primitive qth root of unity, is such that $\det = \xi^{n\rho}$ and $n\rho \not\equiv 0 \pmod{m}$ since $(p, q) = (q, n) = 1$, and thus

$$\begin{pmatrix} \xi^{\rho} & & \\ & \ddots & \\ & & \xi^{\rho} \end{pmatrix} \notin G(m, m, n).$$

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Since $A(m,m,n) \subset A(m,p,n)$ then $G(m,m,n) \subset G(m,p,n)$. The element

$$\begin{pmatrix} \xi^{\rho} & & \\ & \ddots & \\ & & \xi^{\rho} \end{pmatrix} \in G(m,p,n),$$

since $n\rho \equiv 0 \pmod{q} \forall \rho$ so

$$\begin{pmatrix} \xi^{\rho} & & \\ & \ddots & \\ & & \xi^{\rho} \end{pmatrix} \in Z(G(m,p,n)) \quad \text{for} \quad \rho = 1, \ldots, q.$$

Equating orders establishes the result. ∎

COROLLARY 2. (i) *If* $(m,n) = 1$, *then* $G(m,1,n) = G(m,m,n) \times Z_m$.
(ii) *If* $(p,n) = (p,q) = 1$ *then* $G(m,1,n) = G(m,p,n) \times Z_p$, *where* $q = m/p$.
(iii) *If* n *is odd, then* $G(2,1,n) = G(2,2,n) \times Z_2$.

Proof. (i) Let $p = 1$ in Theorem 1, so that $q = m$.
(ii) Since $(p,q) = 1$, we have $Z_m = Z_p \times Z_q$, then using Theorem 1 and (i), we have $G(m,1,n) = G(m,p,n) \times Z_p$.
(iii) When m is 2 and n is odd we have $(2,n) = 1$ and using (i), we establish the result. ∎

Notation. Let $G(m,1,n)$ be denoted by $W(B_n^m)$ and $G(m,m,n)$ by $W(D_n^m)$.

Section 2

Definition. An m-set of partitions of n, $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ consists of partitions $\lambda^{(1)}, \ldots, \lambda^{(m)}$ such that

$$\lambda^{(i)} = (\lambda_1^{(i)}, \ldots, \lambda_{s_i}^{(i)})$$

is a partition of l_i, where $\Sigma_{j=1}^{s_i} \lambda_j^{(i)} = l_i$ and $\Sigma_{i=1}^{m} l_i = n$.

Let $A = A(m,1,n)$ and let ξ be a primitive mth root of unity.

THEOREM 3. *Let* ζ *be an irreducible character of* A. *Let* $C = C_{\mathscr{S}_n}(\zeta)$, *and* ψ *be an irreducible character of* C. *Then* $C = \Pi_{i=1}^{m} \mathscr{S}_{l_i}$, *where* $\Sigma_{i=1}^{m} l_i = n$, *and* l_i *is the number of matrices of the form*

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \xi^j \end{pmatrix} \in A$$

on which ζ *takes the values* $\xi^{(1-i)j}$ *for* $j = 1, \ldots, m$.

Define a map $\phi\colon AC \to \mathbb{C}$ *by* $\phi(ac) = \zeta(a)\psi(c)$. *Then* ϕ *is an irreducible character of* AC.

Let $\phi = \zeta\psi$. *Then* (a) ϕ^G *is an irreducible character of* G; (b) *if* $\phi_1 = \zeta_1\psi_1$, $\phi_2 = \zeta_2\psi_2$, *then* $\phi_1^G = \phi_2^G$ *iff both* $\zeta_1 = {}^h\zeta_2$, $\psi_1 = {}^h\psi_2$ *for some* $h \in \mathscr{S}_n$; (c) *every irreducible character of* G *may be obtained in this way.*

Proof. We make use of the fact that A is abelian, so that ζ is a linear character.

Relabel the symbols $\{1, \ldots, n\}$ so that ζ takes the value

ξ^m on $\{1, \ldots, l_1\}$ called symbols of 1st type
ξ^{m-1} on $\{l_1+1, \ldots, l_1+l_2\}$ symbols of 2nd type
. . . . . .
ξ on $\{\Sigma_{i=1}^{m-1} l_i + 1, \ldots, n\}$ symbols of mth type.

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A matrix

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^j \end{pmatrix}$$

is of the ith type if ζ takes the value $\xi^{(1-i)j}$ on this matrix.

The l_i matrices of the ith type generate an abelian group A_i. Then $A = \Pi_{i=1}^m A_i$ and $C = \mathscr{S}_{l_1} \times \mathscr{S}_{l_2} \times \ldots \times \mathscr{S}_{l_m}$. The proof of (a), (b) follows from Serre ([5], p. 62).

ϕ^G determines uniquely integers l_i, such that $\Sigma_{i=1}^m l_i = n$, and determines up to conjugation by an element of $\mathscr{S}_n$, an irreducible character ψ of $\Pi \mathscr{S}_{l_i}$, which is a product of m irreducible characters $\chi^{\lambda^{(i)}}, \lambda^{(i)} \vdash l_i$. Thus ϕ^G determines uniquely an m-set of partitions of n.

ϕ^G is written as $(\chi^{\lambda^{(1)}}, \ldots, \chi^{\lambda^{(m)}})$.

Hence the irreducible characters of G may be obtained in this way. ∎

G has $2m$ linear characters

$$1, \sigma_1, \sigma_2, \ldots, \sigma_{m-1}, \eta, \epsilon_1, \ldots, \epsilon_{m-1},$$

where

(a) $1(g) = 1, \quad \forall g \in G,$

(b) $\sigma_i(\text{permutation}) = 1, \quad \sigma_i \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^{-j} \end{pmatrix} = \xi^{ij},$

(c) $\eta(\text{permutation}) = \text{sign}, \quad \eta \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^{-j} \end{pmatrix} = 1,$

(d) $\epsilon_i(\text{permutation}) = \text{sign}, \quad \epsilon_i \begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^{-j} \end{pmatrix} = \xi^{ij},$

where sign means sign of permutation and ξ is a primitive mth root of unity.

Note that $1, \eta$ are real characters (called the principal and sign characters respectively) and if m is even, then $\sigma_{m/2}$, $\epsilon_{m/2}$ are also real characters.

Definition. Let $P(\lambda) = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ be an m-set of partitions of n, then for $j = 1, \ldots, m$ define

$$P_j(\lambda) = (\lambda^{(j+1)}, \lambda^{(j+2)}, \ldots, \lambda^{(m)}, \lambda^{(1)}, \ldots, \lambda^{(j)}),$$
$$P'_j(\lambda) = (\lambda^{(j+1)'}, \lambda^{(j+2)'}, \ldots, \ldots, \ldots, \lambda^{(j)'}),$$
$$P_j(\overline{\lambda}) = (\lambda^{(j+1)}, \lambda^{(j)}, \lambda^{(j-1)}, \ldots, \lambda^{(1)}, \lambda^{(m)}, \ldots, \lambda^{(j+2)}).$$

where $\lambda^{(h)'}$ is the usual dual partition of $\lambda^{(h)}$.

LEMMA 4.

(i) $\sigma_i \eta = \epsilon_i$,

(ii) $\sigma_i \chi^{P_j(\lambda)} = \chi^{P_{j-i}(\lambda)}$,

(iii) $P'_j(\lambda) = P_j(\lambda')$,

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(iv) $\epsilon_t \chi^{P_j(\lambda)} = \chi^{P_{j-i}(\lambda)}$,
(v) $\eta\chi^{P_j(\lambda)} = \chi^{P_j'(\lambda)}$,
(vi) $\overline{\chi^{P_j(\lambda)}} = \chi^{P_j(\bar{\lambda})}$.

Proof. (i) and (iii) are trivial, (iv) follows from (ii), (v) and (vi) from (i), (ii), (iii), so we need therefore only prove (ii).

Let

$$\chi^{P_j(\lambda)} = \phi^G, \quad \text{so} \quad \sigma_i \chi^{P_j(\lambda)} = ((\sigma_i)_{AC} \cdot \phi)^G$$

and

$$(\phi_i)_{AC}\phi = (\sigma_i)_A \zeta (\sigma_i)_C \psi = (\sigma_i)_A \zeta \psi.$$

Because $(\sigma_i)_A$ takes the value ξ^{-ij} on

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^{-j} \end{pmatrix}$$

t permutes the symbols of type k to $k+i$. By definition ζ takes the value $\xi^{(1-k)j}$ on

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^{-j} \end{pmatrix}$$

for type k, so when we multiply by σ_i then $\sigma_i\zeta$ takes the value $\xi^{(1-k-i)j}$ on

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & \xi^{t} \end{pmatrix}$$

thus is of type $k+i$. Also $\psi = \Pi_i \chi^{\lambda_i}$, so $\sigma_i \chi^{P_j(\lambda)} = \chi^{P_{j-i}(\lambda)}$. ∎

In order to obtain the Weyl subgroups of G we use the algorithm given in [2] which involves extending the graph for B_n^m and deleting nodes, in a similar way that we obtain the subsystems from a Dynkin diagram.

Let $m_i | m$, where $m_i \in \mathbb{Z}$, and order the divisors of m as follows:

$$m = m_1 > m_2 > \ldots > m_r = 1.$$

From [2] we have that the 'Weyl' subgroups of G are of the form

$$\prod_{i=1}^{r} \left(W(B_{\lambda_1^{m_i}(i)}^{m_i}) \times \ldots \times W(B_{\lambda_{s_i}^{m_i}(i)}^{m_i}) \right),$$

which is written as $W_{(\lambda^{(1)}, \ldots, \lambda^{(r)})}$.

The irreducible characters of G are parametrized by m-sets of partitions of n, but in general $r \leqslant m$.

Given an r-set of partitions $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(r)})$ then the ith position corresponds to the divisor m_i of m and so we extend an r-set to an m-set as follows: place the partition in the ith position from the r-set into the m_ith position in the m-set. The m-set is then completed by placing an empty partition in the jth position, whenever $j \nmid m$.

For each $\lambda^{(i)}$ $(i = 1, \ldots, m)$ define a diagram $D_{\lambda^{(i)}}$ which is obtained by filling in the

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frame associated with $\lambda^{(i)}$ with the symbols $\{i\xi^s | i = 1, \ldots, n;\ s = 1, \ldots, m\}$ such that the moduli of all the numbers appearing are distinct. We obtain a set of diagrams $D_{\lambda^{(1)}}, \ldots, D_{\lambda^{(m)}}$. Write $D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}$ for $D_{\lambda^{(1)}} \cup \ldots \cup D_{\lambda^{(m)}}$.

A row permutation of a diagram $D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}$ is an element p of G such that p permutes the symbols in each row of $D_{\lambda^{(i)}}$ and multiplies any number of symbols in $D_{\lambda^{(i)}}$ by a power of ξ_i, where ξ_i is a primitive ith root of unity and $i|m$.

The row stabilizer $R(D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})})$ is the group of row permutations of $D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}$.

$$R(D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}) = \prod_{i=1}^{r} (W(B^{m_i}_{\lambda^1_i(i)}) \times \ldots \times W(B^{m_i}_{\lambda^{s_i}_i(i)}))$$
$$\simeq W_{(\lambda^{(1)}, \ldots, \lambda^{(m)})};$$

Thus each 'Weyl' subgroup of G can be considered as the row stabilizer of some diagram $D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}$.

THEOREM 5. $((\sigma_{m-1})^G_{W_{P_m(\lambda)}}, \chi^{P_m(\lambda)}) = 1$.

Proof. Write W for $W_{P_m(\lambda)}$ and σ for σ_{m-1}. Thus

$$(\sigma^G_W, \chi^{P_m(\lambda)}) = (\sigma^G_W, \phi^G)$$
$$= \sum_{y \in \{y_i\}} (\sigma_{W \cap (AC)y}, ({}^y\phi)_{W \cap (AC)y}),$$

where $\{y_i\}$ is a set of (W, AC)-double coset representatives. If

$$(\sigma_{W \cap (AC)y}, ({}^y\phi)_{W \cap (AC)y}) \neq 0 \quad \text{then} \quad (\sigma_{W \cap A}, ({}^y\phi)_{W \cap A}) \neq 0.$$

Now

$$W \leqslant AC \quad \text{and} \quad W \cap A = \prod_{i=1}^{r-1} A_{m_i} \quad \text{since} \quad A_{m_i} = A^{m_i}_{\lambda^1_i(i)} \times \ldots \times A^{m_i}_{\lambda^{s_i}_i(i)}.$$

Also

$$W \cap \mathscr{S}_n = \prod_{j=1}^{r} \prod_{i=1}^{s_i} \mathscr{S}_{ji} = \prod_{j=1}^{r} W_{\lambda_j},$$

where W_{λ_j} are Weyl subgroups of $\mathscr{S}_{l_j}$ (as for symmetric groups, see Mayer [3]).

Let $\prod_{i=1}^{r-1} A_{m_i} = A' \leqslant A$, then $({}^y\phi)_{A'} = ({}^y\zeta)_{A'} \psi(1)$. Thus $(\sigma_{A'}, ({}^y\zeta)_{A'} \psi(1)) \neq 0$ so $(\sigma_{A'}, ({}^y\zeta)_{A'}) \neq 0$ and therefore $\sigma_{A'} = ({}^y\zeta)_{A'}$.

Now $\sigma = \sigma_{m-1}$ takes the value ξ^j on

$$\begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ & & & \xi^j \end{pmatrix}$$

in A, thus $\zeta = {}^y\zeta$ because ζ takes the value 1 on A_1. Therefore $y \in C_G(\zeta) = AC$ and so only the first Mackey summand is non-zero. We have

$$(\sigma^G_W, \chi^{P_m(\lambda)}) = (\sigma_{W \cap AC}, \phi_{W \cap AC})$$
$$= (\sigma_{W \cap A}, \zeta_{W \cap A})(\sigma_{W \cap \mathscr{S}_n}, \psi_{W \cap \mathscr{S}_n})$$
$$= (\sigma_{A'}, \zeta_{A'})(1_{\pi W_{\lambda_j}}, \psi_{\pi W_{\lambda_j}})$$
$$= 1 \prod_{j=1} (1_{W_{\lambda_j}}, (\chi^{\lambda_j})_{W_{\lambda_j}})$$
$$= \prod_{j=1} ((1_{W_\lambda})^{\mathscr{S}_{l_j}}, \chi^{\lambda_j})$$
$$= 1 \quad \text{(by Mayer ([3], p. 20).} \blacksquare$$

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Section 3

The algorithm for $W(B_n^m)$

Let $(\lambda^{(1)}, \ldots, \lambda^{(m)})$, $(\alpha^{(1)}, \ldots, \alpha^{(m)})$ be m-sets of partitions of n. Write

$$(\lambda^{(1)}, \ldots, \lambda^{(m)}) \to (\alpha^{(1)}, \ldots, \alpha^{(m)})$$

if $(\alpha^{(1)}, \ldots, \alpha^{(m)})$ may be obtained from $(\lambda^{(1)}, \ldots, \lambda^{(m)})$ by:

(a) removing connected squares from a row of $\lambda^{(i)}$ and placing them (in the same order) at the bottom of $\lambda^{(zi)}$ for $z = 2, \ldots, m/i$);

(b) repeating (a) with different rows of $\lambda^{(i)}$;

(c) then re-ordering the resulting rows so as to give frames of an m-set of partitions $(\gamma^{(1)}, \ldots, \gamma^{(m)})$;

(d) moving up inside $\gamma^{(i)}$ – i.e. take squares from the ends of rows of $\gamma^{(i)}$ and add them onto the ends of rows higher up – so as to obtain $\alpha^{(i)}$, $\gamma^{(i)} \leqslant \alpha^{(i)}$.

LEMMA 6. *Let $W = R(D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})})$.*

(a) *$W = (A \cap W)(\mathscr{S}_n \cap W)$ and $(A \cap W) \cap (\mathscr{S}_n \cap W) = 1$.*

If $g \in \mathscr{S}_n$, $C = C_{\mathscr{S}_n}(\zeta)$ for some irreducible character ζ of A;

(b) *$W^g = (A \cap W^g)(\mathscr{S}_n \cap W^g)$, $(A \cap W^g) \cap (\mathscr{S}_n \cap W^g) = 1$;*

(c) *$AC \cap W^g = (A \cap W^g)(C \cap W^g)$, $(A \cap W^g) \cap (\mathscr{S}_n \cap W^g) = 1$.*

Proof. Mayer ([3], p. 60). |

THEOREM 7. *Let $P_m(\lambda)$, $P_m(\alpha)$ be m-sets of partitions of n, obtained from extending an r-set as in* § 2. *Then*

$$((\sigma_{m-1})^G_{W_{P_m(\lambda)}}, \chi^{P_m(\alpha)}) \neq 0 \Leftrightarrow P_m(\lambda) \to P_m(\alpha).$$

Proof. Suppose the inner product is non-zero. Write W for the subgroup $W_{P_m(\lambda)}$, and $\sigma = \sigma_{m-1}$. Hence

$$(\sigma_W^G, \phi^G) = \sum_{g \in \{g_i\}} ({}^g\sigma_{AC \cap W^g}, \phi_{AC \cap W^g}),$$

where $\{g_i\}$ is a set of (W, AC)-double coset representatives and since $G = A\mathscr{S}_n$, suppose $g_i \in \mathscr{S}_n$.

Thus $\exists g \in \{g_i\}$ such that $({}^g\sigma_{AC \cap W^g}, \phi_{AC \cap W^g}) \neq 0$.

Let $|\alpha_i| = l_i$ and let $A = A_1 \times \ldots \times A_n$. Now

$$0 \neq ({}^g\sigma_{AC \cap W^g}, \phi_{AC \cap W^g}) = ({}^g\sigma_{A \cap W^g}, \zeta_{A \cap W^g})({}^g\sigma_{C \cap W^g}, \psi_{C \cap W^g}).$$

Hence both inner products are non-zero and

$${}^g\sigma_{A \cap W^g} = \zeta_{A \cap W^g}.$$

Now $\sigma = \sigma_{m-1}$ takes the value ξ^j on

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & & \xi^j \end{pmatrix}$$

in G and hence on these in $A \cap W^g$.

Thus $A \cap W^g \leqslant \prod_{i=1}^{r-1} A_{m_i}$.

W^g defines a diagram $D_{P(\lambda)}$, so since $A \cap W^g \leqslant A'$, all the symbols in $D_{\lambda^{(j)}}$ are of types $a_j \equiv 0 \pmod{m_j}$.

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Assume that all the symbols of these types in $D_{\lambda(j)}$ lie at the end of rows (since W^g defines $D_{P(\lambda)}$ up to a permutation of rows). Hence we may remove squares from $D_{\lambda(j)}$ containing symbols of the kth type to $D_{\lambda(k)}$ (these can only be of type $zj, z = 2, \ldots, m/j$). Re-order to obtain a diagram $D_{P(\beta)}$ such that $D_{\beta(i)}$ contains all the symbols of the ith type. This corresponds to operations (a), (b) and (c).

We must now show $\beta^{(i)} \leqslant \alpha^{(i)} \forall i$. By construction

$$|\alpha^{(i)}| = l_i = |\beta^{(i)}| \,\forall i \quad \text{and} \quad (\sigma_{C \cap W^g}, \psi_{W^g \cap C}) = (1_{C \cap W^g}, \psi_{C \cap W^g}) \neq 0.$$

Now

$$C \cap W^g = \prod_{i=1}^{m} (\mathscr{S}_{l_i} \cap W^g)$$

and so $C \cap W^g$ permutes the symbols of each type independently. Hence

$$\prod_{i=1}^{m} (1_{\mathscr{S}_{l_i} \cap W^g}, \chi^{\alpha(i)}_{\mathscr{S}_{l_i} \cap W^g}) \neq 0.$$

But $\mathscr{S}_{l_i} \cap W^g$ is the group of row permutations of the symbols of the ith type in $D_{P(\lambda)}$ and thus the group of row permutations of $D_{\alpha(i)}$.

Therefore $\mathscr{S}_{l_i} \cap W^g = R(D_{\alpha(i)}) = W_{\alpha(i)} < \mathscr{S}_{l_i}$.

Hence $\Pi_{i=1}^{m}(1_{W_{\beta(i)}}, \chi^{\alpha(i)}_{W_{\beta(i)}}) \neq 0$.

So by Frobenius reciprocity $(1^{\mathscr{S}_{l_i}}_{W_\beta}, \chi^{\alpha^{(i)}}) \neq 0 \,\forall i$ from which it follows that $\beta^{(i)} \leqslant \alpha^{(i)} \forall i$. Thus $P(\lambda) \to P(\alpha)$.

Conversely, suppose $P(\lambda) \to P(\alpha)$.

Therefore we may remove parts of rows of $\lambda^{(i)}$ to $\lambda^{(zi)}$ to obtain an m-set of partitions $P(\beta)$ of n, such that $\beta^{(i)} \leqslant \alpha^{(i)} \forall i$.

Hence

$$|\lambda^{(i)}| \geqslant |\beta^{(i)}| = |\alpha^{(i)}| = l_i.$$

So define a diagram $D_{P(\lambda)}$ to be filled with symbols $\{1, \ldots, n\}$ such that

$$\{\Sigma_{j=1}^{i-1} l_j + 1, \Sigma_{j=1}^{i-1} l_j + 2, \ldots, \Sigma_{j=1}^{i} j\} \text{ all occur in } D_{\lambda_{m_i}}.$$

Let $W = W_{(\lambda(1), \ldots, \lambda(m))} = R(D_{(\lambda(1), \ldots, \lambda(m))})$. Then $A \cap W \leqslant \Pi_{i=1}^{r-1} A_{m_i}$. Hence $\sigma_{A \cap W} = \zeta_{A \cap W}$; therefore $(\sigma_{A \cap W}, \zeta_{A \cap W}) \neq 0$. Now

$$\beta^{(i)} \leqslant \alpha^{(i)} \Rightarrow ((1_{W_{\beta(i)}})^{\mathscr{S}_{l_i}}, \chi^{\alpha(i)}) \neq 0 \,\forall i.$$

So

$$(\sigma_{A \cap W}, \zeta_{A \cap W}) \prod_{i=1}^{m} (1^{\mathscr{S}_{l_i}}_{W_{\beta(i)}}, \chi^{\alpha^{(i)}}) \neq 0,$$

and is, by the proof of the first part of the theorem, the first summand in the Mackey formula for $(\sigma^G_W, \chi^{P(\alpha)})$.

Hence $(\sigma^G_W, \chi^{P(\alpha)}) \neq 0$. ∎

See Example 1, which illustrates Theorem 7.

Following an idea by Mayer [3], in order to give a partial ordering on the m-sets of partitions of n, an adaptation of the definition of $\to$ is needed.

Let $P(\lambda), P(\alpha)$ be m-sets of partitions of n. Then $P(\lambda) \leqslant P(\alpha)$ if $P(\alpha)$ may be obtained from $P(\lambda)$ by:

(a) removing a square from the end of a row of $\lambda^{(i)}$ and placing it at the bottom of $\lambda^{(iz)}$ for $z = 2, \ldots, m/i$;

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(*b*) repeating (*a*) as many times as is necessary to obtain an m-set of partitions $P(\gamma)$ of n,

(*c*) moving up inside $\gamma^{(i)}$ to obtain $\alpha^{(i)}$ respectively, so that $\gamma^{(i)} \leqslant \alpha^{(i)}$.

It is clear that $P(\lambda) \to P(\alpha) \Rightarrow P(\lambda) \leqslant P(\alpha)$.

THEOREM 8.

$$(\sigma_{m-1})^G_{W_{P_m(\lambda)}} = \chi^{P_m(\lambda)} + \sum_{P_m(\lambda) < P_m(\alpha)} a_\alpha \chi^{P_m(\alpha)}.$$

Proof. Follows from Theorems 5 and 7.

Remark. If we multiply the character σ_{m-1} by σ_1, then we obtain a similar result to that above for the decomposition of the principal character, using Lemma 4, given in the following:

COROLLARY 9.

(i) $(1_{W_{P_m(\lambda)}}, \chi^{P_{m-1}(\alpha)}) \neq 0 \Leftrightarrow P_m(\lambda) \to P_m(\alpha)$.

(ii) $1^G_{W_{P_m(\lambda)}} = \chi^{P_{m-1}(\lambda)} + \sum_{P_{m-1}(\lambda) < P_{m-1}(\beta)} a_\beta \chi^{P_{m-1}(\beta)}$,

where α_β is non-negative.

Since $(1, \theta) = (1, \bar{\theta})$, where 1 is the principal character and θ is any complex character of any group, the following holds:

(iii) $1^G_{W_{P_m(\lambda)}} = (\chi^{P_{m-1}(\lambda)} + \chi^{P_1(\lambda)}) + \sum_{P_{m-1}(\lambda) < P_{m-1}(\beta)} a_\beta (\chi^{P_{m-1}(\beta)} + \chi^{P_1(\beta)})$, *if* $\chi^{P_{m-1}(\alpha)} \neq \chi^{P_1(\alpha)}$ *for* $\alpha = \lambda, \beta$.

Section 4

Read [4] gives the irreducible components of the restriction to $G(m, p, n)$ of a given irreducible character of $G(m, 1, n)$.

Let $H_p = G(m, p, n)$ where $p|m$. Then H_p is normal in $G = G(m, 1, n)$. $H_p = A(m, p, n)\mathcal{S}_n$ (semi-direct product).

Let $A_p = A(m, p, n)$ so that $H_p = A_p . \mathcal{S}_n$ and let $A = A_1$. Let $P_i(\lambda) = P_j(\lambda) \pmod q$ if $P_i(\lambda) = P_j(\lambda)$ whenever $(i - j) \equiv 0 \pmod q$, where $q = m/p$.

THEOREM 10. (i) $\chi^{P_i(\lambda)} \downarrow H_p = \chi^{P_j(\lambda)} \downarrow H_p$ *for all j such that* $P_i(\lambda) = P_j(\lambda) \pmod q$;

(ii) $\chi^{P_i(\lambda)} \downarrow H_p$ *is irreducible if* $P_i(\lambda) \neq P_j(\lambda) \pmod q$, *for any* $j \neq i$.

Proof. Let $\chi^{P_i(\lambda)} = \phi_i^G$ for $i = 1, \ldots, m$, where $\phi_i = \zeta_i \psi_i$.

By definition

$$\zeta_1 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \xi \end{pmatrix} = \xi^{i-1} \zeta_i \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \xi \end{pmatrix}$$

in $A(m, 1, n)$. So $\zeta_1(*) = \xi^{pi-p} \zeta_i(*)$, where $(*) \in A_p$ since A is abelian.

Now

$$(\zeta_i) \downarrow_{A_p} = (\zeta_j)_{A_p} \text{ iff } \xi^{pi-p} = \xi^{pj-p},$$

i.e.

$$\text{iff } (pi - pj) \equiv 0 \pmod m,$$

i.e.

$$(i - j) \equiv 0 \pmod q.$$

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So, by construction, $(\zeta_i)_{A_p} = (\zeta_j)_{A_p}$. Now

$$C = C_{\mathscr{S}_n}(\zeta_i) \simeq \prod_i \mathscr{S}_{l_i}.$$

Now

$$\psi_i = \psi_j = \psi \quad \text{say} \quad \forall i, j.$$

Let $L_p = A_p.C$. Then $(\phi_i)_{L_p} = (\zeta_i)_{A_p}\psi = (\zeta_j)_{A_p}\psi = (\phi_j)_{L_p}\ \forall i, j$ as above.

Hence, $\chi^{P_i(\lambda)}\downarrow_{H_p} = ((\phi_i)_{L_p})^{H_p} = ((\phi_j)_{L_p})^{H_p} = \chi^{P_j(\lambda)}\downarrow_{H_p}$ if $P_i(\lambda) = P_j(\lambda) \pmod q$.

This ends the proof of (i).

The proof of (ii) follows from Read (*loc. cit.*). |

Let $H = G(m, m, n)$.

Then $P_i(\lambda) \equiv P_j(\lambda) \pmod 1$, iff $P_i(\lambda) = P_j(\lambda)$.

This implies that $\lambda^{(i)} = \lambda^{(\overline{i+k})}$ for $i = 1, \ldots, m$, for some k, where $\bar{j}$ denotes the residue of $j \pmod m$. If $(m, n) = 1$, then $P_i(\lambda) \neq P_j(\lambda)$ for any $i \neq j$. Therefore assume $(m, n) > 1$, and let the divisors of m be, as usual, $m = m_1 > m_2 > \ldots > m_r = 1$. Take a divisor m_i of m and let $m = m_i d_i$. Then $P_j(\lambda) = P_{\overline{j+d_i}}(\lambda)$, $j = 1, \ldots, m$, for this d_i.

Let $n = t_i d_i$.

Now $|\lambda^{(\overline{j+zd_i})}| = t_i$ for $j = 1, \ldots, d_i$; $z = 1, \ldots, m_i - 1$, and denote the number of partitions of t_i by D_i. Now, as in Read (*loc. cit.*), let $l = m/k$; we have $\chi \downarrow H$ is the sum of l irreducible characters with multiplicity 1. We can see that we have chosen m_i to be this l. Let C be the number of $P_i(\lambda)$ which are not congruent to any $P_j(\lambda)$, for $i \neq j$. Then, by Read, $\chi^{P_i(\lambda)}\downarrow_H$ is irreducible. An upper limit for the number of characters of H is $C + \Sigma_{i=1}^{c} m_i D_i$, where C, m_i, D_i are as above and c is as in Read (*loc. cit.*). We have not been able to determine a lower bound in order to establish that this is indeed the number of irreducible characters of H.

An algorithm for $G(m, m, n)$

The 'Weyl' subgroups of $W(D_n^m)$ have the form

$$\prod_{i=1}^{r} (W(D^{m_i}_{\lambda_1^{(i)}} \times \ldots \times W(D^{m_i}_{\lambda_{s_i}^{(i)}})),$$

where $\Sigma_{i,j} \lambda_j^{(i)} = n$ and such that no part of $\lambda_j^{(i)}$ is 1 except when $i = r$. This subgroup is written as $W_{(\lambda^{(1)}, \ldots, \lambda^{(r)})}$ and the r-set is transformed into an m-set as before and so is written as $W_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}$, which we can consider as the row stabilizer of a diagram $D_{(\lambda^{(1)}, \ldots, \lambda^{(m)})}$, where in this case a row permutation is an element of $W(D_n^m)$ which permutes the symbols of each row of $D_{\lambda^{(i)}}$ and also changes the power of ξ to ξ^{ρ_j} in each diagram $D_{\lambda^{(i)}}$ subject to

$$\Sigma \rho_j \equiv 0 \pmod{m_i}.$$

Let $P(\mu) = (\mu^{(1)}, \ldots, \mu^{(m)})$ be an m-set of partitions of n. We write $P(\mu) \underset{D}{\rightarrow} P(\lambda)$ if $P(\lambda)$ can be obtained from $P(\mu)$ by:

(a) removing connected squares from the end of a row of $\mu^{(i)}$ and placing them in the same order, at the bottom of $\mu^{(zi)}$, for any $z = 2, \ldots, m/i$;

(b) repeating (a) with squares from different rows of $\mu^{(i)}$ (and at the same time, with squares that have not been moved already);

(c) transferring complete rows of $\mu^{(j)}$ and placing them at the bottom of $\mu^{(k)}$, $k \leqslant j$, and

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(*c* i) removing connected squares from a row in $\mu^{(k)}$ and placing them in the same order, at the bottom of $\mu^{(\overline{k+jz})}$, $z \in \mathbb{Z}$;

(*c* ii) repeating (*c* i) with squares from different rows of $\mu^{(k)}$;

(*d*) re-ordering the resulting rows to give frames of $P(\gamma)$ say;

(*e*) moving up inside $\gamma^{(i)}$ (usual partial ordering) to obtain $\lambda^{(i)}$,

$$\gamma^{(i)} \leqslant \lambda^{(i)} \ (\forall i = 1, \ldots, m).$$

See Example 2, which illustrates this algorithm.

THEOREM 11.

$$(1^H_{W_{P(\mu)}}, \chi^{P(\lambda)}_H) \neq 0 \Leftrightarrow P(\mu) \underset{D}{\rightarrow} P(\lambda).$$

Proof. Similar to Mayer [3] and the proof of Theorem 7. ∎

Finally we give a remark on the groups $G(m, p, n)$, $p|m$ and $p \neq 1, m$. These groups are generated by $n+1$ reflections. We use the algorithm given in [2] and thus the 'Weyl' subgroups of G_p have the form

$$\prod_{i=1}^{r-1} (W(D^{m_i}_{\lambda^{(i)}_1}) \times \ldots \times W(D^{m_i}_{\lambda^{(i)}_{s_i}})) \times \prod_{i=1}^{u} (W(B^{q_i}_{\rho^{(i)}_1}) \ldots W(B^{q_i}_{\rho^{(i)}_{t_i}})),$$

where $m = m_1 > m_2 > \ldots > m_r = 1$, $q = q_1 > q_2 > \ldots > q_u = 1$, where $q = m/p$ and

$$\sum_{i=1}^{r-1} \sum_{j=1}^{s_i} \lambda^{(i)}_j + \sum_{i=1}^{u} \sum_{j=1}^{t_i} \rho^{(i)}_j = n,$$

such that no part of $\lambda^{(i)}_j$ is 1.

Note that, using the algorithm, we obtain subgroups $G(m, p, r)$ of $G(m, p, n)$, but since these are r dimensional and require $r+1$ generating reflections, we do not consider these.

Define $W(D^{m_i}_x)$ to be *regular* in G_p iff $W(B^{m_i}_x)$ is not a subgroup of G_p. We consider the Weyl subgroups of the above form with $W(D^{m_i}_x)$ regular. The subgroups are denoted by $W_{(\gamma^{(1)}, \ldots, \gamma^{(r)})}$ where

$$\gamma^{(i)} = \begin{cases} \rho^{(i)} & \text{if } i|q, \\ \lambda^{(i)} & \text{otherwise,} \end{cases}$$

such that the partitions $\rho^{(i)}$ are associated with type B, and $\lambda^{(i)}$ with type D. The r-set is then transformed into an m-set as usual.

We adapt the row stabilizer for $G(m, m, n)$ to be subject to $\Sigma\rho_j = 0 \pmod p$ and similarly we adapt the algorithm for $P(\mu) \rightarrow P(\lambda)$ by replacing (*c*) by

(*c'*) transferring complete rows of $\mu^{(j)}$ and placing them at the bottom of $\mu^{(k)}$, where $|k-j| = 0 \pmod q$ and

(*c'* i) removing connected squares from a row of $\mu^{(k)}$ and placing them in the same order, at the bottom of $\mu^{(\overline{k+jz})}$, $z \in \mathbb{Z}$;

(*c'* ii) repeating (*c'* i) with squares from different rows of $\mu^{(k)}$.

We are then able to give corresponding results for the results for $G(m, m, n)$.

Examples. Notation as in the character tables in [2].

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Example 1.

$W = G(3,1,3)$	Decomposition
W_1	χ_1
W_2	$\chi_1+\chi_7$
W_3	$\chi_1+\chi_2+2\chi_7$
W_4	$\chi_1+\chi_7+(\chi_{10}+\chi_{13})$
W_5	$\chi_1+\chi_7+(\chi_{10}+\chi_{13})+(\chi_{12}+\chi_{15})+(\chi_{16}+\chi_{19})+\chi_{22}$
W_6	$\chi_1+(\chi_3+\chi_4)+(\chi_{10}+\chi_{13})+(\chi_{11}+\chi_{14})+(\chi_{12}+\chi_{15})+\chi_{22}$
W_7	$\chi_1+\chi_2+2\chi_7+(\chi_{10}+\chi_{13})+(\chi_{16}+\chi_{19})$
W_8	$\chi_1+\chi_2+2\chi_7+2(\chi_{10}+\chi_{13})+(\chi_{12}+\chi_{15})+2(\chi_{16}+\chi_{19})+(\chi_{18}+\chi_{21})+2\chi_{22}$
W_9	$\chi_1+(\chi_3+\chi_4)+\chi_7+(\chi_8+\chi_9)+2(\chi_{10}+\chi_{13})+2(\chi_{11}+\chi_{14})+2(\chi_{12}+\chi_{15})+(\chi_{16}+\chi_{19})+(\chi_{17}+\chi_{20})+(\chi_{18}+\chi_{21})+3\chi_{22}$
W_{10}	regular character

From the algorithm

W_1: $(0,0,3) \to (0,0,3)$
W_2: $(0,0,21) \to (0,0,21), (0,0,3)$
W_3: $(0,0,1^3) \to (0,0,1^3), (0,0,21), (0,0,3)$
W_4: $(1,0,2) \to (1,0,2), (0,1,2), (0,0,21), (0,0,3)$
W_5: $(2,0,1) \to (2,0,1), (1,1,1,), (1,0,1^2), (1,0,2), (0,2,1), (0,1,1^2), (0,1,2), (0,0,21), (0,0,3)$
W_6: $(3,0,0) \to (3,0,0), (2,1,0), (2,0,1), (1,1,1), (1,0,2), (0,2,1), (0,1,2), (0,3,0), (0,0,3)$
W_7: $(1,0,1^2) \to (1,0,1^2), (1,0,2), (0,1,1^2), (0,1,2), (0,0,1^3), (0,0,21), (0,0,3)$
W_8: $(1^2,0,1) \to (1^2,0,1), (2,0,1), (1,1,1), (1,0,1^2), (1,0,2), (0,1^2,1), (0,2,1), (0,1,1^2), (0,1,2), (0,0,1^3), (0,0,21), (0,0,3)$
W_9: $(21,0,0) \to (21,0,0), (3,0,0), (1^2,1,0), (2,1,0), (1^2,0,1), (2,0,1), (1,1^2,0), (1,1,1), (1,2,0), (1,0,1^2), (1,0,2), (0,21,0), (0,3,0), (0,1^2,1), (0,2,1), (0,1,1^2), (0,1,2), (0,0,21), (0,0,3)$
W_{10}: $(1^3,0,0) \to$ all (α,β,γ).

From Theorem 3, the one–one correspondence with the irreducible characters and 3-sets of partitions of 3 is as follows:

$\chi_1-(0,0,3)$, $\chi_2-(0,0,1^3)$, $\chi_3-(3,0,0)$, $\chi_4-(0,3,0)$, $\chi_5-(1^3,0,0)$, $\chi_6-(0,1^3,0)$, $\chi_7-(0,0,21)$, $\chi_8-(21,0,0)$, $\chi_9-(0,21,0)$, $\chi_{10}-(1,0,2)$, $\chi_{11}-(2,1,0)$, $\chi_{12}-(0,2,1)$, $\chi_{13}-(0,1,2)$, $\chi_{14}-(1,2,0)$, $\chi_{15}-(2,0,1)$, $\chi_{15}-(1,0,1^2)$, $\chi_{17}-(1^2,1,0)$, $\chi_{18}-(0,1^2,1)$, $\chi_{19}-(0,1,1^2)$, $\chi_{20}-(1,1^2,0)$, $\chi_{21}-(1^2,0,1)$, $\chi_{22}-(1,1,1)$.

Example 2. Let $G = G(4,1,2)$, $H = G(4,4,2)$, $T = G(4,2,2)$

From Theorem 3, the one–one correspondence with the irreducible characters and 4-sets of partitions of 2 is as follows:

$\chi_1-(0,0,0,2)$, $\chi_2-(0,0,0,1^2)$, $\chi_3-(0,1^2,0,0)$, $\chi_4-(0,2,0,0)$, $\chi_5-(1^2,0,0,0)$, $\chi_6-(0,0,1^2,0)$, $\chi_7-(2,0,0,0)$, $\chi_8-(0,0,2,0)$, $\chi_9-(1,0,1,0)$, $\chi_{10}-(0,1,0,1)$, $\chi_{11}-(1,0,0,1)$, $\chi_{12}-(0,0,1,1)$, $\chi_{13}-(0,1,1,0)$, $\chi_{14}-(1,1,0,0)$.

For H:

The Weyl subgroups of H are $W(D_2^4)$, $W(D_2^2)$, $\mathscr{S}_1 \times \mathscr{S}_1$ denoted by W_1, W_2, W_3, resp.

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From the algorithm

$$W_1(0,0,0,2) \to (0,0,0,2),\ (0,0,2,0),\ (0,2,0,0),\ (2,0,0,0)$$

and

$$W_2(0,2,0,0) \to (0,2,0,0),\ (0,1,0,1),\ (0,0,0,2),\ (2,0,0,0),\ (1,0,1,0),\ (0,0,2,0).$$

Now $(\chi_i)_H = \theta_1$ for $i = 1, 4, 7, 8$ (illustrating Theorem 10 (i)) and

$$1^H_{W_2} = \theta_1 + \theta_4 \quad \text{and} \quad ((\chi_9)_H, \theta_4) = ((\chi_{10})_H, \theta_4) \neq 0,$$

illustrating Theorem 11.

For T:

The regular Weyl subgroups of T are $W(B_2^2)$, $W(B_1^2) \times W(B_1^2)$, $W(D_2^4)$, $W(B_1^2) \times \mathscr{S}_1$, $\mathscr{S}_2$, $\mathscr{S}_1 \times \mathscr{S}_1$ denoted by $W_1, \ldots, W_6$ resp.

$$W_1(0,2,0,0) \to (0,2,0,0),\ (0,1,0,1),\ (0,0,0,2)$$

and

$$W_2(0,1^2,0,0) \to (0,1^2,0,0),\ (0,2,0,0),\ (0,1,0,1),\ (0,0,0,1^2),\ (0,0,0,2).$$

Now

$$1^T_{W_1} = \eta_1 + \eta_7 \quad \text{and} \quad 1^T_{W_2} = \eta_1 + \eta_2 + \eta_7 + \eta_8$$

and

$$(\chi_1)_T = (\chi_4)_T = \eta_1, \quad (\chi_{10})_T = \eta_1 + \eta_7, \quad (\chi_2)_T = (\chi_3)_T = \eta_2,$$

which illustrates the remark given after Theorem 11.

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