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MONOCHROMATIC SUMS AND QUOTIENTS NEAR ZERO

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Abstract

Goswami ['Monochromatic translated product and answering Sahasrabudhe's conjecture', *Adv. Combin.* (2026), https://doi.org/10.19086/aic.2026.2] proved that whenever the set $\mathbb{N}$ of natural numbers is finitely coloured, the set $\{a, b, ab, b(a+1)\}$ is monochromatic. This also established a variant of the long-standing Hindman's conjecture, which asks for a monochromatic set of the form $\{a, b, ab, a+b\}$, and disproved a conjecture proposed by Sahasrabudhe that $\{a, b, a(b+1)\}$ is not partition regular. In this paper, we prove that $\{a, b, ab, b(a+1)\}$ is monochromatic near zero, which means that for every finite colouring of a dense subsemigroup of $((0,\infty), +)$, the set $\{a, b, ab, b(a+1)\}$ is monochromatic near zero. In other words, given a dense subsemigroup of $((0,\infty), +)$ with a finite colouring, we can find a, b in the subsemigroup as small as we want such that the set $\{a, b, ab, b(a+1)\}$ is monochromatic. We also show that the pattern $x, y, x+y, y/x$ is partition regular near zero.

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1. Introduction

A typical result in Ramsey theory asserts that a given configuration will be completely contained in one of the partition classes for any finite partition of some sufficiently large or rich structure. In the language of graph theory, Ramsey theory asserts that a given configuration will be monochromatic for any finite colouring of some sufficiently large or rich structure. Ramsey theory is the study of the preservation of configurations under set partitions, that is, Ramsey theoretic properties are partition regular.

In 1916, Schur [22] proved that there exist $a, b \in \mathbb{N}$ such that $\{a, b, a+b\}$ is partition regular and it is easy to prove that $\{a, b, a \cdot b\}$ is partition regular by the mapping $n \mapsto 2^n$. These two results are commonly referred to as the additive Schur theorem and the multiplicative Schur theorem, respectively. It was unknown for more than a century whether these two phenomena could be unified in a single statement. In 1979, Graham and Hindman independently proved such a result for two-colourings (see [11, pages 68–69], [13, Section 4]). Hindman conjectured that the statement should hold for any finite colouring, now known as the Hindman conjecture. In 2017,

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Moreira [18] answered this conjecture partially by showing that there exist $a, b \in \mathbb{N}$ such that $\{a, a \cdot b, a + b\}$ is partition regular (for a concise proof, see [1]). In 2017, Sahasrabudhe conjectured that $\{a, b, a(b + 1)\}$ is not partition regular (see [21, Question 31]). Recently, Goswami [10] disproved this conjecture and established the stronger result that there are always a and b such that the larger set $\{a, b, ab, b(a + 1)\}$ is monochromatic for any finite colouring of $\mathbb{N}$. In [19], Di Nasso *et al.* discussed the partition regularity of the configuration $x, y, x + y, y/x$ in $\mathbb{N}$.

In 1999, Hindman and Leader introduced the semigroup consisting of ultrafilters on $(0, 1)$ converging to zero and studied some strong results in Ramsey theory near zero (see [15] for more details). Many results near zero have been obtained in [2, 3, 7, 20]. Motivated by these, we establish Goswami's theorem near zero, that is, $\{a, b, ab, b(a + 1)\}$ is monochromatic near zero for dense subsemigroups S of $((0, \infty), +)$. This means that for any finite partition of $S \cap (0, 1)$, at least one partition cell contains $a, b, ab, b(a + 1)$. We also show that the pattern $x, y, x + y, y/x$ is partition regular near zero.

In Section 2, we present basic results which are used to prove our main results. In Section 3, we prove Goswami's theorem near zero using combinatorial arguments. In the final section, we prove that the pattern $x, y, x + y, y/x$ is partition regular near zero using algebraic arguments and ultrafilters.

2. Basic results

In this section, we present some preliminary concepts, definitions, theorems, conventions and results that will be used frequently later in this paper.

First, we give a brief introduction of the Stone–Čech compactification of a discrete semigroup. For any discrete semigroup $(S, \cdot)$, the Stone–Čech compactification βS of the discrete semigroup $(S, \cdot)$ is the set of all ultrafilters on S. The principal ultrafilters are identified with the points of S. For a subset A of S, we denote by $\bar{A}$ the set $\bar{A} = \{p \in \beta S : A \in p\}$. Then, the set $\{\bar{A} : A \subseteq S\}$ forms a clopen basis for a topology on βS. One can extend the operation $\cdot$ on S to the Stone–Čech compactification βS of S, which makes $(\beta S, \cdot)$ a compact, right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological centre (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). For $p, q \in \beta S$ and $A \subseteq S$, we have $A \in p \cdot q$ if and only if $\{x \in S : x^{-1}A \in q\} \in p$, where $x^{-1}A = \{y \in S : x \cdot y \in A\}$.

A nonempty subset I of a semigroup $(T, \cdot)$ is a *left ideal* of S if $T \cdot I \subseteq I$, a *right ideal* of S if $I \cdot T \subseteq I$, and a *two-sided ideal* (or simply an *ideal*) if it is both a left and a right ideal. A left ideal is *minimal* if it does not contain any proper left ideal. In a similar way, one can define a minimal right ideal and the smallest ideal. For any compact Hausdorff right topological semigroup $(T, \cdot)$, there is a smallest two-sided ideal $K(T)$:

$$K(T) = \bigcup \{L : L \text{ is a minimal left ideal of } T\}$$
$$= \bigcup \{R : R \text{ is a minimal right ideal of } T\}.$$

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For a given minimal left ideal L and minimal right ideal R, $L \cap R$ is a group and, hence, it contains an idempotent (an element $a \in S$ is an *idempotent* if $a = a \cdot a$). An idempotent belonging to the smallest ideal is minimal and conversely. We use $\mathcal{P}_f(X)$ to denote the set of all finite nonempty subsets of a set X and $E(S)$ to denote the set of all idempotents of S. For more details, see [16, Section 2.2].

Our results are derived for an arbitrary subsemigroup of $\mathbb{R}$ that is dense in $(0, \infty)$, for example, the set of positive rational numbers. In [15], Hindman and Leader introduced the semigroup consisting of ultrafilters converging to zero. For a dense subsemigroup S of $((0, \infty), +)$, define $0^+(S) = \{p \in \beta S : (0, \epsilon) \cap S \in p \text{ for all } \epsilon > 0\}$. It is necessary to take S as a dense subsemigroup of $((0, \infty), +)$; for otherwise, $0^+(S)$ will be empty.

THEOREM 2.1 [16, Lemma 13.29(f)]. *Let S be a dense subsemigroup of $((0, \infty), +)$ such that $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$. For each $y \in S \cap (0, 1)$ and each $x \in S$, assume that x/y and $yx \in S$. Then, $0^+(S)$ is a two-sided ideal of $\beta(S \cap (0, 1), \cdot)$.*

As $0^+(S)$ is a compact Hausdorff right topological semigroup, by [5, Theorem 1.3.11], $0^+(S)$ contains a smallest two-sided ideal, denoted by $K(0^+(S))$, and it must contain an idempotent by [8, Corollary 2.10]. As $0^+(S)$ is a two-sided ideal of $(\beta(0, 1)_d, \cdot)$, we have $K(\beta(0, 1)_d, \cdot) \subseteq 0^+(S)$ and, therefore, by [16, Theorem 1.65], $K(0^+(S), \cdot) = K(\beta(0, 1)_d, \cdot)$. Further, $0^+(S) \cap K(\beta S) = \emptyset$ and, hence, $0^+(S)$ provides some new information that is not available from $K(\beta S)$. After introducing the notion of a semigroup of ultrafilters near zero or converging to 0, Hindman and Leader studied some Ramsey theoretic results near zero in [15]. In [20], Patra and Shaikh gave the name HL semigroups to the semigroups satisfying the hypotheses of Theorem 2.1.

DEFINITION 2.2 [20, Definition 5]. Let S be a dense subsemigroup of $((0, \infty), +)$. Then, S is an *HL semigroup* if and only if $S \cap (0, 1)$ is a subsemigroup of $((0, 1), \cdot)$ and, for each $y \in S \cap (0, 1)$ and for each $x \in S$, x/y and $yx \in S$.

In an HL semigroup, the requirement $y < 1$ is not essential, but makes the proofs simpler since under that assumption, if $x < 1/n$, then $yx < 1/n$.

In [9], Furstenberg introduced a number of classes of large sets originating from topological dynamics. Alternative characterisations of such sets are also available in terms of the algebraic structure of $\beta\mathbb{N}$. Next, we introduce the IP sets and central sets which have rich combinatorial properties. There is an intimate relation between IP sets and idempotents. We need the following definitions to move forward.

DEFINITION 2.3. Let $\langle x_n \rangle_{n=1}^{\infty}$ be an infinite sequence of positive real numbers. Then, $\mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in F} x_n : F \in \mathcal{P}_f(\mathbb{N})\}$.

DEFINITION 2.4. A subset A of $\mathbb{N}$ is said to be an *IP-set* if and only if there exists a sequence $\langle x_n \rangle_{n=1}^{\infty}$ in $\mathbb{N}$ such that $\mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) \subseteq A$.

DEFINITION 2.5. Let $(S, \cdot)$ be a semigroup and let $A \subseteq S$. The set A is called *central* in $(S, \cdot)$ if there is some idempotent $p \in K(\beta S)$ such that $A \in p$.

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The next definition gives the near-zero version of the IP-set in $\mathbb{N}$. The IP-set near zero is an extremely important object that will be used to prove our results in the next section.

DEFINITION 2.6. Let S be a dense subsemigroup of $((0,\infty),+)$. A subset A of S is an *IP-set near zero* if and only if there exists a sequence $\langle x_n\rangle_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} x_n$ converges and $\mathrm{FS}(\langle x_n\rangle_{n=1}^{\infty}) \subseteq A$.

The following theorem shows that IP sets near zero are members of idempotents near zero.

THEOREM 2.7 [15, Theorem 3.1]. *Let S be a dense subsemigroup of $((0,\infty),+)$ and let $A \subseteq S$. Then, A is an IP-set near zero if and only if there is some idempotent p in $0^+(S)$ such that $A \in p$.*

The next definition gives the near-zero version of a central set.

DEFINITION 2.8. Let S be an HL semigroup. A subset A of S is a *central set near zero* if there is some idempotent $p \in K(0^+(S))$ such that $A \in p$.

In [4], Bergelson and Glasscock investigated the interplay between additive and multiplicative largeness. The following theorem shows that combined additive and multiplicative structure can be found in one partition cell for any finite partition of $(S \cap (0,1),\cdot)$.

THEOREM 2.9 [15, Theorem 5.6]. *Let S be an HL semigroup. Let $r \in \mathbb{N}$ and let $S \cap (0,1) = \bigcup_{i=1}^{r} B_i$. Then, there is some $i \in \{1,2,\ldots,r\}$ such that B_i is central near zero and B_i is central in $(S \cap (0,1),\cdot)$.*

As a consequence of the above theorem, we have the following corollary.

COROLLARY 2.10 [20, Corollary 1]. *Let S be an HL semigroup. Let $r \in \mathbb{N}$ and let $S \cap (0,1) = \bigcup_{i=1}^{r} B_i$. Then, there is some $i \in \{1,2,\ldots,r\}$ such that B_i is an IP-set near zero and B_i is central in $(S \cap (0,1),\cdot)$.*

3. Product and translated product near zero

In [10], Goswami proved that $\{a, b, ab, b(a+1)\}$ is monochromatic for every finite colouring of $\mathbb{N}$. His proof is similar to the argument of Moreira [18], who employed van der Waerden's theorem [23], while Goswami made iterative use of Hindman's theorem. In this section, our main target is to show that $\{a, b, ab, b(a+1)\}$ is monochromatic near zero for dense subsemigroups of $((0,\infty),+)$, which means we have to show that for any finite partition of $S \cap (0,1)$, at least one partition cell contains $a, b, ab, b(a+1)$. We shall follow similar arguments to those used by Goswami in [10]. Before going to our main theorem, we need the following definition.

DEFINITION 3.1. Let S be an HL semigroup and let $A \subseteq S$. Then, we denote

$$-y + A = \{x \in S : x + y \in A\} \quad \text{and} \quad y^{-1}A = \{x \in S : xy \in A\}.$$

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We need the following three lemmas to prove our main theorem. The first is a near-zero version of a reformulated Hindman's theorem [12].

LEMMA 3.2. *Let S be an HL semigroup. Let $A \subseteq S$ be an IP-set near zero. Then, there exists $y \in A$ such that $(-y + A) \cap A$ is also an IP-set near zero.*

PROOF. Since $A \subseteq S$ is an IP-set near zero, there exists $p \in E(0^+(S))$ such that $A \in p$. Then, $A^* = \{x \in A : -x + A \in p\} \in p$. Take $y \in A^*$. Then, $-y + A^* \in p$, so that $(-y + A^*) \cap A \in p$ and, thus, $(-y + A) \cap A \in p$. Therefore, $(-y + A) \cap A$ is an IP-set near zero. □

LEMMA 3.3. *Let S be an HL semigroup. Let $A \subseteq S$ be an IP-set near zero. Then, for any $y \in S \cap (0, 1)$, $y^{-1}A$ is also an IP-set near zero.*

PROOF. Let p be an idempotent of $0^+(S)$ such that $A \in p$. The function $f : S \to S$ given by $f(x) = y^{-1}x$ is an injective homomorphism and so is its continuous extension $\tilde{f} : \beta S \to \beta S$ by [16, Corollary 4.22 and Exercise 3.4.1]. So, $y^{-1}p$ is an idempotent of $0^+(S)$ and $y^{-1}A \in y^{-1}p$. Thus, $y^{-1}A$ is an IP-set near zero. □

LEMMA 3.4. *Let S be an HL semigroup. Let $r \in \mathbb{N}$ and $S = \bigcup_{i=1}^{r} A_i$ be a finite r-colouring of S. Let $A \subseteq S$ be an IP-set near zero. Then, there exists $i \in \{1, 2, 3, \ldots, r\}$ such that $A \cap A_i$ is also an IP-set near zero.*

PROOF. Choose $p \in E(0^+(S))$ such that $A \in p$. Now, $\bigcup_{i=1}^{r} A_i = S \in p$ and so $A_i \in p$ for some $i \in \{1, 2, 3, \ldots, r\}$. Thus, $A \cap A_i \in p$ for some $i \in \{1, 2, 3, \ldots, r\}$. Therefore, $A \cap A_i$ is an IP-set near zero. □

Next, we state Goswami's theorem and its near-zero version.

THEOREM 3.5 [10, Theorem 1.1]. *For any finite colouring of $\mathbb{N}$, there exist $a, b \in \mathbb{N}$ such that the set $\{a, b, ab, b(a + 1)\}$ is monochromatic.*

THEOREM 3.6. *Let S be an HL semigroup. For any finite colouring of S, there exist $a, b \in S$ such that the set $\{a, b, ab, b(a + 1)\}$ is monochromatic near zero.*

PROOF. Let $\epsilon > 0$ be given. Without loss of generality, assume that $0 < \epsilon < 1$. Suppose that S is finitely coloured, say $S = \bigcup_{i=1}^{r} A_i$. Without loss of generality, assume that A_1 is an IP-set near zero and set $D_0 = A_1$. By Lemma 3.2, we can choose an element $y_1 \in A_1 \cap (0, \epsilon/2)$ such that $(-y_1 + A_1) \cap A_1$ is an IP-set near zero. Then, by Lemma 3.3, $C_1 = y_1^{-1}((-y_1 + A_1) \cap A_1)$ is an IP-set near zero. Finally, by Lemma 3.4, there exists $i \in \{1, 2, 3, \ldots, r\}$ such that $D_1 = C_1 \cap A_{i_1} = y_1^{-1}((-y_1 + A_1) \cap A_1) \cap A_{i_1}$ is an IP-set near zero.

Using Lemmas 3.2 and 3.3, we can choose $y_2 \in D_1 \cap (0, \epsilon/2)$ such that $C_2 = y_2^{-1}((-y_2 + D_1) \cap D_1)$ is an IP-set near zero. Then, by Lemma 3.4, there exists $i \in \{1, 2, 3, \ldots, r\}$ such that $D_2 = C_2 \cap A_{i_2}$ is an IP-set near zero. Proceeding inductively, for each $n \in \mathbb{N}$, choose $y_n \in D_{n-1} \cap (0, \epsilon/2)$ and $i_n \in \{1, 2, 3, \ldots, r\}$ such that the set $D_n = y_n^{-1}((-y_n + D_{n-1}) \cap D_{n-1}) \cap A_{i_n}$ is an IP-set near zero.

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There exists $k \in \{1, 2, 3, \ldots, r\}$ such that $A_{i_j} = A_{i_n} = A_k$ for some indices $j < n$. Now, choose $x \in D_n \cap (0, \epsilon/2) \subseteq A_{i_n} = A_k$.

From the construction of D_n, we have $xy_n \in D_{n-1} \subseteq y_{n-1}^{-1} D_{n-2} \subseteq \cdots \subseteq y_{n-1}^{-1} \cdots y_{j+1}^{-1} D_j$. Thus, $xy_n y_{n-1} \cdots y_{j+1} \in D_j \subseteq A_{i_j} = A_k$. Moreover, since $y_n \in D_{n-1}$, by the same argument, $y_n y_{n-1} \cdots y_{j+1} \in D_j \subseteq A_{i_j} = A_k$. Thus, both $y_n y_{n-1} \cdots y_{j+1}$ and $xy_n y_{n-1} \cdots y_{j+1}$ lie in the same colour class A_k. Since $xy_n + y_n \in D_{n-1}$, by the same argument again, $(xy_n + y_n) y_{n-1} \cdots y_{j+1} \in D_j \subseteq A_{i_j}$.

Now, define $a = x$ and $b = y_n y_{n-1} \cdots y_{j+1}$. Then, $a, b \in A_k$ and, moreover, $xy_n y_{n-1} \cdots y_{j+1} = ab \in A_k$ and $(xy_n + y_n) y_{n-1} \cdots y_{j+1} = (a+1)b \in A_k$. Also, $0 < x < \epsilon/2$ and $0 < y_t < \epsilon/2$, for $j + 1 \leq t \leq n$, imply $0 < a < \epsilon/2 < \epsilon$ and $0 < b < \epsilon/2 < \epsilon$, and hence, $0 < ab < \epsilon/2 < \epsilon$ and $0 < (a+1)b < \epsilon$.

Therefore, the set $\{a, b, ab, b(a+1)\}$ is monochromatic near zero. This completes the proof. □

4. Sums and quotients near zero

In the previous section, we proved our results using combinatorial arguments. In this section, we give stronger results by an algebraic approach using tensor products of ultrafilters. We begin by recalling the definition of a tensor product.

DEFINITION 4.1 [17, Definition 1.1]. Let X and Y be discrete spaces. Let $p \in \beta X$ and $q \in \beta Y$. Then, the tensor product of p and q is defined by

$$p \otimes q = \{A \subseteq X \times Y : \{x \in X : \{y \in Y : (x, y) \in A\} \in q\} \in p\}.$$

It is easy to see that $A \times B \in p \otimes q$ if and only if $A \in p$ and $B \in q$. Next, we define a mapping that connects the tensor product with $0^+(S)$ for an HL semigroup.

DEFINITION 4.2. Let S be an HL semigroup and $s_0 \in S$ be a fixed element of S. Define a map

$$D : S \times S \to S \quad \text{by} \quad D(x, y) = \begin{cases} y/x & \text{if } x \in S \cap (0, 1), \\ s_0 & \text{otherwise.} \end{cases}$$

The following lemma shows that the image of a tensor product of $u \in 0^+(S)$ with itself under D is in $0^+(S)$.

LEMMA 4.3. *Let S be an HL semigroup. If $u \in 0^+(S)$, then $D(u \otimes u) \in 0^+(S)$.*

PROOF. Let $\epsilon > 0$ be given. Now,

$$\begin{aligned} D^{-1}[S \cap (0, \epsilon)] &= \{(x, y) \in S \times S : D(x, y) \in S \cap (0, \epsilon)\} \\ &\supseteq \{(x, y) \in S \times S : y/x \in S \cap (0, \epsilon)\} = \{(x, y) \in S \times S : y \in x(S \cap (0, \epsilon))\}. \end{aligned}$$

From the definition, $\{(x, y) \in S \times S : y \in x(S \cap (0, \epsilon))\} \in u \otimes u$ if and only if

$$\{x \in S : \{y \in S : y \in x(S \cap (0, \epsilon))\} \in u\} \in u,$$

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and this holds if and only if $\{x \in S : \{x(S \cap (0,\epsilon))\} \in u\} \in u$. Since $x(S \cap (0,\epsilon)) \in u$ for each $x \in S \cap (0,1)$, we have $D^{-1}[S \cap (0,\epsilon)] \in u \otimes u$. Thus, $S \cap (0,\epsilon) \in D(u \otimes u)$ for all $\epsilon > 0$ and so $D(u \otimes u) \in 0^+(S)$. □

We write $F < G$ for any $F, G \in P_f(\mathbb{N})$ if $\max F < \min G$.

LEMMA 4.4. *Let S be an HL semigroup and $A \subseteq S \times S$. Then, the following statements are equivalent.*

(1) *There is an idempotent $u \in 0^+(S)$ such that $A \in u \otimes u$.*
(2) *There is a convergent sequence $\langle x_n\rangle_{n=1}^{\infty}$ in S such that $\langle x_{n+1}/x_n\rangle_{n=1}^{\infty}$ is convergent, and $\{(\sum_{n\in F} x_n, \sum_{n\in G} x_n) : F, G \in P_f(\mathbb{N})$ and $F < G\} \subseteq A$.*

PROOF. This lemma is [6, Theorem 4.13] for $m = 2$. □

We are now in a position to prove the main result of this section. The proof is adapted from the proof of [19, Theorem 1.4].

THEOREM 4.5. *Let S be an HL semigroup and $u \in 0^+(S)$ be such that $u = u + u$. Then, for any $C \in D(u \otimes u)$, there exist sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that:*

(1) $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup \{\sum_{n\in G} x_n/\sum_{n\in F} x_n : F, G \in P_f(\mathbb{N})$ *and* $F < G\} \subseteq C$*; and*
(2) $\{\sum_{l\in F}\prod_{n=k}^{l} y_n : F \in P_f(\mathbb{N}), k \le \min F\} \subseteq C$.

PROOF. Since $C \in D(u \otimes u)$, we have $D^{-1}[C] \in u \otimes u$. By Lemma 4.4, there is a convergent sequence $\langle z_n\rangle_{n=1}^{\infty}$ with $\langle z_{n+1}/z_n\rangle_{n=1}^{\infty}$ convergent such that whenever $F, G \in P_f(\mathbb{N})$ and $F < G$, we have $(\sum_{n\in F} z_n, \sum_{n\in G} z_n) \in D^{-1}[C]$ and so $D(\sum_{n\in F} z_n, \sum_{n\in G} z_n) = \sum_{n\in G} z_n/\sum_{n\in F} z_n \in C$.

Now, set $x_n = z_{n+1}/z_1$ and $y_n = z_{n+1}/z_n$. Both the sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ are convergent. Also, for every $F, G \in P_f(\mathbb{N})$, we have $\sum_{n\in F} x_n = \sum_{n\in F} z_{n+1}/z_1$ and

$$\frac{\sum_{n\in G} x_n}{\sum_{n\in F} x_n} = \frac{\sum_{n\in G} z_{n+1}/z_1}{\sum_{n\in F} z_{n+1}/z_1} = \frac{\sum_{n\in G} z_{n+1}}{\sum_{n\in F} z_{n+1}}.$$

It follows that $\mathrm{FS}(\langle x_n\rangle_{n=1}^{\infty}) \cup \{\sum_{n\in G} x_n/\sum_{n\in F} x_n : F, G \in P_f(\mathbb{N})$ and $F < G\} \subseteq C$. Moreover, if $F \in P_f(\mathbb{N})$ and $k \le \min F$, analogously,

$$\sum_{l\in F}\prod_{n=k}^{l} y_n = \sum_{l\in F}\prod_{n=k}^{l}\frac{z_{n+1}}{z_n} = \sum_{l\in F}\frac{z_{l+1}}{z_k} = \frac{\sum_{l\in F} z_{l+1}}{z_k} \in C.$$ □

COROLLARY 4.6. *Let S be an HL semigroup. Then, for every finite colouring of $S = \bigcup_{i=1}^{r} C_i$, $r \in \mathbb{N}$, and $\epsilon > 0$, there are $m \le r$ convergent sequences $\langle x_n\rangle_{n=1}^{\infty}$ and $\langle y_n\rangle_{n=1}^{\infty}$ such that:*

(1) $FS(\langle x_n\rangle_{n=1}^{\infty}) \cup \{\sum_{n\in G} x_n/\sum_{n\in F} x_n : F, G \in P_f(\mathbb{N})$ *and* $F < G\} \subseteq C \cap (0,\epsilon)$*;*
(2) $\{\sum_{l\in F}\prod_{n=k}^{l} y_n : F \in P_f(\mathbb{N}), k \le \min F\} \subseteq C \cap (0,\epsilon)$.

PROOF. Take $u \in 0^+(S)$. By Lemma 4.3, $D(u \otimes u) \in 0^+(S)$. Therefore, choose $m < r$ such that $C_m \cap (0,\epsilon) \in D(u \otimes u)$ and use Theorem 4.5. □

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The following corollary shows that the pattern $x, y, x+y, y/x$ is partition regular near zero.

COROLLARY 4.7. *Let S be an HL semigroup. Then, for every finite colouring of $S = \bigcup_{i=1}^{r} C_i$, $r \in \mathbb{N}$, and $\epsilon > 0$, there exists $m < r$, $x, y \in S$ such that $\{x, y, x+y, y/x\} \subseteq C_m \cap (0, \epsilon)$.*

Note that Goswami's theorem near zero follows from Corollary 4.7, by taking $a = x, b = y/x$.

In 2011, Hindman proved that for every finite colouring of $\mathbb{N}$, there must exist a, b, c and d in $\mathbb{N}$ such that a, b, c and d are distinct and the colour of $a + b = cd$ is the same as that of a, b, c and d [14]. In 2020, Patra and Shaikh proved this result near zero [20]. In 2017, J. Moreira proved that for every finite colouring of $\mathbb{N}$, there exist $a, b \in \mathbb{N}$ such that the set $\{a, a \cdot b, a + b\}$ is monochromatic [18]. We end this work with the following question.

QUESTION 4.8. Let S be an HL semigroup. For any finite colouring of S, do there exist $a, b \in S$ such that the set $\{a, a \cdot b, a + b\}$ is monochromatic near zero?

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