J. Plasma Physics (2012), vol. 78, part 3, pp. 295–302. © Cambridge University Press 2012
doi:10.1017/S002237781200013X

Nonlinear waves and shocks in relativistic two-fluid hydrodynamics

L. HAIM[1], M. GEDALIN[1], A. SPITKOVSKY[2], V. KRASNOSELSKIKH[3] and M. BALIKHIN[4]

[1]Department of Physics, Ben-Gurion University of the Negev, Beer-Sheva, Israel (gedalin@bgu.ac.il)
[2]Princeton University, Princeton, NJ, USA
[3]LPCE/CNRS, Orleans, France
[4]ACSE, University of Sheffield, Sheffield, UK

(Received 23 November 2011; revised 9 January 2012; accepted 10 January 2012; first published online 10 February 2012)

Abstract. Relativistic shocks are present in a number of objects where violent processes are accompanied by relativistic outflows of plasma. The magnetization parameter $\sigma = B^2/4\pi nmc^2$ of the ambient medium varies in wide range. Shocks with low σ are expected to substantially enhance the magnetic fields in the shock front. In non-relativistic shocks the magnetic compression is limited by nonlinear effects related to the deceleration of flow. Two-fluid analysis of perpendicular relativistic shocks shows that the nonlinearities are suppressed for $\sigma \ll 1$ and the magnetic field reaches nearly equipartition values when the magnetic energy density is of the order of the ion energy density, $B_{eq}^2 \sim 4\pi nm_ic^2\gamma$. A large cross-shock potential $e\phi/m_ic^2\gamma_0 \sim B^2/B_{eq}^2$ develops across the electron–ion shock front. This potential is responsible for electron energization.

1. Introduction

Collisionless shocks are one of the most ubiquitous phenomena in the Universe and the only known to us is universal space accelerator of charged particles. Relativistic shocks are thought to be present in a number of objects where violent processes occur and are accompanied with relativistic outflows (Mimica *et al.* 2007), pulsar winds (Gallant and Arons 1994; Arons *et al.* 2005), gamma-ray bursts (Mochkovitch *et al.* 1995; Gallant *et al.* 1999; Daigne and Mochkovitch 2000; Mszros 2000; Piran 2004; Waxman 2006), quasars (Wardle *et al.* 1998), active galactic nuclei (Ulrich *et al.* 1997), relativistic jets (Bogovalov and Tsinganos 2005), and double radio sources (Kong *et al.* 1982). Relativistic shocks are expected to accelerate particles to ultra-relativistic energies (Hoshino *et al.* 1992; Gallant *et al.* 1999; Achterberg *et al.* 2001; Spitkovsky 2005; Nishikawa *et al.* 2006; Katz *et al.* 2007). There are plenty of *in situ* measurements of heliospheric shocks, while relativistic shocks in remote astrophysical objects can only be observed indirectly due to the emission of energized particles. The latter depends on the structure of the shock front, in particular, on magnetic enhancement. Heliospheric shocks are conveniently characterized according to their Alfvenic Mach number, $M = V_u/V_A$, where V_u is the speed of the upstream plasma flow in the shock rest frame, and $V_A = (B_u^2/4\pi n_u m_p)^{1/2}$ is the upstream Alfven speed. It is widely accepted to characterize relativistic shocks using the magnetization parameter, $\sigma = B^2/4\pi nmc^2$, where B and n are the magnetic field and plasma density in the plasma rest frame, respectively. The above expression for the Alfven speed is valid for low magnetization, $\sigma \ll 1$, or $V_A/c \ll 1$. The proper relativistic expression for the Alfven speed at arbitrary magnetization is $V_A/c = [\sigma/(1+\sigma)]^{1/2}$ (Lichnerowicz 1967; Kennel and Pellat 1976; Gedalin 1993). Therefore, the conventional Mach number of a relativistic shock would be $M \approx c/V_A = (1 + 1/\sigma)^{1/2}$. Low-magnetized, $\sigma \ll 1$, relativ-istic shocks are very high Mach number shocks with $M \approx 1/\sqrt{\sigma}$, while strongly magnetized, $\sigma \gg 1$, shocks are low Mach number shocks $M \approx 1 + 1/2\sigma$.

Nonlinear waves and shocks were extensively studied in the framework of magnetohydrodynamics (MHD) and two-fluid hydrodynamics. MHD theory of relativistic shocks is well known (Lessen and Deshpande 1967; Lichnerowicz 1967, 1970; Webb *et al.* 1987; Boillat and Ruggeri 1989; Komissarov 2003). The two-fluid approach is less developed, relative to what has been done for non-relativistic shocks (Sagdeev 1966; Kennel and Sagdeev 1967; Kennel *et al.* 1985; Gedalin 1998; Treumann 2009). A typical analysis sequence is as follows: one-dimensionality assumption, model equation construction, and stationary solution analysis with particular attention to solitons (Kennel and Pellat 1976; Alsop and Arons 1988; Langdon *et al.* 1988; Chiueh 1989; Chiueh and Lai 1991; Gallant *et al.* 1992; Lakhina and Verheest 1997). Although deviations from one dimensionality may be significant and are actually observed at heliospheric shocks with Mach numbers $M \gtrsim 5$ (Burgess 2006), it is widely believed that in

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

most cases one-dimensional waves are a good first-step approximation.

Basics of two-fluid analysis of nonlinear waves in relativistic plasmas were laid down by Kennel and Pellat (1976), who derived the general equations for stationary nonlinear waves in the two-fluid approach within a cold plasma limit. Alsop and Arons (1988) proposed that particles (pairs) make a loop inside a nonlinear wave, thus introducing a special kind of a hot distribution. Both Kennel and Pellat (1976) and Alsop and Arons (1988) have concluded that fast waves propagating perpendicular to the magnetic field are restricted to low amplitudes in relativistic limit and have shown that compressive solitons can exist in electron–positron plasma as well as ion–electron plasma, respectively, in the regime where the flow of cold plasma is perpendicular to the magnetic field. Kennel and Pellat (1976) suggested that in a highly nonlinear regime, the electron and ion inertia is determined by wave amplitude and not by rest mass so that electron–ion plasmas become similar to pair plasmas. They have, respectively, found a critical Mach number, that is $M_C = \sqrt{2(1+\gamma_0)}$, and the largest possible magnetic field amplitude, $B_{\max} = 1 + 2/\gamma_0$, above which a soliton solution does not exist. Generalization onto relativistic hot electron–ion plasmas was done by Chiueh (1989) and Chiueh and Lai (1991) by assuming the ion dominance. This approach is appropriate for oblique propagation but fails for perpendicular propagation, since in this case the electron current dominates (Sagdeev 1966). Yet, in the relativistic regime the perpendicular geometry is much more probable due to the γ-times enhancement of the perpendicular magnetic field when switching to the shock frame from the upstream plasma frame.

With the understanding of the role of filamentation instability in the magnetic field generation (Medvedev and Loeb 1999), the emphasis shifted from early simulations of magnetized pair shocks (Langdon *et al.* 1988; Gallant *et al.* 1992; Hoshino *et al.* 1992) to numerical simulations of shocks in very low-σ plasmas (Spitkovsky 2005; Wiersma and Achterberg 2005; Nishikawa *et al.* 2006; Chang *et al.* 2008; Spitkovsky 2008; Keshet *et al.* 2009). These very low-σ regimes (roughly $\sigma \lesssim 10^{-2}$, as found in numerical simulations) are profoundly non-stationary and do not possess any ordered structure similar to that of heliospheric shocks. Yet, low but not *very* low-σ shocks, which can still be expected to be well structured and are still of interest, fall outside the recent activity.

The objective of the present research is to study the properties of nonlinear, stationary, one-dimensional waves, with implications to shocks, in relativistic plasmas with emphasis on high and moderately low-σ regime, $\sigma \gtrsim 10^{-2}$. We shall treat only the case where the plasma flow is perpendicular to the magnetic field in the wave (shock) frame. We will be interested in the maximum density and magnetic compression, which can be achieved in solitons and shocks. The traditional approach is that the soliton amplitude determines the maximum achievable magnetic compression in the shock front. However, it has been shown (Gedalin 1998) that the structure of the shock front is not necessarily related to the structure of the soliton. It was also argued (Gedalin *et al.* 2008) that the magnetic compression and the shock thickness should *not* be determined by a soliton solution. Indeed, a soliton solution describes the regime when nonlinearity is balanced by dispersion while the hydrodynamical nature of the flow is maintained. However, when steepening proceeds to the scales smaller than the convective ion gyroradius, the ion motion is no longer adiabatic and kinetic effects become crucial. Thus, the achievable magnetic compression can be estimated at the onset of kinetic behavior.

The paper is organized as follows. We derive basic equations for stationary one-dimensional waves in multi-fluid plasmas in Sec. 2. Pair plasma is analyzed in Sec. 3 and electron–ion plasma is analyzed in Sec. 4.

2. Basic equations for one-dimensional stationary waves

We start with a full set of hydrodynamical equations for a relativistic fluid (Lichnerowicz 1967) for each species:

$$\partial_t(n\gamma) + \nabla(n\gamma \boldsymbol{v}) = 0, \tag{2.1}$$

$$\frac{n\gamma}{c^2}\left[\partial_t(w\gamma\boldsymbol{v}) + \boldsymbol{v}\cdot\nabla(w\gamma\boldsymbol{v})\right] = -\nabla p + qn\gamma[\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}/c] = 0, \tag{2.2}$$

$$n\gamma\left[\partial_t(w\gamma) + \boldsymbol{v}\cdot\nabla(w\gamma)\right] = \partial_t p + qn\gamma\boldsymbol{v}\cdot\boldsymbol{E} = 0. \tag{2.3}$$

Here n is the rest frame density, $w(n)$ is the specific enthalpy, and $p(n)$ is pressure. The latter two are related by the state equation, $dw = dp/n$. In what follows, we shall use the polytropic state equation of the form $p = p_0(n/n_0)^K$ so that

$$w = mc^2 + \frac{K}{K-1}\frac{p}{n}, \tag{2.4}$$

where m is the particle rest mass. The lab frame density n_l is related to the rest frame density as follows: $n_l = n\gamma$. Here the rest frame is the frame where the fluid is at rest, whereas lab frame denotes any other frame moving with the velocity v such that $\gamma = (1 - v^2/c^2)^{-1/2}$.

These equations should be completed with Maxwell's Equations:

$$\nabla\times\boldsymbol{B} = \frac{4\pi}{c}\sum_s q_s n_s \gamma_s \boldsymbol{v}_s + \frac{1}{c}\frac{\partial\boldsymbol{E}}{\partial t}, \tag{2.5}$$

$$\nabla\times\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{B}}{\partial t}, \tag{2.6}$$

$$\nabla\cdot\boldsymbol{B} = 0, \tag{2.7}$$

$$\nabla\cdot\boldsymbol{E} = 4\pi\sum_s q_s n_s \gamma_s. \tag{2.8}$$

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

Here s denotes plasma species. For brevity, we omit this index in what follows. In above equations magnetic field, electric field, plasma velocity, and charge density are measured in an arbitrary reference frame which will henceforth designate as the laboratory frame.

Let everything depends only on x, $(\partial/\partial t) = (\partial/\partial y) = (\partial/\partial z) = 0$. In what follows, $c = 1$ for brevity in all intermediate calculations and is restored in the end. Then, for each species,

$$\boldsymbol{u}_\perp = \gamma(v_y, v_z), \quad u_x = \gamma v_x, \tag{2.9}$$

$$\frac{d}{dx}(nu_x) = 0, \Rightarrow nu_x \equiv J = \text{const}, \tag{2.10}$$

$$\frac{d}{dx}\left(nwu_x^2\right) = -\frac{d}{dx}p + qn\gamma E_x + qn(u_y B_z - u_z B_y), \tag{2.11}$$

$$\frac{d}{dx}(nwu_x\boldsymbol{u}_\perp) = qn\gamma\boldsymbol{E}_\perp + qn(\boldsymbol{u}_\perp \times \hat{\boldsymbol{x}})B_x + qnu_x(\hat{\boldsymbol{x}} \times \boldsymbol{B}_\perp), \tag{2.12}$$

$$\frac{d}{dx}(nw\gamma u_x) = qnu_x E_x + qn\boldsymbol{u}_\perp \cdot \boldsymbol{E}_\perp, \tag{2.13}$$

$$\frac{d}{dx}\boldsymbol{B}_\perp = -\frac{4\pi}{c}\sum qn\hat{\boldsymbol{x}} \times \boldsymbol{u}_\perp, \tag{2.14}$$

$$E_x = -\frac{d}{dx}\phi, \quad \boldsymbol{E}_\perp = \text{const}, \tag{2.15}$$

$$\frac{d}{dx}E_x = \sum 4\pi qn\gamma, \tag{2.16}$$

$$\sum qnu_x = 0. \tag{2.17}$$

The last equation means that there is no current in the unperturbed (asymptotic) state. Combining these equations, one has

$$\frac{d}{dx}\left[\sum nw\gamma u_x + \frac{1}{4\pi}\hat{\boldsymbol{x}} \cdot (\boldsymbol{E}_\perp \times \boldsymbol{B}_\perp)\right] = 0 \tag{2.18}$$

or

$$\sum nw\gamma u_x + \frac{1}{4\pi}\hat{\boldsymbol{x}} \cdot (\boldsymbol{E}_\perp \times \boldsymbol{B}_\perp) \equiv Q = \text{const}. \tag{2.19}$$

Further, one has

$$\frac{d}{dx}\left(\sum nwu_x\boldsymbol{u}_\perp\right) = \left(\sum qn\gamma\right)\boldsymbol{E}_\perp + \left(\sum qn\boldsymbol{u}_\perp\right) \times \hat{\boldsymbol{x}}B_x + \left(\sum qnu_x\right)\hat{\boldsymbol{x}} \times \boldsymbol{B}_\perp, \tag{2.20}$$

$$\frac{d}{dx}\left(\sum nwu_x\boldsymbol{u}_\perp\right) = \frac{\boldsymbol{E}_\perp}{4\pi}\frac{d}{dx}E_x + \frac{B_x}{4\pi}\frac{d}{dx}\boldsymbol{B}_\perp, \tag{2.21}$$

$$\sum nwu_x\boldsymbol{u}_\perp - \frac{E_x\boldsymbol{E}_\perp}{4\pi} - \frac{B_x\boldsymbol{B}_\perp}{4\pi} \equiv \boldsymbol{S} = \text{const}. \tag{2.22}$$

Similarly, one also has

$$\sum\left(nwu_x^2 + p\right) - \frac{E_x^2 - B_\perp^2}{8\pi} \equiv P = \text{const}. \tag{2.23}$$

Now multiply equation

$$\frac{d}{dx}\left(nwu_x^2 + p\right) = qn\gamma E_x + qn\hat{\boldsymbol{x}} \cdot (\boldsymbol{u}_\perp \times \boldsymbol{B}_\perp) \tag{2.24}$$

with wu_x/q and sum up to get

$$\sum\frac{wu_x}{q}\frac{d}{dx}\left(nwu_x^2 + p\right) = \left(\sum nw\gamma u_x\right)E_x + \left(\sum nw\gamma\boldsymbol{u}_\perp\right) \cdot (\boldsymbol{B}_\perp \times \hat{\boldsymbol{x}}). \tag{2.25}$$

Using (2.19) one gets

$$\sum\frac{wu_x}{q}\frac{d}{dx}\left(nwu_x^2 + p\right) = QE_x. \tag{2.26}$$

Taking into account $dp = ndw$ one has

$$\phi = -\frac{1}{2Q}\sum nu_x w^2\left(u_x^2 + 1\right), \tag{2.27}$$

where $E_x = -d\phi/dx$.

In what follows, we restrict ourselves with the case of perpendicular propagation, $B_x = 0$, in a two-component (pair or electron–ion) plasma, where $q_1 = -q_2 = q$. Then, $nu_x = n_0u_0$ is same for both species. We are especially interested in the asymptotically homogeneous solutions, $d/dx \to 0$ when $x \to -\infty$. In this case we may set asymptotically $B_y = 0$, $B_z = B_0$, $u_y = u_z = 0$ for both species, and $u_x = u_0$, $\gamma = \gamma_0$ for both species. Then, $E_y = u_0B_0\gamma_0 = \beta_0B_0$, and $E_z = 0$ and $u_z = 0$ everywhere.

3. Pair plasma

We start the analysis with the case of a symmetric pair plasma, where $n_1 = n_2 = n$, $\gamma_1 = \gamma_2 = \gamma$, $w_1 = w_2 = w$, $p_1 = p_2 = p$, and $q_1 = -q_2 = q$. As a result, $E_x = 0$, $u_{x1} = u_{x2} = u_x$, and $u_{y1} = -u_{y2} = u_y$. It is convenient to normalize the variables as follows:

$$N = n/n_0, \quad W = w/w_0, \quad V_x = u_x/u_0, \quad V_y = u_y/u_0, \tag{3.1}$$

$$b = B/B_0, \quad P = p/p_0 = N^K. \tag{3.2}$$

We shall also define $T = p_0/n_0mc^2$ so that

$$w_0 = mc^2(1 + aT), \quad a = \frac{K}{K-1}, \tag{3.3}$$

$$W = \frac{1 + aTN^{K-1}}{1 + aT}. \tag{3.4}$$

With all this, $V_x = 1/N$ and the rest of the equations take the form

$$\gamma = \frac{\gamma_0}{W}\left[1 + \sigma(1-b)\right], \tag{3.5}$$

$$\left(1 - \frac{W}{N}\right) + \chi(1 - N^K) + \frac{\sigma}{2\beta_0^2}(1 - b^2) = 0, \tag{3.6}$$

$$\left(\frac{db}{dX}\right)^2 = \frac{N^2(\gamma^2 - 1)}{\gamma_0^2} - \beta_0^2 \equiv U(b), \tag{3.7}$$

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

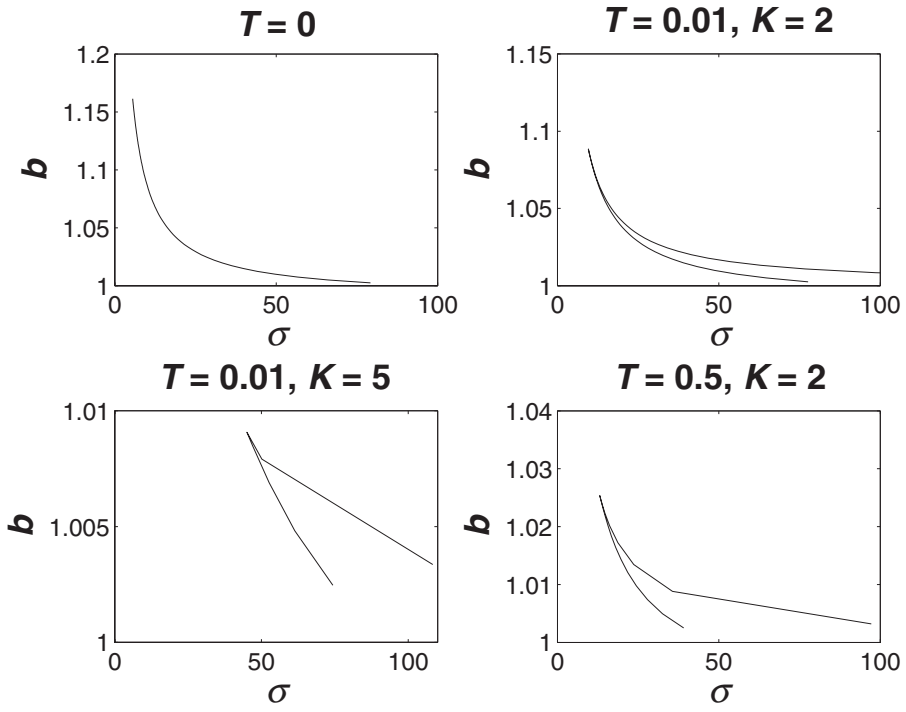


Figure 1. Maximum magnetic compression $b_{\max}$ as a function of σ for $\gamma_0 = 10$ and various T and K.

$$X = x/L, \quad L = \sigma R_g, \quad R_g = \frac{w_0\gamma_0}{qB_0}, \tag{3.8}$$

where

$$\sigma = \frac{B_0^2}{8\pi n_0 w_0 \gamma_0^2}, \quad \chi = \frac{p_0}{n_0 w_0 u_0^2} = \frac{T}{(1+aT)u_0^2}. \tag{3.9}$$

It is worth noting that n_0 is asymptotic density in the plasma rest frame (the one where the plasma is at rest). The shock frame (the one where the shock front is at rest and the plasma flows across the shock) asymptotic density is $n_s = n_0\gamma_0$. On the other hand, the plasma frame magnetic field is $B_r = B_0/\gamma_0$. Therefore, the above definition of σ differs from the one mentioned in the Introduction by the substitution of mc^2 (rest frame energy per particle in a cold plasma) to w_0 (rest frame energy per particle in a hot plasma). The generalized gyroradius in the rest frame of a hot plasma is $R_g = w_0\gamma_0/qB_0 = w_0/qB_r$.

It is easy to see that $U(b=1) = 0$ and $dU/db(b=1) = 0$ so that the asymptotic state is in equilibrium. In order to describe a solitary wave, $db/dx = 0$ and $d^2b/dx^2 = 0$ for $N = 1$ and $b = 1$, this should be an unstable equilibrium, that is $d^2U/db^2 > 0$ for $b = 1$ and $N = 1$. It can be shown that this condition means that the flow velocity should exceed the fast magnetosonic velocity at asymptotic point. The maximum amplitude b_m corresponds to $db/dx = 0$, that is,

$$N^2(\gamma^2 - 1) = u_0^2. \tag{3.10}$$

Together with (3.5) and (3.6), this relation provides algebraic equations for b and N. These equations determine maximum compressions $N_{\max}$ and $b_{\max}$ in the case where nonlinearity *can* be balanced by dispersion so that the derived algebraic relations describe the soliton maximum. For the cold pair plasma the equations are easily solved and give

$$b_{\max} = 2\beta_0\sqrt{1+1/\sigma} - 1. \tag{3.11}$$

The requirement $b_{\max} > 1$ brings back to the requirement $\beta_0 > 1/\sqrt{1+1/\sigma}$ or $\gamma_0 > \sqrt{1+\sigma}$, that is, the asymptotic velocity should exceed the fast magnetosonic speed. On the other hand, the requirements $\gamma \geqslant 1$ and $N > 0$ and $N < \infty$ give $\gamma_0 < 1 + 2\sigma$. For hot plasma it is possible to derive a parametric representation for b and σ as functions of N and γ_0,

$$\gamma = \left(1 + \frac{u_0^2}{N^2}\right), \tag{3.12}$$

$$b = \frac{2\beta_0^2\gamma_0}{\gamma_0 - \gamma W}\left[\left(1 - \frac{W}{N}\right) + \chi(1 - N^K)\right] - 1, \tag{3.13}$$

$$\sigma = \frac{\gamma_0 - \gamma W}{\gamma_0(b-1)}, \tag{3.14}$$

where we also require $b \geqslant 1$, $\gamma \geqslant 1$.

Figure 1 illustrates the dependence of $b_{\max}$ on σ for various values of T and K and $\gamma_0 = 10$. In all cases there is no solution for $\sigma \ll 1$ and the magnetic compression is weak. This is not surprising since for substantial σ the main source of energy is the magnetic field, thus small magnetic variations cause large density variations, which, in turn, are responsible for nonlinearity.

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

Therefore, balance is achieved at low magnetic field amplitudes. If σ is low then the energy source is plasma. Let us again consider the cold plasma case. As it follows from (3.6), the magnetic field is limited from above by $b^2 \lesssim 1/\sigma$, that is, $b \lesssim 1/\sqrt{\sigma}$. Thus, $\sigma b \lesssim \sqrt{\sigma} \ll 1$. Therefore, $\gamma/\gamma_0 - 1 \propto \sqrt{\sigma} \ll 1$, which means that the momentum of the flow does not change noticeably. Expanding N and γ in terms of small parameter σ, one has approximately

$$\frac{db}{dX} = \sqrt{\sigma}(b-1), \tag{3.15}$$

which means that the magnetic field grows in space in the approximately linear regime up to almost equipartition values. If the plasma consists of species with substantially different q/m, it can be expected that steepening can be stopped by strong dispersion in the whistler range (Krasnoselskikh 1985; Krasnosel'Skikh *et al.* 1991; Lobzin *et al.* 2007), at least for not very high Mach numbers. For the pair plasma this scenario does not exist and one can expect that low-σ shocks will be non-stationary.

4. Electron–ion plasma

A low-σ regime is the most interesting one from the point of view of application to shocks. Indeed, in this case most of the energy is in incident ions, and this energy may be, in principle, released to produce substantial electron energization and strong magnetic field enhancement. In most important applications the incident plasma is almost cold, thus we restrict our treatment to the case $T = 0$ leaving the finite temperature effects for elsewhere. In this case $\sigma = B_0^2/4\pi n_0 m_i c^2 \gamma_0^2$.

It is instructive to write down again the equations for the cold electron–ion plasma where the different masses are shown explicitly:

$$J\frac{d}{dx}(m_i u_{iy}) = qn_i\gamma_i E_y - qJB, \tag{4.1}$$

$$J\frac{d}{dx}(m_e u_{ey}) = -qn_e\gamma_e E_y + qJB, \tag{4.2}$$

$$J\frac{d}{dx}(m_i\gamma_i) = qJE_x + qn_i u_{iy} E_y, \tag{4.3}$$

$$J\frac{d}{dx}(m_e\gamma_e) = -qJE_x - qn_e u_{ey} E_y, \tag{4.4}$$

$$\frac{d}{dx}(Jm_i u_{ix}) = qn_i\gamma_i E_x + qn_i u_{iy} B, \tag{4.5}$$

$$\frac{d}{dx}(Jm_e u_{ex}) = -qn_e\gamma_e E_x - qn_e u_{ey} B, \tag{4.6}$$

$$\frac{d}{dx}B = -4\pi q(n_i u_{iy} - n_e u_{ey}), \tag{4.7}$$

$$\frac{d}{dx}E_x = 4\pi q(n_i\gamma_i - n_e\gamma_e), \tag{4.8}$$

$$J = n_i u_{ix} = n_e u_{ex} = n_0 u_0. \tag{4.9}$$

We are interested in the case where $\sigma \ll 1$ but still not within the range where filamentary instability dominates (Spitkovsky 2008). We expect that the cross-shock electric field E_x is substantially smaller than the motion electric field $E_y \approx B$, thus we assume quasi-neutrality $n_e\gamma_e \approx n_i\gamma_i$, which corresponds to the requirement

$$\frac{1}{4\pi q n_i\gamma_i}\left|\frac{d}{dx}E_x\right| \ll 1 \tag{4.10}$$

and immediately gives $v_{ex} \approx v_{ix}$. We shall further assume that electron current dominates, which simple means that electrons are much more easily deflected from straight trajectories than ions:

$$|n_i u_{iy}| \ll |n_e u_{ey}| \Rightarrow |v_{iy}| \ll |v_{ey}|, \tag{4.11}$$

which immediately gives

$$u_{ey} = \left(\frac{1}{4\pi q n_e}\right)\frac{dB}{dx} = \left(\frac{u_{ex}}{4\pi q J}\right)\frac{dB}{dx}. \tag{4.12}$$

According to what has been found to pair plasmas, in the low-σ case the ions are not significantly decelerated even when the magnetic field grows to rather higher values (approaching equipartition) so that $v_{xi} \approx 1$. We also expect that during this stage the electron energy remains substantially smaller than the ion energy, $m_e\gamma_e \ll m_i\gamma_i$. The latter assumption allows to proceed as in the non-relativistic case (see, e.g. Gedalin, 1998):

$$E_x = -v_{ey}B = -\left(\frac{v_{ex}}{4\pi q J}\right)B\frac{dB}{dx}. \tag{4.13}$$

If $|v_{ey}| \ll |v_{ex}|$, which means

$$\left|\left(\frac{1}{4\pi q J}\right)\frac{dB}{dx}\right| \ll 1, \tag{4.14}$$

then $v_{ex} \approx 1$ and

$$E_x = -\left(\frac{1}{8\pi q J}\right)\frac{dB^2}{dx}, \tag{4.15}$$

which means that the cross-shock potential is

$$\phi = \frac{B^2}{8\pi q J}. \tag{4.16}$$

Substituting u_{ey} into the equation for electrons, one has

$$\frac{m_e}{4\pi q J}u_{ex}\frac{d}{dx}\left(u_{ex}\frac{dB}{dx}\right) = -q\gamma_e\beta_0 B_0 + qu_{ex}B. \tag{4.17}$$

Defining $D = u_{ex}(d/dx)$ and taking into account that quasi-neutrality requires $u_{ex}/\gamma_e = u_{ix}/\gamma_i \equiv \beta_i$, one arrives at the following equation,

$$\frac{m_e}{4\pi q^2 J}DDB = \gamma_e(\beta_i B - \beta_0 B_0). \tag{4.18}$$

For $\gamma_e \gg 1$, $u_{xe} \gg 1, u_{ye}$, one has $u_{xe} \approx \gamma_e - 1/2\gamma_e$. Approximating similarly for ions, one has

$$m_i\gamma_i + m_e\gamma_e = (m_i + m_e)\gamma_0 + \frac{\beta_0 B_0(B_0 - B)}{4\pi J}, \tag{4.19}$$

$$\frac{m_i}{\gamma_i} + \frac{m_e}{\gamma_e} = \frac{(m_i + m_e)}{\gamma_0} + \frac{B^2 - B_0^2 - \beta_0 B_0(B - B_0)}{2\pi J}. \tag{4.20}$$

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

In the last expression $m_e\gamma_e \ll m_i\gamma_i$ so that one immediately has

$$\frac{m_i}{\gamma_i} \approx \frac{m_i}{\gamma_0} + \frac{B^2 - B_0^2 - \beta_0 B_0(B - B_0)}{2\pi J}, \tag{4.21}$$

or, with the same precision,

$$\frac{\gamma_i}{\gamma_0} \approx 1 - \frac{B^2 - B_0^2 - \beta_0 B_0(B - B_0)}{2\pi J m_i \gamma_0}. \tag{4.22}$$

Respectively, one has

$$\frac{\gamma_e}{\gamma_0} \approx 1 + \frac{2\left(B^2 - B_0^2\right) - 3\beta_0 B_0(B - B_0)}{4\pi J m_e \gamma_0}. \tag{4.23}$$

Even for $B^2/Jm_i\gamma_0 \ll 1$, as assumed, the ratio $B^2/Jm_e\gamma_0 \ll 1$ is m_i/m_e times greater and can be quite large so that $\gamma_e/\gamma_0 \gg 1$.

Taking into account

$$\beta_i = 1 - 1/2\gamma^2 \approx \beta_0 - \frac{B^2 - B_0^2 - \beta_0 B_0(B - B_0)}{2\pi J m_i \gamma_0^4} \tag{4.24}$$

one eventually has

$$\frac{m_e}{4\pi q^2 J} DDB = f(B) = -\frac{d\Psi}{dB}, \tag{4.25}$$

$$f(B) = \gamma_e(\beta_i B - \beta_0 B_0), \tag{4.26}$$

where the expressions for γ_e and β_i as functions of B are given by (4.23) and (4.24). The obtained equation has the shape of a pseudopotential equation and is a relativistic generalization of the equation for perpendicular fast magnetosonic waves in cold electron–ion plasmas (Sagdeev 1966). Since $d\gamma_e/dB(B = B_0) > 0$ and $d\beta_e/dB$ $(B = B_0) > 0$, we have $df/dB(B = B_0) > 0$, which means that $B = B_0$ is a minimum of pseudopotential. The derived equation is very similar to the equation derived for non-relativistic plasma. In non-relativistic plasma, at large amplitudes of magnetic perturbation, the flow velocity decreases faster than magnetic field increases, which results in the change of sign of $f(B)$ and the existence of a soliton solution. Here, β_i decreases very weakly because $\sigma \ll 1$ so that $f(B)$ is not expected to change sign for any reasonable amplitude of magnetic field. Instead, the magnetic field keeps growing until $B^2 \sim 4\pi n_0 m_i \gamma_0^2$, where the underlying assumptions break strongly.

5. Discussions and conclusions

The central point of our study is magnetic compression, which can be achieved in relativistically weak magnetized plasmas with implications for collisionless shocks in the range where filamentary instability is not of importance. We argue that the widely accepted approach to the shocks by constructing soliton solutions in two-fluid hydrodynamics is somewhat misleading, since a soliton is a reversible solitary structure where nonlinearity is balanced by dispersion, whereas a shock describes an irreversible transition accompanied by a drastic change of the plasma state, that is, non-adiabatic heating. Thus, the parameters of a soliton solution are not necessarily related to the parameters of a collisionless shock, which should be formed whenever the flow velocity exceeds the fast magnetosonic speed. Instead, the shock parameters, such as compression ratio, cross-shock electric field, and typical scale of shock transition, can be estimated from the analysis of the spatial growth of magnetic field up to the point where the underlying assumptions of two-fluid hydrodynamics are no longer valid and the kinetic effects start to play the decisive role. In the spirit of this approach, we have shown that the magnetic compression in a weakly magnetized pair plasma is determined by the asymptotic Lorentz factor of pairs and is not significantly affected by their temperature even if the latter is mildly relativistic. Analysis of electron–ion plasma required making a number of assumptions, most of which are the direct consequence of weak magnetization. The only approximation that breaks down when the magnetic field profile becomes steep is $|v_{ey}| \ll |v_{ex}|$. At this point the ion motion also becomes non-adiabatic and the whole hydrodynamic approach fails. We have shown that the magnetic compression is determined by the ion energy and is high, approaching the equipartition values, while the nonlinearity is still weak. The typical scale length at maximum compression is of the order of the relativistic electron inertial length.

A rough estimate of the condition for transition from steepening to filamentation in electron–ion shocks can be obtained by the comparison of the spatial growth rate of electron filamentation instability with electron gyroradius. The temporal growth rate in the plasma frame is $\Gamma \lesssim \omega_{pe}$, where $\omega_{pe}^2 = 4\pi n_0 e^2/m_e$, n_0 being the rest frame particle density. The Lorentz transformation gives spatial scale in shock frame, $1/L \sim \gamma_0\Gamma/c \lesssim \gamma_0\omega_{pe}/c$. The electron gyroradius is $r_e = m_e c^2\gamma_0/eB_0$. Thus, the transition to filamentation should occur when

$$L \sim \frac{c}{\gamma_0\omega_{pe}} \lesssim r_e = \frac{m_e c^2 \gamma_0}{eB_0} \Rightarrow \sigma \lesssim \gamma_0^2 \left(\frac{m_e}{m_i}\right). \tag{5.1}$$

This is only a rough estimate since it does not take into account possible ion–ion counterstreaming instabilities with magnetized electrons for $L > r_e$. This range is still poorly studied. In the range of σ described by (5.1) we expect that strong magnetic compression, up to almost equipartition values, $B \lesssim B_{eq} = (4\pi n_0 m_i c^2 \gamma_0^2)^{1/2}$, would occur without noticeable change of density. This occurs because decrease of momentum of ultra-relativistic particles is not accompanied by velocity change. The magnetic field increase should be accompanied by acceleration of electrons up to the energies $\gamma_e \sim B^2/B_{eq}^2$ and the cross-shock potential $e\phi/m_i c^2\gamma_0 \sim B^2/B_{eq}^2$. We expect that a low-$\sigma$ pair shock be non-stationary, while electron–ion shocks may exhibit a quasi-stationary structure with a whistler-like precursor. The last conclusion, however, is of somewhat speculative nature. This issue requires a more thorough study.

The work is partially supported by the ISSI grant 'Relativistic reconnection and collisionless shocks'. M.G.

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

and A.S. are partly supported by the BSF grant no. 2006095. A.S. was partially supported by the NSF grant AST-0807381.

References

Achterberg, A., Gallant, Y. A., Kirk, J. G. and Guthmann, A. W. 2001 Particle acceleration by ultrarelativistic shocks: theory and simulations. *MNRAS* **328**, 393–408.

Alsop, D. and Arons, J. 1988 Relativistic magnetosonic solitons with reflected particles in electron-positron plasmas. *Phys. Fluids* **31**, 839–847.

Arons, J., Backer, D. C., Spitkovsky, A. and Kaspi, V. M. 2005 Probing relativistic winds: the case of PSR J0737-3039 A and B. In: *Binary Radio Pulsars* (ed. F. A. Rasio and I. H. Stairs). Astronomical Society of the Pacific Conference Series, Vol. 328. San Francisco, CA: Astronomical Society of the Pacific, pp. 95.

Bogovalov, S. and Tsinganos, K. 2005 Shock formation at the magnetic collimation of relativistic jets. *MNRAS* **357**, 918–928.

Boillat, G. and Ruggeri, T. 1989 Wave and shock velocities in relativistic magnetohydrodynamics compared with the speed of light. *Contin. Mech. Thermodyn.* **1**, 47–52.

Burgess, D. 2006 Interpreting multipoint observations of substructure at the quasi-perpendicular bow shock: simulations. *J. Geophys. Res.* **111**, 10210, American Geophysical Union.

Chang, P., Spitkovsky, A. and Arons, J. 2008 Long-term evolution of magnetic turbulence in relativistic collisionless shocks: electron-positron plasmas. *Astrophys. J.* **674**, 378.

Chiueh, T. 1989 Relativistic solitons and shocks in magnetized e(-)-e(+)-p(+) fluids. *Phys. Rev. Lett.* **63**, 113–116.

Chiueh, T. and Lai, T.-C. 1991 Magnetosonic soliton in a relativistically hot plasma. *Phys. Rev.* A **44**, 6944–6947.

Daigne, F. and Mochkovitch, R. 2000 Gamma-ray bursts from internal shocks in a relativistic wind: a hydrodynamical study. *Astron. Astrophys.* **358**, 1157–1166.

Gallant, Y. A., Achterberg, A. and Kirk, J. G. 1999 Particle acceleration at ultra-relativistic shocks. Gamma-ray burst afterglow spectra and UHECRs. *Astron. Astrophys. Suppl. Ser.* **138**, 549–550.

Gallant, Y. A. and Arons, J. 1994 Structure of relativistic shocks in pulsar winds: a model of the wisps in the Crab Nebula. *Astrophys. J.* **435**, 230–260.

Gallant, Y. A., Hoshino, M., Langdon, A. B., Arons, J. and Max, C. E. 1992 Relativistic, perpendicular shocks in electron–positron plasmas. *Astrophys. J.* **391**, 73–101.

Gedalin, M. 1993 Linear waves in relativistic anisotropic magnetohydrodynamics. *Phys. Rev.* E **47**, 4354–4357.

Gedalin, M. 1998 Low-frequency nonlinear stationary waves and fast shocks: hydrodynamical description. *Phys. Plasmas* **5**, 127–132.

Gedalin, M., Balikhin, M. A. and Eichler, D. 2008 Efficient electron heating in relativistic shocks and gamma-ray-burst afterglow. *Phys. Rev.* E **77**(2), 026403.

Hoshino, M., Arons, J., Gallant, Y. A. and Langdon, A. B. 1992 Relativistic magnetosonic shock waves in synchrotron sources – shock structure and non-thermal acceleration of positrons. *Astrophys. J.* **390**, 454–479.

Katz, B., Keshet, U. and Waxman, E. 2007 Self-similar collisionless shocks. *Astrophys. J.* **655**, 375–390.

Kennel, C. F., Edmiston, J. P. and Hada, T. 1985 *A Quarter Century of Collisionless Shock Research.* Washington DC American Geophysical Union Geophysical Monograph Series Vol. 34. Washington, DC: AGU, pp. 1–36.

Kennel, C. F. and Pellat, R. 1976 Relativistic nonlinear plasma waves in a magnetic field. *J. Plasma Phys.* **15**, 335–355.

Kennel, C. F. and Sagdeev, R. Z. 1967 Collisionless shock waves in high plasmas, 1. *J. Geophys. Res.* **72**(13), 3303–3326.

Keshet, U., Katz, B. and Spitkovsky, A. 2009 Magnetic field evolution in relativistic unmagnetized collisionless shocks. *Astrophys. J.* **693**, L127–L130.

Komissarov, S. S. 2003 Limit shocks of relativistic magnetohydrodynamics. *MNRAS* **341**, 717–720.

Kong, X.-Y., Xu, Y.-H. and Zhong, Z.-X. 1982 Relativistic shock waves in double radio sources. *Acta Astrophys. Sin.* **2**, 81–90.

Krasnosel'skikh, V. V. 1985 The nonlinear motion of a plasma across a magnetic field. *Zhurnal Eksperimental noi i Teoreticheskoi Fiziki* **89**, 498–510.

Krasnosel'Skikh, V. V., Vinogradova, T., Balikhin, M. A., Alleyne, H. S. C., Pardaens, A. K., Woolliscroft, L. J. C., Klimov, S. I., Petrukovich, A., Mier-Jedrzejowicz, W. A. C. and Southwood, D. J. 1991 On the nature of low frequency turbulence in the foot of strong quasi-perpendicular shocks. *Adv. Space Res.* **11**, 15–18.

Lakhina, G. S. and Verheest, F. 1997 Alfvnic solitons in ultrarelativistic electron–positron plasmas. *Astrophys. Space Sci.* **253**, 97–106.

Langdon, A. B., Arons, J. and Max, C. E. 1988 Structure of relativistic magnetosonic shocks in electron-positron plasmas. *Phys. Rev. Lett.* **61**, 779–782.

Lessen, M. and Deshpande, N. V. 1967 Stability of relativistic magnetohydrodynamic shock waves. *Phys. Fluids* **10**, 782–785.

Lichnerowicz, A. 1967 *Relativistic Hydrodynamics and Magnetohydrodynamics*. New York: Benjamin.

Lichnerowicz, A. 1970 Shock waves in relativistic magnetohydrodynamics. *Phys. Scr.* **2**, 221–225.

Lobzin, V. V., Krasnoselskikh, V. V., Bosqued, J.-M., Pinon, J.-L., Schwartz, S. J. and Dunlop, M. 2007 Non-stationarity and reformation of high-Mach-number quasiperpendicular shocks: cluster observations. *Geophys. Res. Lett.* **340**, 5107.

Medvedev, M. V. and Loeb, A. 1999 Generation of magnetic fields in the relativistic shock of gamma-ray burst sources. *Astrophys. J.* **526**(2), 697–706.

Mimica, P., Aloy, M. A. and Mller, E. 2007 Internal shocks in relativistic outflows: collisions of magnetized shells. *Astron. Astrophys.* **466**, 93–106.

Mochkovitch, R., Maitia, V. and Marques, R. 1995 Internal shocks in a relativistic wind as a source for gamma-ray bursts? *Astrophys. Space Sci.* **231**, 441–444.

Mszros, P. 2000 Gamma-ray bursts and bursters. *Nucl. Phys.* B. **80**, 63–77.

Nishikawa, K.-I., Hardee, P., Hededal, C. B., Richardson, G., Preece, R., Sol, H. and Fishman, G. J. 2006 Particle acceleration, magnetic field generation, and emission in relativistic shocks. *Adv. Space Res.* **38**, 1316–1319.

Piran, T. 2004 The physics of gamma-ray bursts. *Rev. Mod. Phys.* **76**, 1143–1210.

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press

Sagdeev, R. Z. 1966 Cooperative phenomena and shock waves in collisionless plasmas. *Rev. Plasma Phys.* **4**, 23.

Spitkovsky, A. 2005 Simulations of relativistic collisionless shocks: shock structure and particle acceleration. In: *Astrophysical Sources of High Energy Particles and Radiation* (ed. T. Bulik, B. Rudak, and G. Madejski). American Institute of Physics Conference Series, Vol. 801. College Park, MD: AIP, pp. 345–350.

Spitkovsky, A. 2008 On the structure of relativistic collisionless shocks in electron-ion plasmas. *Astrophys. J.* **673**(1), L39–L42.

Treumann, R. A. 2009 Fundamentals of collisionless shocks for astrophysical application. 1. Non-relativistic shocks. *Astron. Astrophys. Rev.* **17**(4), 409–535.

Ulrich, M.-H., Maraschi, L. and Urry, C. M. 1997 Variability of active galactic nuclei. *Annu. Rev. Astron. Astrophys.* **35**, 445–502.

Wardle, J. F. C., Homan, D. C., Ojha, R. and Roberts, D. H. 1998 Electron-positron jets associated with the quasar 3C279. *Nature* **395**, 457–461.

Waxman, E. 2006 Gamma-ray bursts and collisionless shocks. *Plasma Phys. Control. Fusion* **48**, B137–B151.

Webb, G. M., Zank, G. P. and McKenzie, J. F. 1987 Relativistic oblique magnetohydrodynamic shocks. *J. Plasma Phys.* **37**, 117–141.

Wiersma, J. and Achterberg, A. 2005 Magnetic field generation in relativistic shocks. *Nuovo Cimento C Geophys. Space Phys. C* **28**, 459.

https://doi.org/10.1017/S002237781200013X Published online by Cambridge University Press