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LINEAR-QUADRATIC OPTIMIZATION FOR MODELS WITH RATIONAL EXPECTATIONS

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We present a method for using rational expectations in a linear-quadratic optimization framework. Following the approach put forward by Sims, we solve the model through a QZ decomposition, which is generally easier to implement than the more widely used Blanchard-Kahn method.

Keywords: Rational Expectations, Stochastic Optimization, Numerical Experiments, Linear Quadratic Optimization

1. INTRODUCTION

Starting in the late 1970's, there was much criticism of the use of control theory in economics. In this view, one of the major drawbacks was that it could not deal with rational expectations (RE). Subsequently, a number of generic methods to solve models with RE were developed. For instance, Fair and Taylor (1983) use an iterative method for solving RE models and, in the tradition of Theil (1968), Fisher et al. (1986) use a method based on stacking the model variables. A hybrid method based on the saddlepoint property is presented by Anderson and Moore (1985).

The Blanchard and Kahn (BK) method (1980) is another well-known method for solving linear models with RE in discrete time. By decomposing the model into stable and unstable parts, the unstable part can be solved backward in time and the stable part forward in time. Although the BK approach is theoretically a powerful method, for practical implementation it has some serious drawbacks. First, not all linear RE models can be put into first-order linear form as required by the BK method and, second, the method is based on the *Jordan canonical form*. The procedure to put the model into the Jordan canonical is not widely available in software libraries and is known to be numerically unstable [see Moler (1994)].

In recent work [Amman et al. (1995), Amman (1996)], we presented a procedure that introduces RE in a linear-quadratic (LQ) control framework based on the BK

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method. Because of the limitations of the BK approach and the fact that we had to rely on the diagonalization of the transition matrix, this work could only deal with a limited set of models. Recently, Sims (1996) proposed a different method for solving linear models with RE, allowing for a broader range of models. That method is not based on the Jordan canonical form, but uses the more widely available QZ form that is based on generalized eigenvalues and which is more numerically stable. We follow the paper of Sims (1996) and incorporate his approach into the LQ framework.

2. PROBLEM STATEMENT AND SOLUTION

Following Kendrick (1981), the standard single-agent linear-quadratic optimization problem is written as described next.

Find the set of admissible instruments $U = \{u_0, u_1, \ldots, u_{T-1}\}$ that minimizes the welfare loss function

$$J_T = \beta^T L_T(x_T) + \sum_{t=0}^{T-1} \beta^t L_t(x_t, u_t) \tag{1}$$

with

$$L_T = \tfrac{1}{2}(x_T - \bar{x}_T)' W_T (x_T - \bar{x}),$$

$$L_t = \tfrac{1}{2}(x_t - \bar{x}_t)' W_t (x_t - \bar{x}_t) + \tfrac{1}{2}(u_t - \bar{u}_t)' R_t (u_t - \bar{u}_t) + (x_t - \bar{x}_t)' F_t (u_t - \bar{u}_t),$$

subject to the model

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t. \tag{2}$$

The vector $x_t \in \Re^n$ is the state of the economy at time t and the vector $u_t \in \Re^m$ contains the policy instruments. The initial state of the economy x_0 is known; $\bar{x}_t$ and $\bar{u}_t$ are target values. W_t, R_t, and F_t are penalty matrices of conformable size, β being a discount factor.

The above model is straightforward to solve and there are a number of packages available. However, a serious drawback for economics is that (2) does not allow for RE. One way of allowing RE to enter the model is to augment (2) in the following fashion:

$$x_{t+1} = A_t x_t + B_t u_t + C_t z_t + \sum_{j=1}^{k} D_{j,t} E_t x_{t+j} + \epsilon_t. \tag{2a}$$

where the matrix $D_{j,t}$ is a parameter matrix, $E_t x_{t+1}$ is the expected state for time $t + j$ at time t, k is the maximum lead in the expectations formation, and ϵ_t is a white-noise vector. To compute the admissible set of instruments, we have to eliminate the RE from the model. In a previous paper [Amman et al. (1995)], we

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used the BK method to solve the RE in the model. However, as mentioned above, the BK method has some features that impede practical implementation. Sims (1996) proposes a different route by applying a *generalized eigenvalue* approach. To apply Sims' method, we first put (2a) in the form[1]

$$\Gamma_{0,t}\tilde{x}_{t+1} = \Gamma_{1,t}\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t, \tag{3}$$

where

$$\Gamma_{0,t} = \begin{bmatrix} I - D_{1,t} & -D_{2,t} & \cdots & -D_{k-1,t} & -D_{k,t} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & & \ddots & 0 & 0 \\ 0 & \cdots & & I & 0 \end{bmatrix},$$

$$\Gamma_{1,t} = \begin{bmatrix} A_t & 0 & 0 & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & & 0 \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & 0 & & I \end{bmatrix}, \quad \Gamma_{2,t} = \begin{bmatrix} B_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \Gamma_{3,t} = \begin{bmatrix} C_t \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \Gamma_4 = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

and the augmented state vector

$$\tilde{x}_t = \begin{bmatrix} x_t \\ Ex_{t+1} \\ Ex_{t+2} \\ \vdots \\ Ex_{t+k-1} \end{bmatrix}. \tag{4}$$

Taking the generalized eigenvalues of (3) allows us to decompose the system matrices $\Gamma_{0,t}$ and $\Gamma_{1,t}$ in the following manner, viz. Coleman and van Loan (1988) or Moler and Stewart (1973):

$$\Lambda_t = Q_t\Gamma_{0,t}Z_t,$$
$$\Omega_t = Q_t\Gamma_{1,t}Z_t,$$

with $Z_t'Z_t = I$ and $Q_t'Q_t = I$. The matrices Λ_t and Ω_t are upper triangular matrices and the generalized eigenvalues are $\forall\, i\ \omega_{i,i}/\lambda_{i,i}$. If we use the transformation $w_t = Z_t'\tilde{x}_t$ and $w_{t+1} = Z_t'\tilde{x}_{t+1}$ we can write (3) as (see Appendix A)

$$\Lambda_t w_{t+1} = \Omega_t w_t + Q_t\Gamma_{2,t}u_t + Q_t\Gamma_{3,t}z_t + Q_t\Gamma_4\epsilon_t. \tag{5}$$

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It is possible to reorder the matrices Z_t and Q_t in such a fashion that the diagonal elements of the matrices Λ_t and Ω_t contain the generalized eigenvalues in ascending order. In that case, we can write (4) as follows:

$$\begin{bmatrix} \Lambda_{11,t} & \Lambda_{12,t} \\ 0 & \Lambda_{22,t} \end{bmatrix} \begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \Omega_{11,t} & \Omega_{12,t} \\ 0 & \Omega_{22,t} \end{bmatrix} \begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{2,t} u_t + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_{3,t} z_t + \begin{bmatrix} Q_{1,t} \\ Q_{2,t} \end{bmatrix} \Gamma_4 \epsilon_t, \tag{6}$$

where the unstable eigenvalues are in the lower right corner, that is, the matrices $\Lambda_{22,t}$ and $\Omega_{22,t}$. By forward propagation and taking expectations, it is possible to derive $w_{2,t}$ as a function of future instruments and exogenous variables Sims (1996, p. 5):

$$\gamma_t = w_{2,t} = -\sum_{j=0}^{\infty} \tilde{M}_{t,j} \Omega_{22,t+j}^{-1} Q_{2,t+j} (\Gamma_{2,t+j} u_{t+j} + \Gamma_{3,t+j} z_{t+j}) \tag{7}$$

(see Appendix B). The matrix $\tilde{M}_{t,j}$ is defined as

$$\tilde{M}_{t,j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for} \quad j > 0$$

and

$$\tilde{M}_{t,j} = I \quad \text{for} \quad j = 0$$

with

$$M_t = \Omega_{22,t}^{-1} \Lambda_{22,t}.$$

Given the fact that $\Omega_{22,t}$ contains the eigenvalues outside the unit circle, we have applied the follow condition in deriving (7):

$$\lim_{j \to \infty} \tilde{M}_{t,j} = 0.$$

In contrast to Sims' method, M_t is not time invariant because we explicity want to allow for time-dependent matrices in the model.[2] Reinserting (7) into (6) gives us

$$\tilde{\Lambda}_t w_{t+1} = \tilde{\Omega}_t w_t + \tilde{\Gamma}_{2,t} u_t + \tilde{\Gamma}_{3,t} z_t + \tilde{\Gamma}_{4,t} \epsilon_t + \tilde{\gamma}_t \tag{8}$$

with

$$\tilde{\Lambda} = \begin{bmatrix} \Lambda_{11,t} & \Lambda_{12,t} \\ 0 & I \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} \Omega_{11,t} & \Omega_{12,t} \\ 0 & 0 \end{bmatrix}, \quad \tilde{\Gamma}_{2,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{2,t},$$

$$\tilde{\Gamma}_{3,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_{3,t}, \quad \tilde{\Gamma}_{4,t} = \begin{bmatrix} Q_{1,t} \\ 0 \end{bmatrix} \Gamma_4, \quad \tilde{\gamma}_t = \begin{bmatrix} 0 \\ \gamma_t \end{bmatrix}.$$

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Knowing that $\tilde{x}_t = Z_t w_t$ and $\tilde{x}_{t+1} = Z_t w_{t+1}$, we can write (8) as

$$\tilde{x}_{t+1} = \tilde{A}_t \tilde{x}_t + \tilde{B}_t u_t + \tilde{C}_t \tilde{z}_t + \tilde{\epsilon}_t \tag{9}$$

with

$$\tilde{A}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Omega}_t Z_t', \quad \tilde{B}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{2,t}, \quad \tilde{C}_t = \begin{bmatrix} Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{3,t} & Z_t \tilde{\Lambda}_t^{-1} \end{bmatrix} \tag{10}$$

and

$$\tilde{\Lambda}_t^{-1} = \begin{bmatrix} \Lambda_{11,t}^{-1} & -\Lambda_{11,t}^{-1} \Lambda_{12,t} \\ 0 & I \end{bmatrix}, \quad \tilde{z}_t = \begin{bmatrix} z_t \\ \tilde{\gamma}_t \end{bmatrix}, \quad \tilde{\epsilon}_t = Z_t \tilde{\Lambda}_t^{-1} \tilde{\Gamma}_{4,t} \epsilon_t. \tag{11}$$

In nonpathological cases [Stewart (1973)], the matrix $\Lambda_{11,t}$ will be nonsingular. With (9), we have transformed (2a) into the form required by (2), enabling us to set up an iterative scheme using the LQ framework in (1) and (2). Knowing that γ_t depends on the future, the basic algorithm works like this[3]:

Step 0. Set the iteration count $\nu = 0$ and set the instruments $u_t^\nu, t = \{0, 1, \ldots, T + s - 1\}$; set $\tilde{x}_0$ (see below).

Step 1. Compute γ_t^ν, $t = \{0, 1, \ldots, T + s - 1\}$, by using

$$\gamma_t^\nu = -\sum_{j=0}^{\infty} \tilde{M}_{t,j} \Omega_{22,t+j}^{-1} Q_{2,t+j} \left(\Gamma_{2,t+j} u_{t+j}^\nu + \Gamma_{3,t+j} z_{t+j} \right).$$

Step 2. Apply the LQ framework in (1) and (2) to compute a new set of optimal instruments $u_t^{\nu+1}$ using the equation below in place of (2):

$$\tilde{x}_{t+1}^{\nu+1} = \tilde{A}_t \tilde{x}_t^{\nu+1} + \tilde{B}_t u_t^{\nu+1} + \tilde{C}_t \tilde{z}_t^\nu.$$

Step 3. Set $\nu = \nu + 1$ and go to step 1 until convergence has been reached.

There is still one remaining issue that we have to take care of. To apply the algorithm for $\nu = 0$, we need to have an initial value of the augmented state vector, which is

$$\tilde{x}_0 = \begin{bmatrix} x_0 \\ Ex_1 \\ Ex_2 \\ \vdots \\ Ex_{k-1} \end{bmatrix}. \tag{12}$$

However, at the beginning of the first iteration step, the expectational variables are not known, and so, we have to give the expectational part of $\tilde{x}_0^0$ an arbitrary value. Once we have completed the first iterations, we can use the computed states to reinitialize $\tilde{x}_0^1$. By iterating on the expectations, the above algorithm produces a expectation-consistent solution of the model formulated in (1) and (2a).[4] In the next section, we apply this algorithm using a simple macro model.

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3. AN EXAMPLE

Consider a simple macro model with output x_t, consumption c_t, investment i_t, government expenditures g_t, and taxes τ_t. The problem then can be stated as follows:

Find for the model

$$x_{t+1} = c_{t+1} + i_{t+1} + g_{t+1}, \tag{13}$$

$$c_{t+1} = 0.8(x_t - \tau_t) + 200, \tag{14}$$

$$i_{t+1} = 0.2E_t x_{t+2} + 100 + \epsilon_t, \tag{15}$$

$$g_{t+1} = u_t, \tag{16}$$

$$\tau_{t+1} = 0.25x_{t+1}. \tag{17}$$

with $x_0 = 1500$, a set of admissible control $U = \{u_0, u_1, \ldots, u_9\}$ to minimize the welfare loss function

$$J_T = \frac{1}{2}(x_{10} - 1600)^2 + \frac{1}{2}\sum_{t=0}^{9}\left[(x_t - 1600)^2 + g_t^2\right]. \tag{18}$$

If we reduce the above model to one equation for output, we get

$$x_{t+1} = 0.6x_t + u_t + 0.2E_t x_{t+2} + 300 + \epsilon_t, \tag{19}$$

which leads to the augmented system

$$\begin{bmatrix} 1 & -0.2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_{t+1} \\ E_t x_{t+2} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_t \\ x_{t+1} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u_t + \begin{bmatrix} 300 \\ 0 \end{bmatrix}z_t + \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}, \tag{20}$$

where $z_t = 1\ \forall t$. Applying the QZ factorization [Coleman and van Loan (1988)], to compute the generalized eigenvalues of the model gives us the time-invariant solution

$$\Lambda = \begin{bmatrix} 1.0822 & -0.9136 \\ 0 & 0.1848 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0.7546 & 0.3979 \\ 0 & 0.7951 \end{bmatrix}, \tag{21}$$

$$Z = \begin{bmatrix} 0.8202 & -0.5719 \\ 0.5719 & 0.8203 \end{bmatrix}, \quad Q = \begin{bmatrix} 0.6523 & 0.7580 \\ -0.7580 & 0.6523 \end{bmatrix}, \tag{22}$$

and so, the eigenvalues are $\{0.7546/1.0822, 0.7952/0.1848\} = \{0.6972, 4.3028\}$ and apparently the ordering of the system is such that the unstable root 4.3028 is

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TABLE 1. Solution of the LQ optimization model with RE

t	0	1	2	3	4	5	6	7	8	9	10
x_t	1500	1556	1576	1584	1587	1588	1589	1589	1587	1584	1578
u_t	40	26	21	19	18	18	18	17	16	11	

in the lower right corner of Λ and Ω. The other components are

$$\tilde{A} = \begin{bmatrix} 0.2966 & 0.5745 \\ 0.2068 & 0.4006 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0.4944 \\ 0.3447 \end{bmatrix}, \tag{23}$$

$$\tilde{C} = \begin{bmatrix} 148.3243 & 0.7580 & 0.1205 \\ 103.4153 & 0.5285 & 1.3031 \end{bmatrix}, \tag{24}$$

$$\tilde{W} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = [1], \quad \tilde{F} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{25}$$

$$\tilde{x}_0 = \begin{bmatrix} 1500 \\ 1500 \end{bmatrix}, \quad \tilde{\bar{x}}_0 = \begin{bmatrix} 1600 \\ 0 \end{bmatrix}, \quad \bar{u} = [0]. \tag{26}$$

Note that we have set $E_0 x_1^0 = 1500$. For this simple model, we can compute the steady state of the system being $x_\infty = 1589.08$ and $u_\infty = 17.81$; see Amman and Kendrick (1998). Hence, $U^0 = \{17.81, \ldots, 17.81\}$ is an educated guess for the instruments. The solution to the model is in Table 1.

4. SUMMARY

We have presented a single-agent linear-quadratic optimization model that allows for rational expectations. Based on Sims's paper, we have used a generalized eigenvalue method for solving the variables that involve the unstable roots. By using an iterative scheme, the reduced model can be fitted into a standard linear-quadratic framework that allows us to derive the optimal policy instruments for the model with rational expectations.

NOTES

1. Note that, in contrast to Sims (1996), the variable z_t contains exogenous variables and not random variables. Hence, the matrix Π in Sims's paper is set to zero.

2. As Lucas (1976) pointed out, the parameters in the model might change because of policy announcements. Hence we would like our method to be able to deal with changing parameters in the system matrices.

3. The MATLAB implementation of this algorithm can be obtained through the corresponding author.

4. It is beyond the scope of this paper to address fully the issue of time consistency. The above algorithm produces a time-consistent, but not necessarily fully optimal, solution of the model. See Kydland and Prescott (1977) or Hall and Henry (1988).

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APPENDIX A: DERIVATION OF EQUATION 5

Begin with (3), that is,

$$\Gamma_{0,t}\tilde{x}_{t+1} = \Gamma_{1,t}\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t. \tag{A.1}$$

From the equation just above (5),

$$\Lambda_t = Q_t\Gamma_{0,t}Z_t, \tag{A.2}$$

$$\Omega_t = Q_t\Gamma_{1,t}Z_t. \tag{A.3}$$

Premultiply (A.2) and (A.3) by Q_t' and postmultiply them by Z_t', keeping in mind that

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$Z_t'Z_t = I$ and $Q_t'Q_t = I$ yields

$$Q_t'\Lambda_t Z_t' = \Gamma_{0,t}, \tag{A.4}$$

$$Q_t'\Omega_t Z_t' = \Gamma_{1,t}. \tag{A.5}$$

Then, substitute (A.4) and (A.5) into (A.1) to obtain

$$Q_t'\Lambda_t Z_t'\tilde{x}_{t+1} = Q_t'\Omega_t Z_t'\tilde{x}_t + \Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t. \tag{A.6}$$

Premultiplication of (A.6) by Q_t and use of the definitions

$$w_{t+1} = Z_t'\tilde{x}_{t+1}, \tag{A.7}$$

$$w_t = Z_t'\tilde{x}_t \tag{A.8}$$

yields

$$\Lambda_t w_{t+1} = \Omega_t w_t + Q_t\Gamma_{2,t}u_t + Q_t\Gamma_{3,t}z_t + Q_t\Gamma_4\epsilon_t, \tag{A.9}$$

which is (5) in the body of the paper.

APPENDIX B: DERIVATION OF EQUATION 7

Begin with the bottom half of (6), that is,

$$\Lambda_{22,t}w_{2,t+1} = \Omega_{22,t}w_{2,t} + Q_{2t}\Gamma_{2,t}u_t + Q_{2t}\Gamma_{3,t}z_t + Q_{2t}\Gamma_4\epsilon_t. \tag{B.1}$$

For the sake of simplification, define

$$\xi_t = Q_{2t}(\Gamma_{2,t}u_t + \Gamma_{3,t}z_t + \Gamma_4\epsilon_t). \tag{B.2}$$

Then, using (B.2) in (B.1) and rearranging terms, we obtain

$$\Omega_{22,t}w_{2,t} = \Lambda_{22,t}w_{2,t+1} - \xi_t \tag{B.3}$$

or

$$w_{2,t} = \Omega_{22,t}^{-1}\Lambda_{22,t}w_{2,t+1} - \Omega_{22,t}^{-1}\xi_t. \tag{B.4}$$

Then, define

$$M_t = \Omega_{22,t}^{-1}\Lambda_{22,t} \tag{B.5}$$

and use (B.5) in (B.4) to obtain

$$w_{2,t} = M_t w_{2,t+1} - \Omega_{22,t}^{-1}\xi_t. \tag{B.6}$$

Next, solve (B.6). Begin this process by using (A.6) to obtain

$$w_{2,t+1} = M_{t+1}w_{2,t+2} - \Omega_{22,t+1}^{-1}\xi_{t+1} \tag{B.7}$$

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Then, substitute (B.7) into (B.6) to obtain

$$w_{2,t} = M_t\left(M_{t+1}w_{2,t+2} - \Omega_{22,t+1}^{-1}\xi_{t+1}\right) - \Omega_{22,t}^{-1}\xi_t \tag{B.8}$$

or

$$w_{2,t} = M_t M_{t+1} w_{2,t+2} - M_t\Omega_{22,t+1}^{-1}\xi_{t+1} - \Omega_{22,t}^{-1}\xi_t. \tag{B.9}$$

Repeat the process above for $w_{2,t+2}$. First, from (B.7),

$$w_{2,t+2} = M_{t+2}w_{2,t+3} - \Omega_{22,t+2}^{-1}\xi_{t+2}. \tag{B.10}$$

Then, substitution of (B.10) into (B.9) yields

$$w_{2,t} = M_t M_{t+1}\left(M_{t+2}w_{2,t+3} - \Omega_{22,t+2}^{-1}\xi_{t+2}\right) - M_t\Omega_{22,t+1}^{-1}\xi_{t+1} - \Omega_{22,t}^{-1}\xi_t \tag{B.11}$$

or

$$w_{2,t} = M_t M_{t+1} M_{t+2} w_{2,t+3} - M_t M_{t+1}\Omega_{22,t+2}^{-1}\xi_{t+2} - M_t\Omega_{22,t+1}^{-1}\xi_{t+1} - \Omega_{22,t}^{-1}\xi_t. \tag{B.12}$$

This process can be continued for s periods until one obtains

$$w_{2,t} = \left(\prod_{i=0}^{s} M_{t+i}\right) w_{2,t+s+1} - \sum_{j=1}^{s}\left(\prod_{i=0}^{j-1} M_{t+i}\right)\Omega_{22,t+j}^{-1}\xi_{t+j} - \Omega_{22,t}^{-1}\xi_t. \tag{B.13}$$

Under the condition that

$$\lim_{s\to\infty}\left(\prod_{i=0}^{s} M_{t+i}\right) = 0, \tag{B.14}$$

(B.13) becomes, in the limit as $s \to \infty$,

$$w_{2,t} = -\sum_{j=1}^{\infty}\left(\prod_{i=0}^{j-1} M_{t+i}\right)\Omega_{22,t+j}^{-1}\xi_{t+j} - \Omega_{22,t}^{-1}\xi_t. \tag{B.15}$$

Then, define

$$\tilde{M}_{t,j} = \prod_{i=0}^{j-1} M_{t+i} \quad \text{for} \quad j \geq 1 \tag{B.16}$$

and

$$\tilde{M}_{t,j} = I, \quad j = 0, \tag{B.17}$$

and write (B.15) as

$$w_{2,t} = -\sum_{j=0}^{\infty}\tilde{M}_{t,j}\Omega_{22,t+j}^{-1}\xi_{t+j}. \tag{B.18}$$

Substitution of (B.2) into (B.18) then yields, after taking expectations,

$$w_{2,t} = -\sum_{j=0}^{\infty}\tilde{M}_{t,j}\Omega_{22,t+j}^{-1}Q_{2,t+j}(\Gamma_{2,t+j}u_{t+j} + \Gamma_{3,t+j}z_{t+j}), \tag{B.19}$$

which is the same as (7) in the body of the paper.

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