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THE POINT VALUE MAXIMIZATION PROBLEM FOR POSITIVE DEFINITE FUNCTIONS SUPPORTED IN A GIVEN SUBSET OF A LOCALLY COMPACT GROUP

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Abstract The century-old extremal problem, solved by Carathéodory and Fejér, concerns a non-negative trigonometric polynomial $T(t) = a_0 + \sum_{k=1}^{n} a_k \cos(2\pi kt) + b_k \sin(2\pi kt) \geqslant 0$, normalized by $a_0 = 1$, where the quantity to be maximized is the coefficient a_1 of $\cos(2\pi t)$. Carathéodory and Fejér found that for any given degree n, the maximum is $2\cos(\pi/n + 2)$. In the complex exponential form, the coefficient sequence $(c_k) \subset \mathbb{C}$ will be supported in $[-n, n]$ and normalized by $c_0 = 1$. Reformulating, non-negativity of T translates to positive definiteness of the sequence (c_k), and the extremal problem becomes a maximization problem for the value at 1 of a normalized positive definite function $c : \mathbb{Z} \to \mathbb{C}$, supported in $[-n, n]$. Boas and Kac, Arestov, Berdysheva and Berens, Kolountzakis and Révész and, recently, Krenedits and Révész investigated the problem in increasing generality, reaching analogous results for all locally compact abelian groups. We prove an extension to all the known results in not necessarily commutative locally compact groups.

Keywords: Carathéodory–Fejér extremal problem; locally compact topological groups; abstract harmonic analysis; Haar measure; modular function; convolution of functions and of measures; positive definite functions; Bochner–Weil theorem; convolution square; Fejér–Riesz theorem

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1. Introduction

In this work we consider locally compact groups. Our aim is to extend a number of results stemming from the classical trigonometric polynomial extremal problem of Carathéodory

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and Fejér to the generality of locally compact (but in general not abelian) groups. As well as the original works of Carathéodory [7] and Fejér [10], our results will contain corresponding results for [1, 4, 18, 20].

A century ago, Carathéodory and Fejér originally addressed the question of maximizing the coefficient of $\cos x$ in a 2π-periodic trigonometric polynomial $T(x) \geqslant 0$ of degree at most N and normalized to have constant term 1. Our starting point is the observation, that the classical problem of Carathéodory and Fejér (and many others) can be formulated in the following fairly general way, even in not necessarily abelian (topological) groups.

Problem 1.1. Let $\Omega \subset G$ be a given set in the group G and let $z \in \Omega$ be fixed. Consider a *positive definite function* $f : G \to \mathbb{C}$ (or $\to \mathbb{R}$), normalized to have $f(e) = 1$ for the unit element $e \in G$, and *vanishing outside* Ω. How large can then $|f(z)|$ be?

Actually, the Carathéodory–Fejér extremal problem, as is briefly discussed below, corresponds to the special case when $G = \mathbb{Z}$ and $\Omega = [-N, N]$. Therefore, a proper generalization to groups is already obtained assuming also the following restriction.

Condition 1.2. Assume that there is a natural number $N \in \mathbb{N}$ satisfying

$$\langle z \rangle \cap \Omega = \{z^n : -N \leqslant n \leqslant N\}, \quad \text{where } \langle z \rangle := \{z^n : n \in \mathbb{Z}\}. \tag{1.1}$$

The analogous problem of maximizing $\int_\Omega f$ under a similar hypothesis was recently well investigated by several authors under the name of 'Turán's extremal problem'—terminology that originated from the paper of Stechkin [25]—although it later turned out that the problem had already been considered well before Turán, see the detailed survey [22]. The problem in our focus, in turn, was also investigated on various classical groups (the Euclidean space, $\mathbb{Z}^d$ and $\mathbb{T}^d$ being the most general ones) and was also termed by some 'the pointwise Turán problem', but one paper [18] traced it back to Boas and Kac [4] in the 1940s and even to the work of Carathéodory [7] and Fejér [10] [11, I, p. 869] as early as in the 1910s.

Note that although [18] extended the problem even to sets Ω in classical groups *not satisfying* Condition 1.2, in the generality of non-abelian groups we cannot yet describe the problem without any extra assumption. Still, the unconditional extension can be handled in locally compact *abelian* (LCA) groups, for which we refer the reader to [20].

On the other hand here we consider locally compact groups lacking any assumption about commutativity. Correspondingly, we write the group operation the usual multiplicative way, and investigate Problem 1.1 either under the Carathéodory–Fejér condition, i.e. Condition 1.2, or, in §§ 7 and 8, under another assumption defined later as *roundness* (see Definition 6.1). As we will explain in the sequel, this latter extension is still general enough to cover all the known extensions, including the unconditional results of [20] for LCA groups.

To the best of our knowledge, this is the first attempt to deal with such extremal problems—including other extremal problems of the 'Turán type', as mentioned above—in the wider generality of *not* necessarily abelian locally compact groups. Yet, the extension is very natural, because all ingredients, in particular positive definiteness, is naturally defined on all groups, not just on abelian groups. It would be interesting to

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decide whether these results extend to arbitrary Ω and $z \in \Omega$, without any additional conditions used, e.g. in this paper.

We termed Problem 1.1, at least under Condition 1.2, the *Carathéodory–Fejér-type extremal problem on G for z and Ω*. Since Carathéodory and Fejér worked on their extremal problem well before the notion of positive definiteness was introduced at all, this may require some explanation. Hewitt and Ross [**15**, p. 325] gives a detailed account of how the development of the notion of positive definiteness was ignited by such extremal problems, and in particular by the work of Carathéodory himself.

Basically, the explanation is that non-negativity of a trigonometric polynomial $T(x) = \sum_k c_k \mathrm{e}^{2\pi i k x}$ can be equivalently spelled out as *positive definiteness* of the coefficient sequence (c_k) on $\mathbb{Z}$, an observation first made by Toeplitz, see [**15**, p. 325] and [**27**]. Also, the information that the degree of the polynomial T is at most N means that $\operatorname{supp} c_k \subset [-N, N]$, and with $z := 1 \in \mathbb{Z}$ this is easily seen to match the condition formulated in (1.1) with $\Omega := [-N, N]$. For more details about this interpretation, we refer the reader to [**20**].

For further use, we also introduce the extremal problems

$$\mathcal{M}(\Omega) := \sup \Bigg\{ a(1) : \exists N \in \mathbb{N},\ \exists a : [1, N] \to \mathbb{R},\ a(n) = 0\ (\forall n \notin \Omega), \\ T(t) := 1 + \sum_{n=1}^{N} a(n) \cos(2\pi n t) \geqslant 0\ (\forall t \in \mathbb{T}) \Bigg\}, \tag{1.2}$$

which is called in [**18**] the *Carathéodory–Fejér-type trigonometric polynomial problem*, and

$$\mathcal{M}_m(\Omega) := \sup \Bigg\{ a(1) : \exists a : \mathbb{Z}_m \to \mathbb{R}, a(0) = 1, a(n) = 0 (\forall n \notin \Omega), \\ T\left(\frac{r}{m}\right) := \sum_{n \mod m} a(n) \cos\left(\frac{2\pi n r}{m}\right) \geqslant 0 (\forall r \mod m) \Bigg\}, \tag{1.3}$$

which is termed in [**18**] the *discretized Carathéodory–Fejér-type extremal problem.*

Remark 1.3. Clearly, we have $\mathcal{M}_m(\Omega) \geqslant \mathcal{M}(\Omega)$, because the restriction on the admissible class of positive definite functions to be taken into account is lighter for the discrete problem: we only require $T(r/m) \geqslant 0$ $(0 \leqslant r < m)$, while for $\mathcal{M}(\Omega)$ the restriction is $T(t) \geqslant 0$ $(\forall t \in \mathbb{T})$.

Note that Problem 1.1 may have various interpretations depending on how we define the exact class of positive definite functions that we consider, what topology we use on G, if any, and how we formulate the restrictions with respect to the function f 'living' in Ω only, or regarding 'nicety' of f. In case of the analogous 'Turán problem' when one maximizes the integral $\int_G f d\mu_G$ rather than just a fixed point value $|f(z)|$, consideration of various classes is more delicate, see [**19**, Theorem 1].

In the Carathéodory–Fejér extremal problem, however, the general approach on LCA groups was found to be largely indifferent to the issues discussed in [**20**]. That will also be the case for not necessarily commutative groups, so, following [**20**], let us restrict our

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approach to the two extremal cases. That is, denoting positive definiteness of a function f by writing $f \gg 0$, we define here only

$$\mathcal{F}_G^{\#}(\Omega) := \{f : G \to \mathbb{C} : f \gg 0,\ f(e) = 1, f(x) = 0\ \forall x \notin \Omega\}, \tag{1.4}$$

$$\mathcal{F}_G^{c}(\Omega) := \{f : G \to \mathbb{C} : f \gg 0,\ f(e) = 1, f \in C(G),\ \operatorname{supp} f \Subset \Omega\}. \tag{1.5}$$

Once again, note that the first formulation is absolutely free of any topological or measurability structure of the group G. On the other hand, equipping G with the discrete topology, the latter gives a formulation close to the former but restricts f to have finite support.

Let us call the attention to the fact that for $f \gg 0$ the support $\operatorname{supp} f$ is always a *symmetric* closed subset of G (as follows immediately from (4.2) and (4.3) below). Therefore, given a subset $\Omega \subset G$, we can equivalently change it to the symmetric subset $\Omega \cap \Omega^{-1}$ in all these formulations. Further, either $e \in \Omega$, or the problem becomes trivial with f not satisfying the normalization condition $f(e) = 1$ and thus the respective function classes becoming $\mathcal{F}_G^{\#}(\Omega) = \mathcal{F}_G^{c}(\Omega) = \emptyset$. Consequently, we will assume throughout that the fundamental sets $\Omega \subset G$ are chosen to be symmetric subsets containing the unit element e.

The respective 'Carathéodory–Fejér constants' are then

$$\mathcal{C}_G^{\#}(\Omega, z) := \sup\{|f(z)| : f \in \mathcal{F}_G^{\#}(\Omega)\}, \qquad \mathcal{C}_G^{c}(\Omega, z) := \sup\{|f(z)| : f \in \mathcal{F}_G^{c}(\Omega)\}. \tag{1.6}$$

Note that for finite G, the conditions are automatically satisfied if f is continuous or $\operatorname{supp} f$ is compact, hence there is no need to distinguish between these classes, and we can just write $\mathcal{F}_G(\Omega)(:= \mathcal{F}_G^{c}(\Omega) = \mathcal{F}_G^{\#}(\Omega))$ and $\mathcal{C}_G(\Omega, z)(:= \mathcal{C}_G^{c}(\Omega, z) = \mathcal{C}_G^{\#}(\Omega, z))$.

In view of (4.4) below, given that for $f \gg 0$, $\|f\|_\infty = f(e)$, the *trivial estimate* or *trivial (upper) bound* for the Carathéodory–Fejér constants is simply $\mathcal{C}_G^{c}(\Omega, z) \leqslant \mathcal{C}_G^{\#}(\Omega, z) \leqslant 1$. Note also that the lower estimation $\mathcal{C}_G^{\#}(\Omega, z) \geqslant \mathcal{C}_G^{c}(\Omega, z) \geqslant 1/2$ can be worked out analogously to [**20**, Proposition 3.2].

By the above general definition, for $G = \mathbb{Z}$ and $G = \mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$, the Carathéodory–Fejér constants (1.6) with $z := 1$ (and denoting by H the fundamental set Ω in this case) are

$$\begin{aligned}
\mathcal{C}^{\#}(H) &:= \mathcal{C}_{\mathbb{Z}}^{\#}(H, 1) := \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}}^{\#}(H)\} \\
&:= \sup\{|\varphi(1)| : \varphi : \mathbb{Z} \to \mathbb{C}, \varphi \gg 0, \varphi(0) = 1, \operatorname{supp}\varphi \subset H\}, \\
\mathcal{C}^{c}(H) &:= \mathcal{C}_{\mathbb{Z}}^{c}(H, 1) := \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}}^{c}(H)\} \\
&:= \sup\{|\varphi(1)| : \varphi : \mathbb{Z} \to \mathbb{C}, \varphi \gg 0, \varphi(0) = 1, \operatorname{supp}\varphi \subset H, \#\operatorname{supp}\varphi < \infty\}, \\
\mathcal{C}_m(H) &:= \mathcal{C}_{\mathbb{Z}_m}^{\#}(H, 1) = \mathcal{C}_{\mathbb{Z}_m}^{c}(H, 1) := \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}_m}(H)\} \\
&:= \sup\{|\varphi(1)| : \varphi : \mathbb{Z}_m \to \mathbb{C}, \varphi \gg 0, \varphi(0) = 1, \operatorname{supp}\varphi \subset H\}.
\end{aligned} \tag{1.7}$$

The issue of whether we consider complex valued functions or real valued functions also occurs naturally. As is discussed in [**20**], for the Carathéodory–Fejér-type problem, the choice of function classes simplifies compared with the 'Turán problem', while the issue of considering real or complex valued functions becomes less simple—in fact, it splits in some cases while remaining equivalent in others.

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So, for any group (and in particular for $G = \mathbb{Z}$ and $G = \mathbb{Z}_m$) we put

$$\mathcal{F}_G^{\#\mathbb{R}}(\Omega) := \{\varphi : G \to \mathbb{R} : \varphi \in \mathcal{F}_G^{\#}(\Omega)\}, \qquad \mathcal{F}_G^{c\mathbb{R}}(\Omega) := \{\varphi : G \to \mathbb{R} : \varphi \in \mathcal{F}_G^{c}(\Omega)\}, \quad (1.8)$$

and, as before, for finite groups we again simplify by putting $\mathcal{F}_G^{\mathbb{R}}(H) := \mathcal{F}_G^{c\mathbb{R}}(H) = \mathcal{F}_G^{\#\mathbb{R}}(H)$. Then, we can write

$$\mathcal{K}_G^{\#}(\Omega, z) := \sup_{\varphi \in \mathcal{F}_G^{\#\mathbb{R}}(\Omega)} |\varphi(z)|, \qquad \mathcal{K}_G^{c}(\Omega, z) := \sup_{\varphi \in \mathcal{F}_G^{c\mathbb{R}}(\Omega)} |\varphi(z)|,$$

$$\mathcal{K}^{\#}(H) := \mathcal{K}_{\mathbb{Z}}^{\#}(H, 1), \quad \mathcal{K}^{c}(H) := \mathcal{K}_{\mathbb{Z}}^{c}(H, 1), \qquad \mathcal{K}_m(H) := \mathcal{K}_{\mathbb{Z}_m}(H, 1) := \sup_{\varphi \in \mathcal{F}_{\mathbb{Z}_m}^{\mathbb{R}}(H)} |\varphi(1)|. \quad (1.9)$$

Proposition 1.4. *We have for arbitrary symmetric $0 \in H \subset \mathbb{Z}$ the following.*

(i) $\mathcal{M}(H \cap \mathbb{N}) = 2\mathcal{K}(H)$ *and for all* $m \in \mathbb{N}$ $\mathcal{M}_m(H \cap [0, m/2]) = 2\mathcal{K}_m(H)$.

(ii) $\mathcal{K}^c(H) = \mathcal{K}^{\#}(H) = \mathcal{C}^c(H) = \mathcal{C}^{\#}(H)$ *(which can thus be denoted by* $\mathcal{C}(H)$ *from now on).*

(iii) *For all* $m \in \mathbb{N}$, $\cos(\pi/m)\mathcal{C}_m(H) \leqslant \mathcal{K}_m(H) \leqslant \mathcal{C}_m(H)$.

(iv) *If* $4 \leqslant m \in \mathbb{N}$, *then in general (iii) is the best possible estimate, with both inequalities being attained for some symmetric subset* $H \subset \mathbb{Z}_m$.

(v) *If* $m = 2, 3$, *then for any admissible* H *we must have* $H = \mathbb{Z}_m$ *and thus* $\varphi(x) \equiv 1$ *shows* $\mathcal{C}_m(H) = \mathcal{K}_m(H) = 1$.

Proof. This is a combination of [**20**, Proposition 3.1] and [**20**, Proposition 3.5]. In (iv) an oral communication of Ruzsa is also used. □

2. Previous results on Carathéodory–Fejér-type extremal problems

As mentioned above, the development of these results started with the extremal problem of Carathéodory and Fejér, originally formulated for positive trigonometric polynomials, see [**7**, **10**] or [**11**, volume 1, p. 869].

Theorem 2.1 (Carathéodory and Fejér). *If* $T(t) := 1 + \sum_{n=1}^{N} a(n)\cos(2\pi nt) \geqslant 0$ $(\forall t \in \mathbb{T})$, *then* $|a_1| \leqslant 2\cos(\pi/N + 2)$, *and the bound is sharp. In other words,* $\mathcal{M}([1, N]) = 2\cos(\pi/N + 2)$.

Boas and Kac [**4**] used the result of Carathéodory and Fejér to prove the following theorem.

Theorem 2.2 (Boas and Kac). *Let* $\Omega \subset \mathbb{R}^d$ *be a convex, symmetric, open, bounded set, i.e. one which generates a corresponding norm* $\|\cdot\| = \|\cdot\|_\Omega$ *on* $\mathbb{R}^d$. *Consider the Carathéodory–Fejér extremal problem, Problem 1.1. Then, with* $\lceil x \rceil$ *denoting the upper*

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integer part of x, we have

$$\mathcal{K}^{\#}_{\mathbb{R}^d}(\Omega, z) = \cos\left(\frac{\pi}{\lceil 1/\|z\|\rceil + 1}\right).$$

Observe that here $N < 1/\|z\| \leqslant N+1$ means by convexity and the definition of the norm that $\langle z\rangle \cap \Omega = \{nz : n \in [-N, N]\}$, that is, Condition 1.2. Given also that $\lceil 1/\|z\|\rceil = N+1$, by Proposition 1.4 (i) the result is indeed the Euclidean space version of the Carathéodory–Fejér result on $\mathbb{Z}$. In view of Proposition 1.4 (ii), this can be complemented by the equalities $\mathcal{K}^{\#}_{\mathbb{R}^d}(\Omega, z) = \mathcal{K}^{c}_{\mathbb{R}^d}(\Omega, z) = \mathcal{C}^{\#}_{\mathbb{R}^d}(\Omega, z) = \mathcal{C}^{c}_{\mathbb{R}^d}(\Omega, z) = \cos(\pi/(\lceil 1/\|z\|\rceil + 1))$.

In particular, the result means that for an interval $(-h, h) \subset \mathbb{R}$ and $z \in (-h, h)$, we have $\mathcal{K}^{\#}_{\mathbb{R}}((-h, h), z) = \mathcal{K}^{c}_{\mathbb{R}}((-h, h), z) = \mathcal{C}^{\#}_{\mathbb{R}}((-h, h), z) = \mathcal{C}_{\mathbb{R}}((-h, h), z) = \cos(\pi/(\lceil h/z\rceil + 1))$.

It is more difficult to deal with the case of the torus. The first results in this respect were obtained by Arestov *et al.* [**1**]. A key observation is that for a symmetric, open set $\Omega \subset (-1/2, 1/2)^d$, we can consider it as a subset of both $\mathbb{R}^d$ and of $\mathbb{T}^d$, and for the latter $\langle z\rangle \cap \Omega$ always contains the one in the previous case. On the other hand, if the order of $z \in \mathbb{T}^d$ is a finite number m—which happens exactly when $z \in \mathbb{Q}^d$—then we also have the effect mentioned in Remark 1.3. Finally, the trigonometric versions and the exponential versions can be exchanged with each other, according to Proposition 1.4 (i), so that finally $\mathcal{K}^{c}_{\mathbb{R}^d}(\Omega, z) \leqslant \mathcal{K}^{c}_{\mathbb{T}^d}(\Omega, z)$ follows (cf. [**18**, Proposition 6.1]). As a corollary, improving upon the estimate of [**1**, Theorem 3 (14)], it was also found that for a symmetric, open convex set $\Omega \subset (-1/2, 1/2)^d$, $\mathcal{K}^{c}_{\mathbb{T}^d}(\Omega, z) \geqslant \cos(\pi/\lceil 1/\|z\|\rceil + 1)$, see [**18**, Corollary 6.2].

As mentioned above, it is difficult to compute exact extremal values if the order of z is finite, say $o(z) = m$. More work was done on this for the case when $\Omega = (-1/2, 1/2)^d$, so in a sense only the 'boundary' (a smaller-dimension subset) is removed from $\mathbb{T}^d$. In this case, it is known [**18**, Theorem 7.3] that for $z \notin \mathbb{Q}^d$ or $z \in \mathbb{Q}^d$ but $z = (p_1/q_1, \dots, p_d/q_d)$ (in simplest terms) containing in the denominator exactly the same power 2^s of 2 in each coordinate, we have $\mathcal{K}^{c}_{\mathbb{T}^d}(\Omega, z) = 1$, while for other $z \in \Omega$ we have $\mathcal{K}^{c}_{\mathbb{T}^d}(\Omega, z) = 1/2(1 + \cos(2\pi/m))$, where $m := [q_1, \dots, q_d]$ is the least common multiple of the denominators in the coordinates of z. (Note that in this case some of the denominators must be even, hence m is even and $m/2 \in \mathbb{N}$.) The key to the calculation is the computation in $\mathbb{Z}_m$ of the extremal value $\mathcal{K}_m(\mathbb{Z}_m \setminus \{m/2\}) = 1/2(1 + \cos(2\pi/m))$.

For a general interval $\Omega = (-h, h) \subset \mathbb{T}$ and the special value $z := p/q$ with $p/q < h \leqslant (p+1)/q$, see the paper [**16**]. (The result is also quoted and explained in [**20**, Theorem 5.2].)

Later developments surpassed the assumption of convexity or restrictions of the kind of Condition 1.2. Kolountzakis and Révész [**18**] already considered open symmetric subsets $\Omega \subset \mathbb{R}^d$ or $\mathbb{T}^d$, and recently Krenedits and Révész [**20**] extended the results to LCA groups as well.

Perhaps the main result of [**18**] is the understanding that the above point value extremal problems in the more complicated groups $\mathbb{R}^d$ and $\mathbb{T}^d$ are in fact equivalent to the trigonometric polynomial extremal problems on $\mathbb{Z}$ or $\mathbb{Z}_m$. Until that work, the equivalence remained unclear in spite of the fact that, e.g., Boas and Kac found ways to deduce the solution of the trigonometric extremal problem (1.2) from their results

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on Problem 1.1. The recent results of [**20**] capitalized on this observation when proving analogous equivalences even in LCA groups.

Theorem 2.3 (Krenedits and Révész). *Let G be any LCA group and let $\Omega \subset G$ be an open (symmetric) neighbourhood of 0. Let also $z \in \Omega$ be any fixed point with $o(z) = \infty$, and denote $H := H(\Omega, z) := \{k \in \mathbb{Z} : kz \in \Omega\}$. Then we have*

$$\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^\#(\Omega, z) = \mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^\#(\Omega, z) = \mathcal{C}(H). \tag{2.1}$$

Recall that in $\mathbb{Z}$ we have $\mathcal{C}(H) := \mathcal{C}^c(H) = \mathcal{C}^\#(H) = \mathcal{K}^c(H) = \mathcal{K}^\#(H) =: \mathcal{K}(H)$.

Theorem 2.4 (Krenedits and Révész). *Let G be any LCA and let $\Omega \subset G$ be an open (symmetric) neighbourhood of 0. Let also $z \in \Omega$ be any fixed point with $o(z) = m < \infty$, and denote $H_m := H_m(\Omega, z) := \{k \in \mathbb{Z}_m : kz \in \Omega\}$. Then we have*

$$\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^\#(\Omega, z) = \mathcal{C}_m(H_m) \quad \text{and} \quad \mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^\#(\Omega, z) = \mathcal{K}_m(H_m). \tag{2.2}$$

These results extended the corresponding Theorems 2.1 and 2.4 of [**18**] to all LCA groups. Note that these theorems are *equivalence statements*, which greatly decrease the complexity of the problems when reducing them to given extremal problems on $\mathbb{Z}$ or $\mathbb{Z}_m$, yet they do not necessarily *solve* them in the sense of providing the exact numerical value of the extremal quantity. Nevertheless, it already follows that for $o(z) = \infty$ all four versions of the Carathéodory–Fejér extremal constants, listed in (2.1), are equal—and thus can be denoted by $\mathcal{C}_G(\Omega)$ in general—while for $o(z) < \infty$ at least the two complex and respective two real versions coincide, and thus can be denoted as $\mathcal{C}_G(\Omega)$ and $\mathcal{K}_G(\Omega)$, respectively. One of the goals of the current work is to see this phenomenon in non-commutative groups too, at least under suitable conditions, see in particular Theorems 7.1 and 7.2. For the numerous applications and exact computations or estimates of the concrete values of these extremal constants, see the original papers and the references therein.

Also let us recall once again that, at least for the time being, we cannot obtain such fully general results for the current non-commutative case: here, we need to assume further conditions, e.g., Condition 1.2 or validity of the condition of roundness, formulated later in Definition 6.1. Still, our new results below will completely cover the theorems of Fejér and Carathéodory, Boas and Kac, Arestov, Berdysheva and Berens, Kolountzakis and Révész, and even Krenedits and Révész.

3. Formulation of the results under the Carathéodory–Fejér Condition

When the order of z is infinite, the result will be an exact generalization of the Carathéodory–Fejér result.

Theorem 3.1. *Let G be any locally compact topological group, with unit element e, and let $\Omega \subset G$ be an open (symmetric) neighbourhood of e. Let also $z \in \Omega$ be any fixed point with $o(z) = \infty$, and assume that Condition 1.2 is satisfied with a certain N. Then we have*

$$\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^\#(\Omega, z) = \mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^\#(\Omega, z) = \cos\left(\frac{\pi}{N+2}\right). \tag{3.1}$$

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This also extends Theorem 2.3 from LCA groups to not necessarily commutative locally compact groups if (1.1) is satisfied.

If $z \in G$ is cyclic (has torsion), then we will show that Problem 1.1 reduces to a well-defined discrete problem of the sort in (1.3). Again, this is an extension of Theorem 2.4 to not necessarily commutative locally compact groups in the case where (1.1) holds.

Theorem 3.2. *Let G be any locally compact topological group, with unit element e, and let $\Omega \subset G$ be an open (symmetric) neighbourhood of e. Let also $z \in \Omega$ be any fixed point with $o(z) = m < \infty$, and assume that Condition 1.2 is satisfied with some $N \leqslant m$. Then we have*

$$\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^{\#}(\Omega, z) = \mathcal{C}_m([-N, N]) \quad \text{and} \quad \mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^{\#}(\Omega, z) = \mathcal{K}_m([-N, N]). \tag{3.2}$$

Remark 3.3. Note the slight difference between the two results. For $o(z) = \infty$ we get the exact value of $\cos(2\pi/(N+2))$ of the extremal constant, while for $o(z) = m < \infty$ we only obtain an equivalence, but not the concrete value. This is a consequence of the fact that the numerical value of the Carathéodory–Fejér extremal problem is known, and also that for $o(z) = \infty$ the real and complex cases agree, while for the finite group analogue the value(s) are not known, nor is equality of the complex and real settings known to hold. It would be interesting to also compute this case, in particular as it depends now on two variables, N and also m, which certainly makes the calculation non-trivial both for the real and the complex case.

Compare also to [**16**], where the 'discrete Fejér-type maximization problem' of maximizing the νth coefficient, i.e. computing $\mathcal{K}_{\mathbb{Z}_m}([0, N], \nu)$, is formulated, and also computed for the special case of $\nu = N$, i.e. the last (and not the first) coefficient.

The proofs of these results can be found in § 5.

4. Preliminaries on positive definite functions on locally compact groups

Positive definite functions were introduced on $\mathbb{Z}$ by Toeplitz [**27**] in 1911 and on $\mathbb{R}$ by Matthias in 1923 [**21**]. For general groups, positive definite functions are defined by the property that

$$\forall n \in \mathbb{N},\ \forall x_1, \dots, x_n \in G,\ \forall c_1, \dots, c_n \in \mathbb{C} \quad \sum_{j=1}^{n} \sum_{k=1}^{n} c_j \overline{c_k} f(x_j x_k^{-1}) \geqslant 0. \tag{4.1}$$

In other words, positive definiteness of a real or complex valued function f on G means that for all n and all choices of n group elements $x_1, \dots, x_n \in G$, the $n \times n$ square matrix $[f(x_j x_k^{-1})]_{j=1,\dots,n}^{k=1,\dots,n}$ is a positive (semi)definite matrix. We will use the notation $f \gg 0$ for a short expression of the positive definiteness of a function $f : G \to \mathbb{C}$ or $G \to \mathbb{R}$.

Definition 4.1 has some immediate consequences[*], the first being that $f(e) \geqslant 0$ is non-negative real (take $n := 1$, $c_1 := 1$ and $x := e$).

[*] These properties are basic and well known, e.g. [**15**, (32.4) Theorem] and [**12**, p. 84]. We prove them just for being self-contained, as they are easy.

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For any function $f : G \to \mathbb{C}$, the *converse* or *reversed* function $\widetilde{f}$ (of f) is defined as

$$\widetilde{f}(x) := \overline{f(x^{-1})}. \tag{4.2}$$

For instance, for the characteristic function χ_A of a set A we have $\widetilde{\chi_A} = \chi_{A^{-1}}$ (where, as usual, $A^{-1} := \{a^{-1} : a \in A\}$), because $x^{-1} \in A$ if and only if $x \in A^{-1}$.

Now let $f : G \to \mathbb{C}$. Then, in the case where f is positive definite, we necessarily have

$$f = \widetilde{f}. \tag{4.3}$$

Indeed, taking the defining formula (4.1) of positive definiteness $x_1 := e$, $x_2 := x$ and $c_1 := c_2 := 1$ and also $c_1 := 1$ and $c_2 := i$, we get both $0 \leqslant 2f(e) + f(x) + f(x^{-1})$ entailing that $f(x) + f(x^{-1})$ is real, and also $0 \leqslant 2f(e) + if(x) - if(x^{-1})$ entailing that $if(x) - if(x^{-1})$ is also real. However, for the two complex numbers $v := f(x)$ and $w := f(x^{-1})$, we have both $v + w \in \mathbb{R}$ and $i(v - w) \in \mathbb{R}$ if and only if $v = \overline{w}$.

Next, observe that for any positive definite function $f : G \to \mathbb{C}$ and any given point $z \in G$

$$|f(z)| \leqslant f(e), \tag{4.4}$$

and so in particular if $f(e) = 0$ then we also have $f \equiv 0$. Indeed, let $z \in G$ be arbitrary: if $|f(z)| = 0$, then we have nothing to prove, and if $|f(z)| \neq 0$, let $c_1 := 1$, $c_2 := -\overline{f(z)}/|f(z)|$ and $x_1 := e$, $x_2 := z$ in (4.1). Then, recalling that according to (4.3) $f(z^{-1}) = \overline{f(z)}$, we obtain $0 \leqslant 2f(e) + c_2 f(z) + \overline{c_2} f(z^{-1}) = 2f(e) - 2|f(z)|$, and (4.4) follows.

Therefore, all positive definite functions are bounded, and $\|f\|_\infty = f(e)$. This is an important property which makes the analysis easier: in particular, we immediately see that $|f(z)|$ cannot exceed 1 for $f \in \mathcal{F}_G^{\#}(\Omega)$, hence the above-mentioned trivial upper bound.

If $u : G \to L(\mathcal{H}, \mathcal{H})$ is a unitary representation of the group G in the Hilbert space $\mathcal{H}$, and $v \in \mathcal{H}$ is a fixed vector, then it is easy to see that the function $x \to \langle u(x)v, v\rangle$ makes a positive definite function. In fact, all positive definite functions can be represented in such a way, see [**12**, (3.15) Proposition and (3.20) Theorem] or, better, [**15**, (32.3) Theorem (iii)]. As an easy consequence, any character $\gamma \in \widehat{G}$ of a LCA group G is positive definite.

We will make use of the following further constructions of positive definite functions.

Lemma 4.1. *Let $f, g : G \to \mathbb{C}$ be arbitrary positive definite functions. Then we have the following.*

(i) *If $H \leqslant G$ is a subgroup of G, and $h := \chi_H g$, that is, $g|_H$ on H and vanishing elsewhere, then also $h \gg 0$.*

(ii) *$\overline{f} \gg 0$, $f^\star(x) := f(x^{-1}) \gg 0$ and $\Re f \gg 0$.*

(iii) *If $\alpha, \beta > 0$ are arbitrary positive constants, then $\alpha f + \beta g \gg 0$.*

(iv) *For arbitrary $n \in \mathbb{N}$, complex numbers $a_j \in \mathbb{C}$ $(j = 1, \dots, n)$ and group elements $y_j \in G$ $(j = 1, \dots, n)$, the derived function $F(x) := \sum_{j=1}^n \sum_{k=1}^n \overline{a_j} a_k f(y_j^{-1} x y_k) \gg 0$.*

(v) *$fg \gg 0$.*

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Proof. The parts (i)–(iv) are simple facts, which the reader will not find difficult to check by a straightforward calculation, but we note that (v) is a non-trivial fact, which follows from the Schur product theorem: if the matrices $A, B \in \mathbb{C}^{n\times n}$ are both positive definite, then so is their entrywise product $[a_{jk}b_{jk}]_{j=1,\dots,n}^{k=1,\dots,n}$. For this latter fact from linear algebra, see [**13**, § 85, Theorem 2] or [**15**, (D.12) Lemma, Appendix D, pp. 683–684].

All the statements (i)–(v) can be found in [**15**, (32.8) (d)] and [**15**, (32.9) Theorem]. □

As the converse to (i), we can recall the following easily proved observation of Hewitt, see [**15**, (32.43) (a)].

Lemma 4.2. *Let $H \leqslant G$ be a subgroup of G and let $h \gg 0$ on H. Furthermore, define $g : G \to \mathbb{C}$ as its trivial extension:*

$$g(x) := \begin{cases} h(x) & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases} \tag{4.5}$$

Then $g \gg 0$ on G.

The only problem with this is that it does not necessarily preserve analytic structure, e.g., continuity, so in a case involving structural restrictions, such a statement in itself may not suffice. Actually, the crux of the proof of our results is to circumvent this technical obstacle and build a further function, still positive definite and (essentially) preserving the values at H, but admitting some continuity (or at least integrability) restrictions, too. For a discussion of the (limited) possibility to use this 'trivial extension' in our constructions and proofs, in particular in case of cyclic or monothetic (sub)groups, see [**20**, § 5].

To fix notations, for locally compact (Hausdorff) topological groups G we will consider their *left-invariant Haar-measure* μ_G normalized the standard way, so that for discrete groups all points have measure 1 and for non-discrete compact groups $\mu_G(G) = 1$. Recall that on G there exist essentially unique left and also right Haar measures, see, e.g., [**12**, § 2.2].

For the fixed left Haar measure μ_G and for any fixed $g \in G$ the measure $H \to \mu_G(Hg)$ ($\forall H \subset G$ Borel measurable) provides another left Haar measure. The essential uniqueness of the left Haar measure thus means that there is a function $\Delta : G \to (0, \infty)$ with $\mu_G(Hg) = \Delta(g)\mu_G(H)$ for all Borel measurable $H \subset G$ and any $g \in G$. Moreover, the so-introduced *modular function* $\Delta(g)$ is positive, continuous and multiplicative, i.e. a continuous group homomorphism from G to $\mathbb{R}_+$, see [**14**, Theorem (15.11)].

Correspondingly, we will consider the (left) convolution of functions with respect to the left Haar measure μ_G, that is,

$$(f \star g)(x) := \int_G f(y)g(y^{-1}x)\,\mathrm{d}\mu_G(y) = \int_G f(xz)g(z^{-1})\,\mathrm{d}\mu_G(z), \tag{4.6}$$

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defined for all functions $f, g \in L^1(\mu_G)$, see, e.g., [12, p. 50]. (The same convolution formula appears in [14] as a *theorem*, namely Theorem (20.10) (i) on p. 291, as Hewitt and Ross introduce convolution in a more general setting.)

As a concrete application, consider now the characteristic functions χ_A, χ_B of two Borel measurable sets A, B with finite left Haar measure. In this particular case definition, (4.6) yields

$$\chi_A \star \chi_B(x) = \int_G \chi_A(y)\chi_B(y^{-1}x)\,\mathrm{d}\mu_G(y) = \int_G \chi_A \chi_{xB^{-1}}\,\mathrm{d}\mu_G = \mu_G(A \cap xB^{-1}). \quad (4.7)$$

In particular, if $B = A^{-1}$, then $(\chi_A \star \chi_{A^{-1}})(x) = \mu_G(A \cap xA)$.

For more on convolution of measures and functions, see, e.g., [12, pp. 49–54] and [14, §§ 19 and 20] (with respect to left Haar measure) or [2, Chapter 9] (with respect to right Haar measure). Note that convolution is in general *not* commutative: more precisely, the convolution is commutative if and only if the locally compact group itself is commutative, e.g. [14, Theorem (20.24)]. We will use the next well-known assertion only in the rather special case of characteristic functions.

Lemma 4.3. *Let $f \in L^2(\mu_G)$ be arbitrary. Then the 'convolution square' of f exists, moreover, it is a continuous positive definite function, that is, $f \star \widetilde{f} \gg 0$ and $f \star \widetilde{f} \in C(G)$.*

Proof. This can be found for LCA groups in [24, § 1.4.2 (a)], and for the general case, e.g., [2, Example 4, p. 220] or [15, (32.43) (e)]. It also follows from a combination of [12, (3.35) Proposition] and [12, (3.16) Corollary]. □

Although it is very useful when it holds, in general this statement cannot be reversed. Even for classical abelian groups, it is a delicate question when a positive definite continuous function has a 'convolution root' in the above sense. For a nice survey on the issue see, e.g., [8]. We will however be satisfied with a very special case, where this converse statement follows from a classical result of Fejér and Riesz, see [26, Theorem 1.2.1], [10] or [11, I, p. 845]. For details about the below reformulation in the language of positive definite sequences, see [20, Lemma 2.2 (i)].

Lemma 4.4. *Let $\psi : \mathbb{Z} \to \mathbb{C}$ be a finitely supported positive definite sequence. Then there exists another sequence $\theta : \mathbb{Z} \to \mathbb{C}$, also finitely supported, such that $\theta \star \widetilde{\theta} = \psi$. Moreover, if* $\operatorname{supp}\psi \subset [-N, N]$, *then we can take* $\operatorname{supp}\theta \subset [0, N]$.

A slightly less strict analogue of the existence of a convolution square root also holds in $\mathbb{Z}_m$. Again, for the (simple and straightforward) proof, see [20, Lemma 2.2 (ii)].

Lemma 4.5. *If $\psi : \mathbb{Z}_m \to \mathbb{C}$, $\psi \gg 0$ on $\mathbb{Z}_m$, then there exists $\theta : \mathbb{Z}_m \to \mathbb{C}$ with $\theta \star \widetilde{\theta} = \psi$.*

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Note the slight loss of precision—we cannot bound the support of θ in terms of a control of the support of ψ. This is natural, as the same finitely supported sequence can be positive definite on $\mathbb{Z}_m$ more easily than on $\mathbb{Z}$, in view of the fact that the equivalent restriction that its Fourier transform (the trigonometric polynomial T) satisfies $T(2\pi n/m) \geqslant 0$ $(\forall n \in \mathbb{Z}_m)$ is met more easily than $T(t) \geqslant 0$ $(\forall t \in \mathbb{T})$.

5. Proofs of the results under Condition 1.2

Proof of Theorem 3.1. Let us write throughout the proof $H := [-N, N]$. According to Proposition 1.4 (i), (ii) and the result of Carathéodory and Fejér (Theorem 2.1), we have $\mathcal{C}^c(H) = \mathcal{C}^\#(H) = \mathcal{K}^c(H) = \mathcal{K}^\#(H) = \cos(\pi/(N+2))$, hence it suffices to prove that $\mathcal{C}^c_G(\Omega, z) = \mathcal{C}^\#_G(\Omega, z) = \mathcal{C}^c(H)$ and $\mathcal{K}^c_G(\Omega, z) = \mathcal{K}^\#_G(\Omega, z) = \mathcal{K}^c(H)$.

The complex and real cases are exactly similar, so we will work out only the complex case here. As obviously we have $\mathcal{C}^c_G(\Omega, z) \leqslant \mathcal{C}^\#_G(\Omega, z)$, we need to prove only two inequalities, the first being that $\mathcal{C}^\#_G(\Omega, z) \leqslant \mathcal{C}^c(H)$.

Since now $H \subset \mathbb{Z}$ is finite, there can be no difference between $\mathcal{C}^c(H)$ and $\mathcal{C}^\#(H)$, so it suffices to show that $\mathcal{C}^\#_G(\Omega, z) \leqslant \mathcal{C}^\#(H)$. Let us now take any $f \in \mathcal{F}^\#_G(\Omega)$. Consider the subgroup $Z := \langle z \rangle \leqslant G$. Noting that according to Lemma 4.1 (i), $g := f|_Z \gg 0$ on Z, we have defined a function $g \in \mathcal{F}^\#_Z((\Omega \cap Z))$. Finally, let us remark that the natural isomorphism $\eta : \mathbb{Z} \to Z$, which maps according to $\eta(k) := z^k$, carries over g, defined on $Z \leqslant G$, to a function $\psi := g \circ \eta : \mathbb{Z} \to \mathbb{C}$, which is therefore positive definite on $\mathbb{Z}$, has normalized value $\psi(0) = g(e) = f(e) = 1$, and $\operatorname{supp}\psi \subset H$ for $\operatorname{supp} g \subset (\operatorname{supp} f \cap Z) \subset (\Omega \cap Z) = \{z^k : |k| \leqslant N\} = \eta(H)$ in view of (1.1). That is, $\psi \in \mathcal{F}^\#_{\mathbb{Z}}(H)$.

From here we see that $|f(z)| = |g(z)| = |\psi(1)| \leqslant \sup\{|\varphi(1)| : \varphi \in \mathcal{F}^\#_{\mathbb{Z}}(H)\} = \mathcal{C}^\#(H)$. Taking $\sup_{f \in \mathcal{F}^\#_G(\Omega)}$ on the left-hand side concludes the proof of the first part.

It remains to show that $\mathcal{C}^c(H) \leqslant \mathcal{C}^c_G(\Omega, z)$.

Let us consider an arbitrary function $\psi \in \mathcal{F}^c(H)$. Since $\psi \gg 0$, its Fourier transform, the trigonometric polynomial $T(t) := \check{\psi}(t) = \sum_{n=-N}^{N} \psi(n) \mathrm{e}^{2\pi i n t}$, is non-negative, and we can invoke the classical theorem of Fejér and Riesz to represent it as a square: $T(t) = |P(t)|^2$. Using the above-described preliminaries precisely, we can apply Lemma 4.4. Thus we find another sequence $\theta : [0, N] \to \mathbb{C}$ such that $\psi(n) = \sum_{k=0}^{N} \theta(k)\overline{\theta(k-n)} = (\theta \star \widetilde{\theta})(n)$.

Let us consider some compact neighbourhood U of e. Observe that for any given value $-N \leqslant n \leqslant N$, we have at most $N+1$ different pairs $0 \leqslant k, j \leqslant N$ with $j - k = n$ (while for $|n| > N$ we have none). If $j - k = n$, then $z^j U z^{-k}$ is the continuous image of the compact set U under the continuous homeomorphism $x \to z^j x z^{-k}$. These continuous homeomorphisms take e to z^n, hence $z^j U z^{-k}$ is a compact neighbourhood of z^n, and so is the set $U_n := \bigcup\{z^j U z^{-k} : 0 \leqslant j, k \leqslant N, k - j = n\}$. Note that by Condition 1.2, all of these $z^n \in \Omega$.

Clearly, then, if $U \Subset \Omega$ is chosen to be small enough, we can ensure that $U_n \Subset \Omega$, hence also $\bigcup_{n=-N}^{N} U_n \Subset \Omega$. Also, the finitely many z^n with $n \in H = [-N, N]$ are all different (for $o(z) = \infty$), therefore, for U chosen small enough, we can ensure $z^n \notin U$ for all $0 < |n| \leqslant N$.

Next, let f be any positive definite function, supported compactly on the chosen $U \Subset \Omega$. Then, by Lemma 4.1 (iv), the function $F(x) := \sum_{j=0}^{N} \sum_{k=0}^{N} \theta(k)\overline{\theta(j)} f(z^{-j} x z^k)$ is also

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positive definite; moreover, we have $\operatorname{supp} F \Subset \bigcup_{j,k=0}^{N} z^j U z^{-k} = \bigcup_{n=-N}^{N} U_n$. Therefore, by the above choice of U, we have $\operatorname{supp} F \Subset \Omega$.

Now, let us compute $F(e)$ and $F(z)$. By the above choice of U and F, we have

$$F(e) = \sum_{j,k=0}^{N} \theta(k)\overline{\theta(j)} f(z^{-j+k}) = \sum_{j=0}^{N} \theta(j)\overline{\theta(j)} f(e) + \sum_{j=0}^{N} \sum_{\substack{0 \leqslant k \leqslant N \\ k \neq j}} \theta(k)\overline{\theta(j)} f(z^{-j+k}) = f(e),$$

because the second sum vanishes in view of $z^n \notin \operatorname{supp} f \subset U$ $(n \neq 0)$ by the choice of U, while $\sum_{j=0}^{N} |\theta(j)|^2 = \psi(0) = 1$. Similarly, we compute

$$\begin{aligned} F(z) &= \sum_{j,k=0}^{N} \theta(k)\overline{\theta(j)} f(z^{-j+1+k}) \\ &= \sum_{j=1}^{N} \theta(j-1)\overline{\theta(j)} f(e) + \sum_{j=0}^{N} \sum_{\substack{0 \leqslant k \leqslant N \\ k \neq j-1}} \theta(k)\overline{\theta(j)} f(z^{-j+1+k}) = \overline{\psi(1)} f(e), \end{aligned}$$

because again the second sum vanishes and $\sum_{j=1}^{N} \theta(j-1)\overline{\theta(j)} = (\theta \star \widetilde{\theta})(-1) = \psi(-1) = \overline{\psi(1)}$.

Then, if there exists a function $f \gg 0$, continuous and with $\operatorname{supp} f \Subset U$, then we also have another continuous $F \gg 0$ with $\operatorname{supp} F \Subset \bigcup_{n=-N}^{N} U_n \Subset \Omega$ having the properties that $F(e) = \psi(0) f(e)$ and $F(z) = \overline{\psi(1)} f(e)$. Of course, $f \equiv 0$ would qualify, but we want also $f(e) \neq 0$, which, in view of (4.4), is equivalent to $f \not\equiv 0$. So, with any such not identically vanishing function f, we can construct the above F, and even normalize it to $F_0 := (1/f(e))F$, to obtain $F_0 \in \mathcal{F}^c(\Omega)$ with $F_0(z) = \overline{\psi(1)}$. This entails that

$$|\psi(1)| = |F_0(z)| \leqslant \sup\{|\phi(z)| : \phi \in \mathcal{F}_G^c(\Omega)\} = \mathcal{C}_G^c(\Omega, z).$$

Taking the supremum on the left-hand side over all $\psi \in \mathcal{F}_{\mathbb{Z}}^c(H)$ thus leads to $\mathcal{C}^c(H) \leqslant \mathcal{C}_G^c(\Omega, z)$, provided we can find a continuous, not identically zero $f \gg 0$, supported in U.

It only remains to construct a continuous $f \gg 0$ with $\operatorname{supp} f \Subset U$ for the given compact neighbourhood U of e.

As U is a neighbourhood of e and group multiplication is continuous, there exists an open neighbourhood V of e such that $VV^{-1} \subset U$. Let now $W \Subset V$ be chosen such that $\mu_G(W) \neq 0$ (for which it suffices, e.g., that W is a closure of an open, non-void set). Then the characteristic function $g := \chi_W$ of W is certainly in $L^\infty(G) \subset L^2(G)$, and we can apply the above Lemma 4.3 to obtain $f := g \star \widetilde{g} \gg 0$ and $f \in C(G)$. Note that $\widetilde{g} = \chi_{W^{-1}}$ and $f(x) = (\chi_W \star \chi_{W^{-1}})(x) = \mu_G(W \cap xW)$, see (4.7). In particular, $f(e) = \mu_G(W) \neq 0$, and also $\operatorname{supp} f \subset \overline{WW^{-1}} = WW^{-1} \subset VV^{-1} \subset U$, as needed.

Plugging this f in the above construction thus completes the proof of $\mathcal{C}^c(H) \leqslant \mathcal{C}_G^c(\Omega, z)$, and hence the theorem. □

Proof of Theorem 3.2. The direct proof is rather similar to the preceding one, once we carefully change all references from $\mathbb{Z}$ to $\mathbb{Z}_m$, $\mathcal{C}(H)$ to $\mathcal{C}_m(H)$ and $\mathcal{F}_{\mathbb{Z}}(\Omega, z)$ to $\mathcal{F}_{\mathbb{Z}_m}(\Omega, z)$, and note that $Z := \langle z \rangle$ is now only a finite subgroup with $Z \cong \mathbb{Z}_m$, so the natural isomorphism $\eta(k) := z^k$ acts between $\mathbb{Z}_m$ and Z.

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Note that here we have $H(\Omega, z) = \{k \in \mathbb{Z}_m : z^k \in \Omega\}$ in view of $z^m = e$ and Condition 1.2, hence $\mathcal{C}_m(H) = \mathcal{C}_m([-N, N])$ and $\mathcal{K}_m(H) = \mathcal{K}_m([-N, N])$. □

However, instead of describing such a slightly varied proof, we can postpone the proof until the later Theorem 7.2 is proven. As that result is more general, from there only the specialization of $H(\Omega, z) = [-N, N]$ is needed to infer Theorem 3.2. In this regard, it is of importance that the condition of roundness, defined and used later in the corresponding theorem for $o(z) = \infty$, is not needed in the fully general, unconditional Theorem 7.2 for the finite order case. The reason for this formal difference between the corresponding results will also be explained in the next section.

6. A new condition on topological groups and their elements

Now, instead of Condition 1.2 we introduce a different one. Recall that $e \in G$ stands for the unit element of the group G, and the left invariant Haar measure is denoted by μ_G. Also note that the *symmetric difference* of two subsets of G are defined by $A \triangle B := (A \setminus B) \cup (B \setminus A)$ $(A, B \subset G)$.

Definition 6.1. We say that $z \in G$ is a *round* element of the locally compact group G, if for all open neighbourhoods U of the unit e there exists another open neighbourhood $V \subset U$ of e such that $\mu_G(zVz^{-1} \triangle V) = 0$.

Furthermore, the *group* G itself is called *round*, if all elements $z \in G$ are round according to the above.

Remark 6.2. The seemingly one-sided definition does not depend on using the left or right Haar measure. Indeed, if, for instance, $z \in G$ is *left-round*, and U is any open neighbourhood of G, then consider $U^{-1} \subset G$, which is another open neighbourhood of $e \in G$, and take another neighbourhood $V \subset U^{-1}$ of e with the postulated property that zVz^{-1} agrees with V apart from a μ_G-null set. Taking inverses, this means that for the right Haar measure $\mu_G^*(A) := \mu_G(A^{-1})$ and $W := V^{-1} \subset (U^{-1})^{-1} = U$ we have $0 = \mu_G(zVz^{-1} \triangle V) = \mu_G^*((zVz^{-1} \triangle V)^{-1}) = \mu_G^*(zWz^{-1} \triangle W)$, proving that z is also right-round.

Remark 6.3. Note that the condition is similar, but not equivalent, to the Følner condition, which is a condition for amenability of the topological group G.

The Følner condition postulates that for every finite or compact set $F \subset G$, and for any $\varepsilon > 0$, there exists a measurable set U of positive finite Haar measure, such that $\mu_G(U \triangle gU) < \varepsilon \mu_G(U)$ holds for all $g \in F$. On the one hand, this seemingly requires more, as the set U must be *uniformly* given for all $g \in F$. On the other hand, it does not require vanishing, but only relative smallness, of the measure of the symmetric difference. This seemingly small deviation means that in fact $\mu_G(U \triangle gU)$ is typically large; however, $\mu_G(U)$ is much larger still. An simple example is any compact interval in $\mathbb{R}$, with U a much larger interval. Such large sets U also exist in all abelian groups in view of a well-known result of LCA groups, see, e.g., Theorem 2.6.7 on p. 52 of [**24**]. (This also means, as is well known, that all LCA groups are amenable.)

However, the major difference between the two notions is the one-sided multiplication in amenability, and conjugation in roundedness. Inner automorphisms (i.e. conjugations)

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may distort sets, and may even change the Haar measure (depending on the value of the modular function); however, the definition of roundedness means that some (arbitrarily small) neighbourhoods stay invariant. While amenability is related in the first instance to the possibility of introducing an invariant mean on the group, roundedness is kind of a topological substitution to commutativity. If elements themselves do not always commute, roundedness formulates the property that elements and some arbitrarily small neighbourhoods (a topological basis system) can still be interchanged. Therefore, one may perhaps call this property 'topological commutativity'.

Proposition 6.4. *A point $z \in G$ is round if and only if for any neighbourhood U of e there exists a nonzero, continuous, positive definite function $f \gg 0$ such that* supp $f \Subset U$ *and f is invariant under conjugation by z: $f(z^{-1}xz) = f(x)$ for all $x \in G$.*

Proof. Let us suppose that there exists such a function f. Recall that $f(0) = \|f\|_\infty = \max\{|f(x)| : x \in G\}$, which must be positive in view of f being not identically zero. Let $V := \{x : |f(x)| > 0\}$. Then V is an open set, and $e \in V$, as $|f|$ attains the maximum at e. Furthermore for all $x \in V$, $f(zxz^{-1}) = f(x) \neq 0$, hence also $zxz^{-1} \in V$. It follows that $(zVz^{-1} \triangle V) = \emptyset$ and $\mu_G(zVz^{-1} \triangle V) = 0$, so z is a round element.

On the other hand, take a compact neighbourhood W of e with $WW^{-1} \subset U$, which is possible if $(x, y) \to xy^{-1}$ is continuous from $G \times G$ to G. Moreover, by continuity $U_0 := WW^{-1}$ is also compact, hence $U_0 \Subset U$. Now we use the roundedness assumption on z: we select an open neighbourhood $V \subset W$ of e with $z^{-1}Vz$ matching V apart from a μ_G-null set of their symmetric difference. Then we define $f := \chi_V \star \widetilde{\chi_V} = \chi_V \star \chi_{V^{-1}}$, having value $f(x) = \mu_G(V \cap xV)$ according to the calculations in (4.7). Note that as before, by Lemma 4.3, f is a continuous, compactly supported positive definite function, with support supp $f \Subset \overline{VV^{-1}} \Subset U_0 \Subset U$. □

Example 6.5. An *LCA* group G is round.

Indeed, commutativity implies $zVz^{-1} = V$ for any points $z \in G$ and sets $V \subset G$.

Example 6.6. In any locally compact group G, the elements of the *centrum* $C(G) := \{a \in G : ag = ga \ \forall g \in G\}$ are round elements.

Indeed, for any U taking $V := U$, there is an open neighbourhood of e within U, and $z \in C(G)$ obviously has the property $zUz^{-1} = \{zuz^{-1} : u \in U\} = U$.

Note that in abelian groups $C(G) = G$, hence this is more general than Example 6.5.

Example 6.7. If the centrum $C := C(G)$ is open in G, then G is round.

Indeed, for an open neighbourhood U of e, take $V := U \cap C \subset C$: then for any $z \in G$ we have $zVz^{-1} = \{zvz^{-1} : v \in V\} = V$, as z and v commute for all $v \in V \subset C$.

Again, in an LCA group this holds, as the centrum is the full group G, and hence open.

Example 6.8. If $z \in G$ is a cyclic element (has torsion), then it is also round.

Indeed, let $o(z) = m < \infty$ and $e \in U \subset G$ be open. Then the conjugates $z^j U z^{-j}$ are also open, and even the intersection $V := \cap_{j=0}^{m-1} z^j U z^{-j}$ will be an open neighbourhood of e. Moreover, $zVz^{-1} = \cap_{j=1}^{m} z^j U z^{-j} = V$, for $z^m U z^{-m} = eUe^{-1} = U$.

Example 6.9. Let G be any group, equipped with discrete topology. Then G is round.

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Indeed, $V := \{e\}$ works for all choices of U and for all z.

Example 6.10. Suppose G has a topological basis $\{V_\alpha : \alpha \in A\}$ at e to generate the topology of G and consists of normal subgroups. Then G is round.

Indeed, for any U there is $V := V_\alpha \subset U$, with $V \lhd G$ (i.e. a normal subgroup of G), to which all conjugates $z^{-1}Vz = V$.

Note that this covers the case of an arbitrarily large cardinality box or topological product of finite, discrete or otherwise round locally compact groups.

Of course, it is also interesting to see examples of not-round groups and elements. To find such $z \in G$ becomes easy once we note the next simple observation.

Lemma 6.11. *Let $z \in G$ be a round element. Then for the modular function $\Delta(z) = 1$.*

Proof. First, let us take any compact neighbourhood U of e, and some open $e \in V \subset U$, essentially invariant under the conjugation by z as guaranteed by the roundedness of z. The left Haar measure of the set V is finite (as U is compact), but positive (as V is open). According to roundedness, $\mu_G(z^{-1}Vz \triangle V) = 0$, thus we must also have $\mu_G(V) = \mu_G(z^{-1}Vz)$. Using the left invariance of μ_G and the definition of the modular function we find $\mu_G(V) = \mu_G(z^{-1}Vz) = \mu_G(Vz) = \mu_G(V)\Delta(z)$, so using $\mu_G(V) \neq 0$ the assertion follows. □

As a result, any locally compact topological group G which is not unimodular is not round either. Moreover, any $z \in G$ with $\Delta(z) \neq 1$ is not a round element.

Corollary 6.12. *If a group G is round, then it must be unimodular.*

In other words, finding a not-round group or element is easy—just take any group element with the modular function not assuming 1 at it.

Recall[*] that in a locally compact group G an element $g \in G$ is called:

- *equicontinuous* if there is a basis of identity neighbourhoods in G that are all invariant under conjugation by g;
- *distal* if for every $x \in G \setminus \{e\}$, the closure $\overline{\{g^n x g^{-n} | n \in \mathbb{Z}\}}$ does not contain e.

Note that (according to Ellis, who refers to Zippin, see the closing paragraph of [9]) distality was considered by Hilbert in an attempt to give a topological characterization of the concept of a rigid group of motions. The closely related notions of distal elements, contraction groups and equicontinuous elements provided tools for exploring the structures of locally compact topological groups, see, e.g., [3, 5, 6, 17, 23] and the references therein.

Proposition 6.13. *An element $g \in G$ is round if and only if it is equicontinuous.*

[*] For all the following nice comments in this section, which connect our notion of roundness to other well-known notions in locally compact groups, we thank Professor Pierre-Emmanuel Caprace, who kindly provided us with these in an e-mail on November 15, 2013.

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Proof. The 'if' part is clear. Assume conversely that g is round and let W be an identity neighbourhood. Then there is an identity neighbourhood $U \subset W$ such that $U \cdot U^{-1} \subset W$. By the roundness of g, we can find an identity neighbourhood $V \subset U$ such that $\mu(V \triangle gVg^{-1}) = 0$. It follows that the set $V' = \bigcap_{n\in\mathbb{Z}} g^n V g^{-n}$ has the same measure as V; in particular $\mu(V') > 0$. By [**15**, Corollary 20.17], the set $V' \cdot (V')^{-1}$ is an identity neighbourhood. By construction, it is g-invariant and contained in W. Thus g is equicontinuous. □

Recall that for any group element $g \in G$ the *contraction group of* G is defined as $\mathrm{con}(g) := \{x \in G : g^n x g^{-n} \to e (n \to \infty)\}$. Note that this set is indeed trivially a subgroup of G, by continuity of group multiplication and taking inverses.

Distal elements can be characterized in terms of their contraction groups, as follows.

Proposition 6.14. *An element $g \in G$ is distal if and only if its contraction group* $\mathrm{con}(g)$ *is trivial.*

Proof. See Corollary 4.13 in [**23**]. □

In the case where the ambient group is totally disconnected, the following result, owing to Baumgartner–Willis and Jaworski, implies that all three properties are equivalent.

Proposition 6.15. *Let G be a totally disconnected locally compact group. An element $g \in G$ is equicontinuous if and only if* $\mathrm{con}(g)$ *is trivial.*

Proof. The 'only if' part is clear, as by the property of the topology on G we have for any $x \in G, x \neq e$ an identity neighbourhood $U \in \mathcal{N}_e$ with $x \notin U$. Then by the (total) roundness of g there is another, g-invariant open identity neighbourhood $e \in V \subset U \not\ni x$, so that for all $n \in \mathbb{N}$ the conjugate $g^n x g^{-n} \notin g^n U g^{-n}$, hence $\notin g^n V g^{-n} = V$, and e cannot be an accumulation point of the sequence $(g^n x g^{-n})_{n=1}^{\infty}$.

The converse follows from Proposition 3.16 in [**3**] in the case where G is metrizable. All the results from [**3**] have, however, been extended to the general case by Jaworski [**17**]. □

7. More general results on the Carathéodory–Fejér problem

Theorem 7.1. *Let G be any locally compact topological group, with unit element e, and let $\Omega \subset G$ be an open (symmetric) neighbourhood of e. Let also $z \in \Omega$ be any torsion-free point (i.e. $o(z) = \infty$) which is* round *in the sense of Definition 6.1. Then with $H := H(\Omega, z) := \{k \in \mathbb{Z} : z^k \in \Omega\}$ we have*

$$\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^{\#}(\Omega, z) = \mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^{\#}(\Omega, z) = \mathcal{C}(H). \tag{7.1}$$

We can denote this common value simply by $\mathcal{C}_G(\Omega, z)$.

Note that in view of Example 6.5 this result extends Theorem 2.3.

If $z \in \Omega$ is a cyclic (torsion) element, then the situation is described by the next result.

Theorem 7.2. *Let G be any locally compact topological group, with unit element e, and let $\Omega \subset G$ be an open (symmetric) neighbourhood of e. Let also $z \in \Omega$ be any cyclic*

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point with $o(z) = m < \infty$, *and let* $H_m := H_m(\Omega, z) := \{k \in \mathbb{Z}_m : z^k \in \Omega\}$. *Then we have*

$$\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^\#(\Omega, z) = \mathcal{C}_m(H_m) \quad \text{and} \quad \mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^\#(\Omega, z) = \mathcal{K}_m(H_m). \tag{7.2}$$

These common values can be denoted by $\mathcal{C}_G(\Omega, z)$ and $\mathcal{K}_G(\Omega, z)$, respectively.

Note that here we did not assume any extra condition, neither (1.1) nor roundedness of z. Still, we obtain the full strength of the result, and thus an unconditional extension of the result of Theorem 2.4. Dropping the roundedness condition, assumed for torsion-free z, is possible here in view of Example 6.8.

8. Proofs of the general results of § 7

Proof of Theorem 7.1. According to Proposition 1.4 (i), (ii) we have $\mathcal{C}^c(H) = \mathcal{C}^\#(H) = \mathcal{K}^c(H) = \mathcal{K}^\#(H)$, hence it suffices to prove $\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^\#(\Omega, z) = \mathcal{C}(H)$ and $\mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^\#(\Omega, z) = \mathcal{K}(H)(= \mathcal{C}(H))$.

The complex and real cases are exactly similar, so we work out only the complex case. Again, in view of the obvious $\mathcal{C}^c(H) \leqslant \mathcal{C}^\#(H)$, it suffices to prove two inequalities, the first being that $\mathcal{C}_G^\#(\Omega, z) \leqslant \mathcal{C}(H) = \mathcal{C}^\#(H)$.

Let us now take any $f \in \mathcal{F}_G^\#(\Omega)$. Consider the subgroup $Z := \langle z \rangle \leqslant G$. By Lemma 4.1 (i), $g := f|_Z \gg 0$ on Z, so we have defined a function $g \in \mathcal{F}_Z^\#(\Omega \cap Z)$. Consider the natural isomorphism $\eta : \mathbb{Z} \to Z$, which maps according to $\eta(k) := z^k$. Then η carries over g, defined on $Z \leqslant G$, to a function $\psi := g \circ \eta : \mathbb{Z} \to \mathbb{C}$, which is therefore positive definite on $\mathbb{Z}$, has normalized value $\psi(0) = g(e) = f(e) = 1$, and $\operatorname{supp} \psi \subset H$ for $\operatorname{supp} g \subset (\operatorname{supp} f \cap Z) \subset (\Omega \cap Z) = \{z^k : z^k \in \Omega\} = \eta(H)$ by definition of H.

From here we see that $|f(z)| = |g(z)| = |\psi(1)| \leqslant \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}}^\#(H)\} = \mathcal{C}^\#(H)$. Taking $\sup_{f \in \mathcal{F}_G^\#(\Omega)}$ on the left-hand side concludes the proof of the first part.

Note that in this part we did not use the condition of 'roundedness', and we could circumvent any compactness (or finiteness on Z) requirement for the support of $f \in \mathcal{F}_G^\#(\Omega)$. This is a key point to finally also obtain $\mathcal{C}_G^\#(\Omega, z) = \mathcal{C}_G^c(\Omega, z)$ as a consequence of the two inequalities proved.

Now we turn to the second inequality, i.e. we show $\mathcal{C}(H)(= \mathcal{C}^c(H)) \leqslant \mathcal{C}_G^c(\Omega, z)$.

Let us consider an arbitrary function $\psi \in \mathcal{F}^c(H)$. Here it is of relevance that we have the finiteness assumption on $S := \operatorname{supp} \psi$, which is thus included in $[-N, N]$ for some $N \in \mathbb{N}$. This finiteness guarantees not only that $\psi \gg 0$, but also that its Fourier transform consists of finitely many terms and so is a trigonometric polynomial $T(t) := \check{\psi}(t) = \sum_{n=-N}^{N} \psi(n) \mathrm{e}^{2\pi i n t}$.

The next idea here is that as $\psi \gg 0$, T is non-negative, and we can invoke the classical theorem of Fejér and Riesz to represent it as a square: $T(t) = |P(t)|^2$. Using the above-described preliminaries precisely, we can apply Lemma 4.4 to $\psi \gg 0$. Thus, we find another sequence $\theta : [0, N] \to \mathbb{C}$ such that $\psi(n) = (\theta \star \widetilde{\theta})(n) = \sum_{k=\max(n,0)}^{\min(N,N+n)} \theta(k)\overline{\theta(k-n)}$ $(|n| \leqslant N)$.

Let us consider some compact neighbourhood U of e. Observe that for any given value $-N \leqslant n \leqslant N$ we have at most $N+1$ different pairs $0 \leqslant k, j \leqslant N$ with $k - j = n$ (while

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for $|n| > N$ we have none). If $k - j = n$, then $z^{-j}Uz^k$ is the continuous image of the compact set U under the continuous homeomorphism $x \to z^{-j}xz^k$. These continuous mappings take e to z^n, hence $z^{-j}Uz^k$ is a compact neighbourhood of z^n, and so are the sets

$$\begin{aligned} U_n &:= \bigcup \{z^{-j}Uz^k : 0 \leqslant j,k \leqslant N, k-j=n\} \\ &= \bigcup_{k=\max(n,0)}^{\min(N,N+n)} z^{n-k}Uz^k \\ &= z^n \bigcup_{k=\max(n,0)}^{\min(N,N+n)} z^{-k}Uz^k \end{aligned}$$

for all $n = -N, \dots, N$.

However, we cannot postulate here that all these $z^n \in \Omega$, and even less that $U_n \Subset \Omega$. The only certain fact is that *whenever $\psi(n) \neq 0$, the point z^n belongs to Ω*. This is because $\psi(n) \neq 0$ implies that $n \in S := \operatorname{supp}\psi \subset H = H(\Omega, z)$, which means by construction that $z^n \in \Omega$. The difficulty here is that even if n can occur as $k - j$ with $k, j \in \operatorname{supp}\theta$, it may still happen that $\psi(n) = (\theta \star \widetilde{\theta})(n) = \sum_{0\leqslant k,j\leqslant N,\ k-j=n} \theta(k)\overline{\theta(j)} = 0$, and so we have no guarantee that $z^n \in \Omega$.

So, before proceeding, let us clarify what conditions we assume when choosing the initial set U. We require the following to hold.

(i) $U^* := \bigcup_{k=0}^N z^{-k}Uz^k \subset \Omega$.

(ii) $U_n^* := z^nU^* \subset \Omega$ whenever $n \in S$.

(iii) If $0 < |n| \leqslant N$, then $z^n \notin U^*$, i.e. $e \notin z^{-n}U^* = U^*_{-n}$; equivalently, $z^n \notin U_k^*$ unless $n = k$.

Choosing U as a sufficiently small neighbourhood of $e \in \Omega$, together with U the homeomorphic images $z^{-k}Uz^k$ of U will also stay within Ω, hence (i) is easy to satisfy.

Now, if $n \in S \subset H$, then also $z^n \in \Omega$, i.e. $e \in z^{-n}\Omega$, and for fixed n a suitably small choice of U can ascertain that the $N+1$ homomorphic images $z^{-k}Uz^k$ of U all stay within $z^{-n}\Omega$; that is, $z^nU^* \subset \Omega$. Considering the necessary restrictions for the finitely many elements $n \in S$, it is also possible to simultaneously satisfy (ii) for all $n \in S$.

Finally, consider (iii). Let us take $\Omega^* := \Omega \setminus \{z^n : 0 < |n| \leqslant N\}$, which is a finite modification of Ω, and is still a symmetric open neighbourhood of e (for $z^n \neq e$ if $0 < |n| \leqslant N$). Arguing as in (i), we can thus ascertain that $U^* \subset \Omega^*$, hence $z^n \notin U^*$, and (iii) holds.

Next, let f be any continuous and positive definite function, supported compactly in the chosen $U \Subset \Omega$, and invariant under the effect of conjugation by z. By Proposition 6.4 such an f exists, as $z \in G$ was assumed to be a round element of G. Therefore, according to Lemma 4.1 (iv), the function $F(x) := \sum_{j=0}^N \sum_{k=0}^N \theta(k)\overline{\theta(j)}f(z^jxz^{-k})$ is also continuous and positive definite; moreover, we obviously have $\operatorname{supp}F \Subset \bigcup_{j,k=0}^N z^{-j}Uz^k = \bigcup_{n=-N}^N U_n \subset \bigcup_{n=-N}^N U_n^*$, hence F is compactly supported.

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However, we also need $\operatorname{supp} F \Subset \Omega$. Therefore, we compute the support of F more precisely, showing that it in fact lies within Ω. To this end, let us calculate $F(x)$ for a general point x, using the invariance of f with respect to conjugation by z. We compute

$$F(x) = \sum_{j,k=0}^{N} \theta(k)\overline{\theta(j)} f(z^j x z^{-k}) = \sum_{j,k=0}^{N} \theta(k)\overline{\theta(j)} f(z^{-k+j}x)$$
$$= \sum_{n=-N}^{N} \sum_{k=\max(n,0)}^{\min(N,N+n)} \theta(k)\overline{\theta(k-n)} f(z^{-n}x) = \sum_{n=-N}^{N} \psi(n) f(z^{-n}x) = \sum_{n\in S} \psi(n) f(z^{-n}x), \tag{8.1}$$

as $n \notin S$ provides $\psi(n) = 0$, hence the term drops out. Thus, we can better describe the support of F: $\operatorname{supp} F \Subset \bigcup_{n\in S} \operatorname{supp} f(z^{-n}x) = \bigcup_{n\in S} z^n \operatorname{supp} f \Subset \bigcup_{n\in S} U_n^* \Subset \Omega$, in view of (ii).

Therefore, with $F_0 := (1/F(e))F$, we obtain $F_0 \in \mathcal{F}^c(\Omega)$, noting that $F(e) \neq 0$, as F is not identically zero and $F \gg 0$. Actually, we can compute the precise values of $F(e)$ and $F(z)$ easily: $F(e) = \sum_{n\in S} \psi(n) f(z^{-n}) = \psi(0) f(e) = f(e)$, as $z^n \notin U$ for $0 < |n| \leqslant N$ by (iii), and $F(z) = \sum_{n\in S} \psi(n) f(z^{-n+1}) = \psi(1) f(e)$, for the same reason. Finally, $F_0(z) = (1/F(e))F(z) = (1/f(e))f(e)\psi(1) = \psi(1)$, hence $|\psi(1)| = |F_0(z)| \leqslant \sup\{|\phi(z)| : \phi \in \mathcal{F}_G^c(\Omega)\} = \mathcal{C}_G^c(\Omega, z)$. Taking the supremum on the left-hand side over all $\psi \in \mathcal{F}_{\mathbb{Z}}^c(H)$ thus leads to $\mathcal{C}^c(H) \leqslant \mathcal{C}_G^c(\Omega, z)$. □

Proof of Theorem 7.2. The proof is rather similar to the preceding one, once we carefully change all references from $\mathbb{Z}$ to $\mathbb{Z}_m$, $\mathcal{C}(H)$ to $\mathcal{C}_m(H)$ and $\mathcal{F}_{\mathbb{Z}}^{\#}(\Omega, z)$ and $\mathcal{F}_{\mathbb{Z}}^c(\Omega, z)$ to $\mathcal{F}_{\mathbb{Z}_m}(\Omega, z)$, and note that $Z := \langle z\rangle$ is now only a finite subgroup with $Z \cong \mathbb{Z}_m$, so the natural isomorphism $\eta(k) := z^k$ acts between $\mathbb{Z}_m$ and Z. Let us briefly work out the main steps of the argument.

Note that $H := H(\Omega, z)$ is finite together with the group $\mathbb{Z}_m$, so of course there is no difference between the formulations with respect to continuity or compactness assumptions, and the notations $\mathcal{C}_m(H)$ and $\mathcal{K}_m(H)$ of (1.7) and (1.9) are in effect.

Again, it suffices to prove $\mathcal{C}_G^c(\Omega, z) = \mathcal{C}_G^{\#}(\Omega, z) = \mathcal{C}_m(H)$ and $\mathcal{K}_G^c(\Omega, z) = \mathcal{K}_G^{\#}(\Omega, z) = \mathcal{K}_m(H)$. As before, the complex and real cases are entirely similar, so again we work out only the complex case. Once again, as $\mathcal{C}_G^c(\Omega, z) \leqslant \mathcal{C}_G^{\#}(\Omega, z)$, we need to prove only two inequalities, the first being that $\mathcal{C}_G^{\#}(\Omega, z) \leqslant \mathcal{C}_m(H)$.

Now take any $f \in \mathcal{F}_G^{\#}(\Omega)$. In view of Lemma 4.1 (i), $g := f|_Z \gg 0$ on Z, thus we have defined a function $g \in \mathcal{F}_Z((\Omega \cap Z))$. Note that as $o(z) = m$, we have $Z \simeq \mathbb{Z}_m$ and, in particular, finite. The natural isomorphism $\eta : \mathbb{Z}_m \to Z$ carries over g, defined on $Z \leqslant G$, to a positive definite function $\psi := g \circ \eta : \mathbb{Z}_m \to \mathbb{C}$ on $\mathbb{Z}_m$, which has normalized value $\psi(0) = g(e) = f(e) = 1$, and satisfies $\operatorname{supp}\psi \subset H$ for $\operatorname{supp} g \subset (\operatorname{supp} f \cap Z) \subset (\Omega \cap Z) = \{z^k : z^k \in \Omega\} = \eta(H)$ by definition of $H := H(\Omega, z)$ and $\eta : \mathbb{Z}_m \to Z$.

From here we see that $|f(z)| = |g(z)| = |\psi(1)| \leqslant \sup\{|\varphi(1)| : \varphi \in \mathcal{F}_{\mathbb{Z}_m}(H)\} = \mathcal{C}_m(H)$. Taking $\sup_{f\in\mathcal{F}_G^{\#}(\Omega)}$ on the left-hand side concludes the proof of the first part.

It remains to show $\mathcal{C}_m(H) \leqslant \mathcal{C}_G^c(\Omega, z)$, so the proof hinges upon the construction from an arbitrary function $\psi \in \mathcal{F}_{\mathbb{Z}_m}(H)$ another function $F_0 \in \mathcal{F}_G^c(\Omega, z)$ with $|\psi(1)| \leqslant |F_0(z)|$. Denote $S := \operatorname{supp}\psi \subset \mathbb{Z}_m$.

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As before, we manipulate using the convolution square root of ψ, provided now by Lemma 4.5. We thus get the sequence $\theta : \mathbb{Z}_m \to \mathbb{C}$ with $\psi(n) = \sum_{k \bmod m} \theta(k)\overline{\theta(k-n)} = (\theta \star \widetilde{\theta})(n)$.

However, without any further condition we can now always find for any given neighbourhood U of e a function $f \gg 0$, continuous and supported compactly in U, with the invariance property with respect to conjugation by z. Indeed, according to Proposition 6.4, such a function exists for any neighbourhood U of e if and only if $z \in G$ is a round element; however, that holds true in view of Example 6.8. So we have such a function f.

At this point, as in the previous proof of Theorem 7.1, we choose a proper neighbourhood U of e, satisfying, similar to (i), (ii), and (iii) in the previous case, the following analogous properties.

(i) $U^* := \bigcup_{k \in \mathbb{Z}_m} z^{-k} U z^k \subset \Omega$;

(ii) $U^*_n := z^n U^* \subset \Omega$ whenever $n \in S$;

(iii) If $0 \neq n \in \mathbb{Z}_m$, then $z^n \notin U^*$, i.e. $e \notin z^{-n}U^* = U^*_{-n}$; equivalently, $z^n \notin U^*_k$ unless $n = k \mod m$.

Choosing U as a sufficiently small neighbourhood of $e \in \Omega$, for any $k \in \mathbb{Z}_m$, the homeomorphic image $z^{-k}Uz^k$ of U will, together with U, stay within Ω, hence (i) holds for U^*.

Now, if $n \in S \subset H$, then also $z^n \in \Omega$, i.e. $e \in z^{-n}\Omega$, and for fixed n a suitably small choice of U can ascertain that the m homomorphic (and also continuous) images $z^{-k}Uz^k$ of U all stay within $z^{-n}\Omega$, that is, $z^nU^* \subset \Omega$. Considering the necessary restrictions for the finitely many elements $n \in S$, it is also possible to simultaneously satisfy (ii) for all $n \in S$.

Finally, consider (iii). Let us take $\Omega^* := \Omega \setminus \{z^n : 0 \neq n \mod m\}$, which is a finite modification of Ω, and is still a symmetric open neighbourhood of e (for $z^n \neq e$ if $0 \neq n \mod m$). Arguing as in (i), we can thus ascertain that $U^* \subset \Omega^*$, hence $z^n \notin U^*$, and (iii) holds.

Finally, we define the new function $F(x) := \sum_{j \bmod m} \sum_{k \bmod m} \theta(k)\overline{\theta(j)} f(z^j x z^{-k})$, with the above f as before. Then, exactly as above, we find $F \gg 0$, $F \in C(G)$, $\operatorname{supp} F \subset \Omega$, $F(e) = f(e)$ and $F(z) = \psi(1)f(e)$, thus $F_0 := (1/F(e))F \in \mathcal{F}^c_G(\Omega, z)$ and $|F_0(z)| = |\psi(1)|$, concluding the argument in this case. □

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References

1. V. Arestov, E. Berdysheva and H. Berens, On pointwise Turán's problem for positive definite functions, *East J. Approx.* **9**(1) (2003), 31–42.
2. G. Bachman, *Elements of abstract harmonic analysis* (Academic Press, New York, London, 1964).

https://doi.org/10.1017/S0013091517000062 Published online by Cambridge University Press

3. U. Baumgartner and G. A. Willis, Contraction groups and scales of automorphisms of totally disconnected locally compact groups, *Israel J. Math.* **142** (2004), 221–248.
4. R. P. Boas, Jr. and M. Kac, Inequalities for Fourier transforms of positive functions, *Duke Math. J.* **12** (1945), 189–206.
5. P.-E. Caprace and N. Monod, Decomposing locally compact groups into simple pieces, *Math. Proc. Cambridge Philos. Soc.* **150**(1) (2011), 97–128.
6. P.-E. Caprace, C. D. Reid and G. A. Willis, Limits of contraction groups and the Tits core, *J. Lie Theory* **24**(4) (2014), 957–967.
7. C. Carathéodory, Über den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen, *Rend. Circ. Mat. Palermo* **32** (1911), 193–217.
8. W. Ehm, T. Gneiting and D. Richards, Convolution roots of radial positive definite functions with compact support, *Trans. Amer. Math. Soc.* **356**(11) (2004), 4655–4685.
9. R. Ellis, Distal transformation groups, *Pacific J. Math.* **8** (1958), 401–405.
10. L. Fejér, Über trigonometrische Polynome, *J. Reine Angew. Math.* **146** (1915), 53–82.
11. L. Fejér, *Gesammelte arbeiten I–II* (Akadémiai Kiadó, Budapest, 1970).
12. G. B. Folland, *A course in abstract harmonic analysis*, Studies in Advanced Mathematics (CRC Press, Boca Raton, FL, 1995).
13. P. R. Halmos, *Finite dimensional vector spaces*, 2nd edition (Van Nostrand, 1958).
14. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, I, Die Grundlehren der mathemtischen Wissenchaften in Einzeldarstellungen, Volume 115 (Springer, 1963).
15. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, II, Die Grundlehren der mathemtischen Wissenchaften, Volume 152 (Springer, 1970).
16. V. I. Ivanov and A. V. Ivanov, Turán problems for periodic positive definite functions, *Ann. Univ. Sci. Budapest. Sect. Comput.* **33** (2010), 219–237.
17. W. Jaworski, On contraction groups of automorphisms of totally disconnected locally compact groups, *Israel J. Math.* **172** (2009), 1–8.
18. M. N. Kolountzakis and Sz. Gy. Révész, On pointwise estimates of positive definite functions with given support, *Canad. J. Math.* **58**(2) (2006), 401–418.
19. M. N. Kolountzakis and Sz. Gy. Révész, Turán's extremal problem for positive definite functions on groups, *J. Lond. Math. Soc.* **74** (2006), 475–496.
20. S. Krenedits and Sz. Gy. Révész, Carathéodory–Fejér type extremal problems on locally compact Abelian groups, *J. Approx. Theory* **194** (2015), 108–131.
21. M. Mathias, Über Positive Fourier-Integrale, *Math. Z.* **16** (1923), 103–125 (in German).
22. Sz. Gy. Révész, Turán's extremal problem on locally compact abelian groups, *Anal. Math.* **37** (2011) 15–50.
23. C. R. E. Raja and R. Shah. Some properties of distal actions on locally compact groups, *Ergodic Theory Dynam. Systems*, to appear.
24. W. Rudin, *Fourier analysis on groups*, Interscience Tracts in Pure and Applied Mathematics, Volume 12 (Interscience Publishers, 1962).
25. S. B. Stechkin, An extremal problem for trigonometric series with nonnegative coefficients, *Acta Math. Acad. Sci. Hung.* **23**(3–4) (1972), 289–291 (in Russian).
26. G. Szegő, *Orthogonal polynomials*, American Mathematical Society Colloquium Publications, Volume XXIII (American Mathematical Society, Providence, RI, 1975).
27. O. Toeplitz, Über die Fourier'sche Entwicklung positiver Funktionen, *Rend. Circ. Mat. Palermo* **32** (1911), 191–192.

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