Probability in the Engineering and Informational Sciences, **20**, 2006, 231–250. Printed in the U.S.A.

SOME NEW BOUNDS FOR THE RENEWAL FUNCTION

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In the literature, most of the bounds for the renewal function $U(x)$ corresponding to a lifetime distribution F are given in terms of the first two moments of F only. The best general upper bound of this type is the one given in Lorden (1970). In the present article, we show that improved bounds can be obtained if one exploits the specific form of the distribution F. We derive a bound that improves upon Lorden's, at least on an interval $[0, a)$ with $a \leq \infty$, and we give both sufficient and necessary conditions for this improvement to hold uniformly for $x \geq 0$. Refined upper as well as lower bounds are given for the case where F belongs to a class of distributions with monotone aging or when the renewal density is monotone.

1. INTRODUCTION

Let F be a probability distribution on $[0,\infty)$ and $U(x) = \sum_{k=0}^{\infty} F^{*k}(x)$ be the associated renewal function; here, and in the sequel, $F^{*k}(x)$ denotes the kth Lebesgue–Stieltjes convolution power of F, whereas for $k = 0$, we conventionally set $F^{*0}(x) = 1_{[0,\infty)}$, the indicator function on the nonnegative half-axis. $U(x)$ is typically seen as the expected number of occurrences of an event in a renewal process, counting the occurrence at time 0.

Lorden [12] gave the following general upper bound for U:

$$U(x) \leq \frac{x}{\mu} + \frac{\mu_2}{\mu^2} := U_L(x), \tag{1}$$

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https://doi.org/10.1017/S0269964806060141 Published online by Cambridge University Press

where

$$\mu = \int_{[0,\infty)} x\, dF(x), \qquad \mu_2 = \int_{[0,\infty)} x^2\, dF(x).$$

Note that a general lower bound for U is given in Barlow and Proschan [2, p. 171], namely

$$U(x) \geq \frac{x}{\mu}, \tag{2}$$

so that we have the two-sided bound $x\mu^{-1} \leq U(x) \leq x\mu^{-1} + \mu_2\mu^{-2}$. Both bounds have been improved for certain nonparametric classes of distributions for F; if F is new better (worse) than used in expectation (NBUE, resp. NWUE), Barlow and Proschan [2, p. 171] stated that

$$U(x) \leq (\geq) 1 + \frac{x}{\mu}. \tag{3}$$

If F is increasing mean residual life (IMRL), then a sharpening of the upper bound is given by [3]

$$U(x) \leq \frac{x}{\mu} + \frac{\mu_2}{2\mu^2}, \tag{4}$$

and, finally, if F has increasing failure rate (IFR), then the following improvement of the lower bound has been given in Brown [4]:

$$U(x) \geq \frac{x}{\mu} + \frac{\mu_2 - \mu^2}{\mu^2}. \tag{5}$$

A common feature of these bounds is that they only require knowledge of the first (and, in some cases, second) moment of the distribution F and not the exact form or even the shape of the distribution itself. One thus anticipates that sharper bounds than the above can be obtained if one utilizes the specific knowledge of F; the following is a simple example of this.

Let $F_e(x) = \int_0^x \bar{F}(t)\, dt/\mu$ be the equilibrium distribution corresponding to F. Its tail is given by

$$\overline{F_e}(x) = 1 - F_e(x) = \frac{1}{\mu} \int_x^\infty \bar{F}(t)\, dt.$$

Consider the renewal equation

$$H(x) = \overline{F_e}(x) + \int_{[0,x]} H(x - t)\, dF(t) \tag{6}$$

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whose unique solution is known to be (see, e.g., [10])

$$H(x) = U(x) - \frac{x}{\mu}.$$

By the nonnegativity of the integral in (6) we obtain immediately that

$$U(x) \geq \frac{x}{\mu} + \overline{F}_e(x), \tag{7}$$

which improves the lower bound for U in (2). Note in particular that the above bound is exact for $x = 0$, whereas that in (2) is not. Moreover, another way to obtain (7) is to *insert* the bound provided by (2) into the well-known renewal equation for U (e.g., [10, XI]),

$$U(x) = 1 + \int_{[0,\,x]} U(x - t)\, dF(t). \tag{8}$$

This approach, which we will use in the following sections, has been used by Chadjiconstantinidis and Politis [6,7] who established new bounds for various quantities of interest in actuarial risk theory. For a variant of this method, see also Politis [14]. The underlying theme of the method is to start with an existing (upper or lower) bound for the solution of a renewal equation; inserting this bound in that equation, in turn, yields another, upper or lower, bound, which in many cases is uniformly better than the original one.

In the following section we present various bounds for the renewal function U. As it turns out, these bounds are particularly useful if one wishes to use a bound for $U(x)$ that performs well for small values of x. In Theorem 2.1 we provide a bound that is always better than Lorden's bound in a neighborhood of zero. To compensate for the fact that the bounds of the present article require knowledge of F, rather than μ and μ_2, we derive bounds for U that can be applied even if the first moment of F is not finite; see, in particular, Theorem 3.2. We demonstrate how refined bounds might be obtained if one assumes that the renewal density is monotone (Section 3) or that F belongs to an aging class (Section 4) and we give numerical examples to illustrate the performance of the bounds. Further, we explain how one might use bounds for U to derive upper or lower bounds for the tail of a compound geometric distribution. The last section has some concluding remarks.

2. LOWER AND UPPER BOUNDS FOR *U*

In this section we use the method described in Section 1 to establish a new general upper bound for the renewal function U corresponding to a probability distribution F on $[0,\infty)$. To avoid unnecessary trivialities, we assume throughout that $F(0) = 0$,

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although it should be clear that the method applies with minor modifications if F has a mass at the origin. Moreover, we will assume that F is not concentrated on one point.

THEOREM 2.1: *For $x \geq 0$, an upper bound for the renewal function is given by*

$$U(x) \leq \frac{x}{\mu} + \frac{\mu_2 F(x)}{\mu^2} + \overline{F_e}(x) := U_1(x). \tag{9}$$

Moreover, this bound is better than Lorden's for all x such that

$$\mu \int_x^\infty \bar{F}(t)\, dt \leq \mu_2 \bar{F}(x). \tag{10}$$

The last relation is always true at least on an interval $[0, a)$ with $0 < a \leq \infty$.

PROOF: Inserting Lorden's bound given in (1) into the renewal equation (8) gives

$$\begin{aligned} U(x) &\leq 1 + \frac{\mu_2 F(x)}{\mu^2} + \int_{[0,x]} \frac{x-y}{\mu}\, dF(y) \\ &= 1 + \frac{\mu_2 F(x)}{\mu^2} + \frac{xF(x)}{\mu} - \frac{1}{\mu} \int_{[0,x]} y\, dF(y). \end{aligned}$$

A simple integration by parts yields

$$\int_{[0,x]} y\, dF(y) = \mu - x\bar{F}(x) - \int_x^\infty \bar{F}(y)\, dy, \tag{11}$$

and the bound in (9) follows immediately from the last two relations.

In order to compare $U_1(x)$ with $U_L(x)$, let $\Delta(x) = U_L(x) - U_1(x)$. Then, for $\Delta(x)$, we have

$$\begin{aligned} \Delta(x) &= \frac{\mu_2}{\mu^2} - \frac{\mu_2 F(x)}{\mu^2} - \frac{1}{\mu} \int_x^\infty \bar{F}(y)\, dy \\ &= \frac{\mu_2 \bar{F}(x)}{\mu^2} - \frac{1}{\mu} \int_x^\infty \bar{F}(y)\, dy \end{aligned} \tag{12}$$

and, therefore, $U_1(x) \leq U_L(x)$ if and only if (10) holds, as asserted.

Finally, it is clear that $\Delta(0) = \mu_2/\mu^2 - 1$, which is strictly positive since F is nondegenerate. Since Δ is a continuous function, we have $\Delta(x) > 0$, at least in a neighborhood near zero. ■

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Marshall [13] used a similar approach to establish an upper bound of the form

$$U(x) \le \frac{x}{\mu} + \sup_t m(t),$$

where $m(t) = \int_t^\infty \bar{F}(y)\,dy/\bar{F}(t)$ is the mean residual life (MRL) function associated with F and the supremum above is taken over all t with $\bar{F}(t) > 0$. A disadvantage of the last bound is that it cannot be used for distributions with unbounded MRL, such as IMRL distributions with $\lim_{t\to\infty} m(t) = \infty$. However, as demonstrated by Marshall [13], one advantage of the method is that it can be used recursively, by inserting the successive bounds in (8), and this also can be done with the bound in (9).

A direct comparison of the upper bound (9) to the lower bound (7) reveals that, on using both of them, we can provide an interval estimation for $U(x)$ of length $\mu_2 F(x)/\mu^2$.

Moreover, it is worth noting that if the distribution F in Theorem 2.1 is NBUE, then

$$\int_x^\infty \bar{F}(t)\,dt \le \mu\bar{F}(x) \le \frac{\mu_2}{\mu}\,\bar{F}(x)$$

(i.e. condition (10) holds for all x). Therefore, in this case, $U_1(x)$ is uniformly better than Lorden's bound $U_L(x)$.

On the other hand, if (10) is true in a neighborhood of zero ($0 \le x < a$), we can write

$$m(x) = \frac{\displaystyle\int_x^\infty \bar{F}(t)\,dt}{\bar{F}(x)} \le \frac{\mu_2}{\mu}, \qquad 0 \le x < a,$$

and taking into account that the mean residual lifetime can be expressed as

$$m(x) = -\left[\frac{d\ln\left(\displaystyle\int_x^\infty \bar{F}(t)\,dt\right)}{dx}\right]^{-1},$$

we obtain

$$\ln\left(\int_a^\infty \bar{F}(t)\,dt\right) - \frac{\mu(x-a)}{\mu_2} \le \ln\left(\int_x^\infty \bar{F}(t)\,dt\right) \le \ln\mu - \frac{\mu x}{\mu_2}.$$

This leads to the double inequality

$$\mu\overline{F_e}(a)\exp\left(-\frac{\mu(x-a)}{\mu_2}\right) \le \int_x^\infty \bar{F}(t)\,dt \le \mu e^{-\mu x/\mu_2},$$

which offers a necessary condition for (10) to hold on the interval $0 \le x < a$.

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Example 1: Let F be exponential with mean 1. Then the renewal function $U(x)$ equals $U(x) = 1 + x$. Lorden's bound turns out to be $U_L(x) = 2 + x$, and the new bound is

$$U_1(x) = x + 2(1 - e^{-x}) + e^{-x} = 2 + x - e^{-x}.$$

Example 2: Consider a Weibull distribution with survival function

$$\bar{F}(x) = e^{-\tau x^{\gamma}}.$$

When $\gamma \geq 1$, the distribution is NBUE, so in view of the above remark, we have $a = \infty$ in the theorem. Thus, we consider an example where $\gamma < 1$, so that the improvement of $U_1(x)$ over Lorden's bound holds on a bounded interval $[0, a)$. Taking $\tau = 3$ and $\gamma = \frac{1}{2}$, we see that $U_L(x) = 6 + 9x/2$, and some straightforward calculations yield that the MRL function equals

$$m(x) = \frac{2}{9} + \frac{2}{3}\sqrt{x}.$$

Note that $\mu_2/\mu = \frac{4}{3}$, so that $m(x)$ is smaller than μ_2/μ in the interval $[0, \frac{25}{9})$ and larger than μ_2/μ in the interval $(\frac{25}{9}, \infty)$.

Table 1 presents the values of the two bounds $U_1(x)$ and $U_L(x)$ and their difference, $\Delta(x) = U_L(x) - U_1(x)$, for various values of x. Lorden's bound performs poorly for small values of x; notice in particular that $\Delta(0) = 5$ and that for small values of x, say $x \leq 1$, there is a substantial improvement of $U_1(x)$ over $U_L(x)$. On the other hand, for values of $x > \frac{25}{9}$, when U_L is better, the magnitude of $\Delta(x)$ is always smaller than 3×10^{-3}.

Table 1. Values of the Two Bounds $U_L(x)$, $U_1(x)$ and Their Difference

	$U_L(x)$	$U_1(x)$	$\Delta(x) = U_L(x) - U_1(x)$
$x = 0$	6.0000	1.0000	5.0000
$x = 0.25$	7.1250	6.3440	0.7810
$x = 0.5$	8.2500	7.9049	0.3451
$x = 0.75$	9.3750	9.1963	0.1787
$x = 1$	10.5000	10.4004	0.0996
$x = 1.5$	12.7500	12.7164	0.0336
$x = 2$	15.0000	14.9891	0.0109
$x = 3$	19.5000	19.5011	−0.0011
$x = 5$	28.5000	28.5021	−0.0021
$x = 8$	42.0000	42.0007	−0.0007
$x = 10$	51.0000	51.0003	−0.0003
$x = 12$	60.0000	60.0002	−0.0002

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Example 3: We have seen earlier that if F is NBUE, then condition (10) holds, so that $a = \infty$ in Theorem 2.1. Is the NBUE condition necessary for $a = \infty$ in the theorem? The answer is negative, as elucidated by the following example.

Consider a lifetime distribution F with density $f(x)$ given by

$$f(x) = \begin{cases} 2xe^{-x^2}, & 0 \le x < 1 \\ xe^{-x}/2, & x \ge 1. \end{cases}$$

It is easily checked that the MRL function corresponding to F is

$$m(x) = \begin{cases} \sqrt{\pi}e^{x^2}(1 - \mathrm{Erf}(x))/2, & 0 \le x < 1 \\ (2 + x)/(1 + x), & x \ge 1, \end{cases}$$

where Erf denotes the error function. Note that although F has two NBUE parts, it is not a NBUE distribution itself. For a NBUE distribution, it holds that $m(x) \le m(0)$ for all x, whereas here, for example, $m(1) = 1.5$ and $m(0) = 0.89$. Further, one can verify that $\mu = 1.2986$ and $\mu_2 = 3.2073$, so that $\mu_2/\mu = 2.47$. On noting that the maximum of the MRL function is achieved at $x = 1$, with $m(1) = 1.5$, we have shown that $m(x) \le \mu_2/\mu$ for all x, which, in turn, yields that condition (10) in Theorem 2.1 holds uniformly on the nonnegative half-axis.

Condition (10) requires $m(x)$ to be bounded above by μ_2/μ, a fact turning $U_1(x)$ to a uniform improvement of Lorden's bound. At this point it is natural to ask whether an alternative upper bound could be established if $m(x)$ was simply bounded, not necessarily by the quantity μ_2/μ. The next proposition gives an affirmative answer to this, providing a much more flexible bound than $U_1(x)$. Note also that in the case of a MRL that is bounded below, a lower bound for $U(x)$ is deduced.

PROPOSITION 2.2: *Let a life distribution F be such that its MRL function is bounded above (below) by a constant r; that is,*

$$m(x) \le (\ge) r \qquad \text{for all } x \in [0, a) \tag{13}$$

for some $0 < a \le \infty$. *Then for the renewal function associated with F and for* $x \in [0, a)$, *it holds that*

$$U(x) \le (\ge) \frac{x}{\mu} + \frac{rF(x)}{\mu} + \overline{F_e}(x).$$

PROOF: We mention first the well-known fact that the solution of (6) is given by

$$U(x) - \frac{x}{\mu} = \int_{[0,x]} \overline{F_e}(x - t)\, dU(x), \tag{14}$$

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(see e.g. Feller [10, XI]). Further, since U has a unit mass at zero, (14) can be rewritten as

$$U(x) - \frac{x}{\mu} = \overline{F}_e(x) + \sum_{k=1}^{\infty} \int_{[0,x]} \overline{F}_e(x-t)\, dF^{*k}(x). \tag{15}$$

Since $m(x) = \mu\overline{F}_e(x)/\bar{F}(x)$, the condition $m(x) \leq (\geq) r$ for $x \in [0,a)$ implies that $\overline{F}_e(x) \leq (\geq) r\bar{F}(x)/\mu$, and applying this on the integral term of the right-hand-side series, we obtain

$$\begin{aligned} U(x) - \frac{x}{\mu} &\leq (\geq)\overline{F}_e(x) + \frac{r}{\mu} \sum_{k=1}^{\infty} \int_{[0,x]} \bar{F}(x-t)\, dF^{*k}(x) \\ &= \overline{F}_e(x) + \frac{r}{\mu} \sum_{k=1}^{\infty} (F^{*k}(x) - F^{*k+1}(x)) \\ &= \overline{F}_e(x) + \frac{r}{\mu} F(x) \end{aligned}$$

for $x \in [0,a)$, as asserted. ■

If the life distribution F is NBUE (NWUE), then the MRL function is bounded above (below) by μ for all $x \in [0,\infty)$. Thus, we arrive at the following result, which is an improvement over (3).

COROLLARY 2.3: *If F is NBUE (NWUE), then*

$$U(x) \leq (\geq) \frac{x}{\mu} + F(x) + \overline{F}_e(x). \tag{16}$$

Note that Proposition 2.2 is of much wider applicability than Corollary 2.3. First, the corollary applies only when F is NBUE (NWUE), whereas Proposition 2.2 is in force for the much larger class of distributions whose mean residual life is bounded above (below). Next, condition (13) is only required to hold on a finite interval, not on the whole of the nonnegative half-axis; therefore, the proposition is in fact applicable to all distributions whose MRL is bounded in an interval $[0,a)$ for some $a > 0$. This implies, in particular, that for most distributions of interest, *both* a lower and an upper bound are available.

Before closing this section, we state two additional lower bounds for $U(x)$ that do not require any assumptions on the underlying distribution F.

Insert first (7) into (8) to obtain

$$\begin{aligned} U(x) &\geq 1 + \int_{[0,x]} \overline{F}_e(x-t)\, dF(t) + \int_{[0,x]} \frac{x-t}{\mu}\, dF(t) \\ &= 1 + \int_{[0,x]} \overline{F}_e(x-t)\, dF(t) + \frac{xF(x)}{\mu} - \frac{1}{\mu} \int_{[0,x]} y dF(y), \end{aligned}$$

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and then use (11) to deduce the more appealing expression

$$U(x) \geq 1 + \int_{[0,x]} \overline{F}_e(x-t)\, dF(t) + \frac{xF(x)}{\mu} - \frac{1}{\mu}\left(\mu - x\bar{F}(x) - \int_x^\infty \bar{F}(t)\, dt\right)$$

$$= \int_{[0,x]} \overline{F}_e(x-t)\, dF(t) + \frac{x}{\mu} + \overline{F}_e(x).$$

This can also be obtained as an immediate consequence of the fact that the solution of (6) is given by (14). From (14), another simple lower bound can be obtained; in particular, since $\overline{F}_e$ is decreasing, we get

$$U(x) - \frac{x}{\mu} \geq \overline{F}_e(x) U(x),$$

and solving for $U(x)$, this gives that for all x such that $F_e(x) > 0$,

$$U(x) \geq \frac{x}{\mu F_e(x)},$$

which is another improvement of (2).

3. BOUNDS FOR RENEWAL FUNCTIONS WITH MONOTONE DENSITY

In this section we provide bounds for the renewal function U under the assumption that a version of the associated renewal density u is monotone. Note that u is related to U via the relation

$$U(x) = 1 + \int_0^x u(t)\, dt.$$

A sufficient condition for u to be decreasing is that F has decreasing failure rate (DFR). If F is IFR, however, it does not follow that u is increasing. As Shaked and Zhu [17] proved, a necessary and sufficient condition for U to be concave (convex), so that the renewal density u is decreasing (increasing), is that the excess lifetime is increasing (decreasing) in t. However, as this might be a difficult condition to check, we note that a necessary (but not sufficient) condition for the concavity (convexity) of the renewal function is that the distribution F is NWU (NBU) [8].

In the analysis that follows, we will use often the following simple result from calculus (see, e.g., [1]).

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LEMMA 3.1: *Let f and g be two functions that are integrable over a finite interval $[a,b]$ and they have different monotonicity within that interval. Then it holds that*

$$\int_a^b f(x)g(x)\,dx \le \frac{1}{b-a}\left(\int_a^b f(x)\,dx\right)\left(\int_a^b g(x)\,dx\right).$$

Moreover, if f and g have the same monotonicity on $[a,b]$, then the sign of the above inequality is reversed.

We now state one of our main results, which gives a two-sided bound for U.

THEOREM 3.2: *Assume that $U(x)$ has a density $u(x)$ that is increasing (decreasing). Then it holds that*

$$U(x) \le (\ge) 1 + \frac{xF(x)}{\int_0^x \bar{F}(t)\,dt}. \tag{17}$$

Provided that F has a finite first moment μ, the above inequality can be expressed as

$$U(x) \le (\ge) 1 + \frac{x}{\mu} + \frac{x[F(x) - F_e(x)]}{\int_0^x \bar{F}(t)\,dt}. \tag{18}$$

Moreover, complementary bounds if u is increasing (decreasing) are given by

$$U(x) \ge (\le) 1 + \frac{x}{\mu} + \frac{x\left[\mu^{-1}\int_0^x \overline{F_e}(t)\,dt - F_e(x)\right]}{\int_0^x F_e(t)\,dt}. \tag{19}$$

PROOF: First we express (8) in terms of the renewal density. Since convolution is commutative between positive (possibly infinite) measures, we get

$$U(x) = 1 + \int_0^x F(x-t)\,dU(t) = 1 + F(x) + \int_0^x F(x-t)u(t)\,dt.$$

Assume now that $u(x)$ is decreasing. Using Lemma 3.1, we then obtain that

$$\begin{aligned} U(x) &\ge 1 + F(x) + \frac{1}{x}\left(\int_0^x F(t)\,dt\right)\left(\int_0^x u(t)\,dt\right) \\ &= 1 + F(x) + \frac{1}{x}\left(\int_0^x F(t)\,dt\right)[U(x) - 1] \end{aligned}$$

and the lower bound in (17) follows by solving for $U(x)$.

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For the upper bound in (19), note that (14) can be written as

$$U(x) - \frac{x}{\mu} = \overline{F}_e(x) + \int_0^x \overline{F}_e(x-t)u(t)\,dt.$$

Since u is decreasing and for a fixed x, $\overline{F}_e(x-t)$ is an increasing function of t, again using Lemma 3.1, we derive that

$$U(x) - \frac{x}{\mu} \le \overline{F}_e(x) + \frac{1}{x}[U(x)-1]\int_0^x \overline{F}_e(t)\,dt,$$

from which

$$U(x)\left[1 - \frac{1}{x}\int_0^x \overline{F}_e(t)\,dt\right] \le \frac{x}{\mu} + \overline{F}_e(x) - \frac{1}{x}\int_0^x \overline{F}_e(t)\,dt.$$

Multiplying both sides by x gives

$$U(x) \le \frac{x^2\mu^{-1} + x\overline{F}_e(x) - \int_0^x \overline{F}_e(t)\,dt}{x - \int_0^x \overline{F}_e(t)\,dt},$$

which, on replacing $\int_0^x \overline{F}_e(t)\,dt$ by $x - \int_0^x F_e(t)\,dt$ and $\overline{F}_e(x)$ by $1 - F_e(x)$, reduces to

$$U(x) \le 1 + \frac{x^2\mu^{-1} - xF_e(x)}{\int_0^x F_e(t)\,dt}.$$

It is readily checked that the last inequality is the same as the upper bound in (19).

The proof of the complementary bounds in (17) and (19) for the case where u is increasing is similar. Finally, to show that the lower (upper) bound in (17) is the same as the corresponding bound in (18) when μ is finite, note that

$$\frac{xF(x)}{\int_0^x \bar{F}(t)\,dt} - \frac{x}{\mu} = \frac{xF(x) - x\mu^{-1}\int_0^x \bar{F}(t)\,dt}{\int_0^x \bar{F}(t)\,dt} = \frac{xF(x) - xF_e(x)}{\int_0^x \bar{F}(t)\,dt},$$

and this concludes the proof. ■

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It is worth noting that for (17) to hold, we do not require the existence of *any* moments for F.

As already mentioned, if F is DFR, then the density u is decreasing and therefore the bound in (17) or (18) can be used. Since F will also belong to the NWUE class, the lower bound (3) is also applicable. In this case, the lower bounds provided by Theorem 3.2 are preferable over (3), as the next proposition indicates.

PROPOSITION 3.3: *If F is DFR, then the lower bound in Theorem 3.2 is sharper than the corresponding bound in (3) for all $x \geq 0$.*

PROOF: For the result to hold, we require that $F(x) - F_e(x) \leq 0$ for all $x \geq 0$, which is the same as

$$\mu \bar{F}(x) \leq \int_x^{\infty} \bar{F}(t)\, dt.$$

However, this is obviously true; F being DFR implies that it is also NWUE. ■

Assume now that F has a density f and let

$$u(x) = \sum_{k=1}^{\infty} f^{*k}(x) \tag{20}$$

be the associated renewal density. Here $*$ denotes the usual convolution between integrable functions, defined by $(f * g)(x) = \int_0^x f(x-t)g(t)\, dt$, and f^{*k} is the kth convolution power of f.

If the distribution F is IMRL, it follows from the results of Brown [3] that there exists a version of the renewal density such that $u(x) \geq \mu^{-1}$ for $x \geq 0$. Using this inequality on the right-hand side of the renewal equation

$$u(x) = f(x) + \int_0^x u(x-t) f(t)\, dt, \tag{21}$$

we obtain

$$u(x) \geq f(x) + \frac{1}{\mu} F(x),$$

and inserting this bound again into (21) yields

$$u(x) \geq f(x) + f^{*2}(x) + \frac{1}{\mu} F^{*2}(x).$$

By induction, we get that for all $m \geq 1$, the following inequality holds:

$$u(x) \geq \sum_{k=1}^{m} f^{*k}(x) + \frac{1}{\mu} F^{*m}(x).$$

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This gives a new bound for U via the relation $U(x) = 1 + \int_0^x u(t)\,dt$; more explicitly,

$$U(x) \geq \sum_{k=0}^{m} F^{*k}(x) + \frac{1}{\mu}\int_0^x F^{*m}(t)\,dt.$$

In view of the above-mentioned inequality, it is clear that

$$U(x) - \sum_{k=0}^{m} F^{*k}(x) = \sum_{k=m+1}^{\infty} F^{*k}(x) \geq \frac{1}{\mu}\int_0^x F^{*m}(t)\,dt.$$

One possible use of this bound is the following. A very simple and commonly used approximation for U is based on truncating the series of convolution powers of F, so that $U \approx \sum_{k=0}^{m} F^{*k}$ for some positive integer m. The bound above thus yields a lower bound for the truncation error of that approximation.

Finally, we mention that if F is DFR, then it is also concave (see, e.g., Sengupta and Nanda [16]), so that, in particular, it possesses a decreasing density, $f(x)$. Thus, we have from (21),

$$\frac{1}{\mu} \leq u(x) = f(x) + \int_0^x f(x-t)u(t)\,dt$$

and applying Lemma 3.1, we derive that

$$\frac{1}{\mu} \leq f(x) + \frac{1}{x}F(x)[U(x) - 1].$$

Solving this for U, we obtain the following lower bound:

$$U(x) \geq 1 + \frac{x}{\mu F(x)} - \frac{xf(x)}{F(x)}.$$

4. BOUNDS UNDER AGING PROPERTIES FOR THE INTERARRIVAL TIMES

In this section we obtain refined bounds under various additional assumptions on the distribution F, which represents the interarrival time between renewals. More specifically, we consider the case where F belongs to an aging class of distributions, such as those mentioned in Section 1. Our first result concerns the IFR class of distributions.

PROPOSITION 4.1: *Let F be an IFR life distribution with finite mean μ and variance σ^2. Then for the renewal function associated with F, we have the lower bound*

$$U(x) \geq \frac{x}{\mu} + \frac{\sigma^2 F(x)}{\mu^2} + \overline{F_e}(x). \tag{22}$$

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PROOF: Inserting Brown's result (5) into (8) we get, by virtue of (11),

$$U(x) \geq 1 + \frac{\sigma^2 F(x)}{\mu^2} + \int_{[0,x]} \frac{x-y}{\mu}\, dF(y)$$

$$= 1 + \frac{\sigma^2 F(x)}{\mu^2} + \frac{xF(x)}{\mu} - \frac{1}{\mu}\left[\mu - x\bar{F}(x) - \int_x^\infty \bar{F}(y)\, dy\right],$$

and the result follows immediately. ■

Note that the new bound is better than that in (5) for all x such that

$$\frac{\overline{F_e}(x)}{\bar{F}(x)} \geq \frac{\sigma^2}{\mu^2},$$

which is the same as $m(x) \geq \sigma^2/\mu$. Since $m(0) = \mu > \sigma^2/\mu$ for any IFR distribution with finite mean, the previous condition is true at least in an interval $[0, b)$ with $0 < b \leq \infty$. Further, when F is IFR with a finite mean, it is also DMRL, so that $m(x)$ is decreasing. Thus, $b = \infty$ for all IFR distributions whose MRL function has a limit greater than or equal to σ^2/μ (i.e., $\lim_{x\to\infty} m(x) \geq \sigma^2/\mu$).

As an illustration, consider the case where F is a Gamma distribution with shape parameter $\alpha > 1$ and scale parameter λ. Then it holds that

$$\frac{\sigma^2}{\mu} = \frac{1}{\lambda},$$

whereas for the MRL function $m(x)$, we have

$$\lim_{x\to\infty} m(x) = \frac{1}{\lambda}, \qquad m(x) \geq \frac{1}{\lambda} \qquad \text{for all } x \geq 0.$$

Thus, in this case, the lower bound for $U(x)$ in Proposition 4.1 is an improvement over that in (5) for all $x \geq 0$.

This is illustrated further in the following example where the two bounds in (5) and (22) are also compared with the lower bound given in Theorem 3.2. Note first that the lower bounds for $U(x)$ from (19) apply when the renewal density u is increasing, and the lower bounds from (5) and Proposition 4.1 can be used when the underlying distribution F is IFR. Since, in general, F is IFR does not imply that u is increasing, we use a well-known example of a Gamma distribution with shape parameter greater than one (so that it is IFR) for which the renewal density is known to be increasing. This has the additional advantage that the renewal function can be found explicitly, so that the absolute as well as the relative performance of the various bounds can be assessed.

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Example 4: Consider the IFR Gamma density $f(x) = xe^{-x}$, with associated renewal density

$$u(x) = \frac{1}{2} - \frac{1}{2}e^{-2x}.$$

Brown's [4] bound for $U(x)$ (cf. (5)) is, in this case,

$$U(x) \geq \frac{x+1}{2},$$

whereas the lower bound from Proposition 4.1 takes the form

$$U(x) \geq \frac{x+1+e^{-x}}{2}.$$

Call $L_B(x)$ and $L_1(x)$ the bounds from (5) and Proposition 4.1, respectively, and let $L_2(x)$ be the lower bound for the renewal function from (19). Table 2 presents these bounds, as well as the true values for $U(x)$, for various choices of x.

As expected from the above expressions, L_1 is uniformly better than L_B; however, the improvement becomes negligible for large values of x. On the other hand, L_2 is much better than L_B for all values of x, and its relative performance over L_1 increases as x gets larger; we mention in particular, that, in contrast with the other two bounds, L_2 seems to possess the correct asymptotic behavior.

The last remark can be made rigorous; a standard result in renewal theory (see, e.g. [10]) is that

$$\lim_{x\to\infty}\left(U(x) - \frac{x}{\mu}\right) = \frac{\mu_2}{2\mu^2},$$

Table 2. Bounds from (5), (19), and Proposition 4.1 for Example 4

	$U(x)$	$L_B(x)$	$L_1(x)$	$L_2(x)$
$x = 0.5$	1.092	0.750	1.053	1.066
$x = 1$	1.284	1.000	1.184	1.220
$x = 1.5$	1.512	1.250	1.362	1.420
$x = 2$	1.755	1.500	1.568	1.646
$x = 3$	2.251	2.000	2.025	2.136
$x = 5$	3.250	3.000	3.003	3.160
$x = 8$	4.750	4.500	4.500	4.693
$x = 12$	6.750	6.500	6.500	6.714
$x = 20$	10.750	10.500	10.500	10.730

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whereas a simple application of l'Hospital's rule in (19) yields

$$\lim_{x\to\infty}\left(L_2(x)-\frac{x}{\mu}\right)=\frac{\mu_2}{2\mu^2}.$$

Note that this is valid for any distribution where the associated renewal density is increasing. It is worth noting that the correct asymptotic behavior of L_2 is also useful in the DFR case, when u is decreasing and L_2 is used as an upper bound.

Example 5: Consider now the case where the interarrival distribution F in a renewal process has a Weibull density

$$f(x)=6xe^{-3x^2},$$

so that F is again IFR. It is easy to see that in this case, $\lim_{x\to\infty}m(x)=0$ and

$$\frac{\sigma^2}{\mu}=\frac{\sqrt{3}(4-\pi)}{6\sqrt{\pi}}.$$

Let $L_B(x)$, $L_1(x)$, and $L_2(x)$ be the bounds from (5), Proposition 4.1, and (19) respectively. Table 3 presents these bounds for various choices of x. Compared with Table 2, we see that now $L_2(x)$ is not uniformly superior to the other two bounds, as it performs poorly for very small values of x. For such values, $L_1(x)$ improves significantly over $L_B(x)$ and, as expected, when x becomes larger, the difference between these two values is negligible. For $x\geq 1$, the bound from (19) exceeds the other two bounds by a steady amount of approximately 0.72.

Table 3. Bounds from (5), (19), and Proposition 4.1 for Example 5

	$L_B(x)$	$L_1(x)$	$L_2(x)$
$x=0.25$	0.7618	1.0756	0.0000
$x=0.5$	1.2504	1.3420	1.0693
$x=1$	2.2276	2.2284	2.8841
$x=1.5$	3.2049	3.2048	3.9300
$x=2$	4.1821	4.1821	4.9089
$x=3$	6.1365	6.1365	6.8632
$x=5$	10.0453	10.0453	10.7721
$x=8$	15.9086	15.9086	16.6353
$x=12$	23.7262	23.7262	24.4529
$x=20$	39.3614	39.3614	40.0882

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The rest of this section deals with upper and lower bounds for IMRL distributions. First, we recall that if F is IMRL, then $H(x) = U(x) - x/\mu$ is increasing (see, e.g., [3]). Then (6) gives

$$H(x) \le \overline{F}_e(x) + H(x)F(x),$$

and for all x such that $F(x) < 1$, the next bound results for $U(x)$:

$$U(x) \le \frac{x}{\mu} + \frac{\overline{F}_e(x)}{\bar{F}(x)}.$$

Another upper bound is obtained by inserting Brown's bound (4) into (8); this gives

$$U(x) \le \frac{x}{\mu} + \frac{\mu_2}{2\mu^2} F(x) + \frac{1}{\mu} \overline{F}_e(t). \tag{23}$$

Observe that this is a refinement of (4) for all x such that

$$\int_x^\infty \bar{F}(t)\, dt \le \frac{\mu_2}{2\mu} \bar{F}(x).$$

A lower bound for U in the IMRL case can also be obtained using the next simple result, which is easily verified. Let $F_e^{(2)}$ be the equilibrium distribution corresponding to F_e. Then for $x \ge 0$, the following identity holds:

$$\overline{F}_e * H = H - \frac{\mu_2}{2\mu^2} F_e^{(2)}. \tag{24}$$

Put now $H_0(x) = H(x) - 1$. Then from Brown [3], we have that $H_0(x)$ is nonnegative and increasing. Thus, we can write

$$\overline{F}_e * H_0 = H - \frac{\mu_2}{2\mu^2} F_e^{(2)} - \overline{F}_e,$$

and since the left-hand side above is nonnegative, so is the right; therefore, we get $H(x) \ge (\mu_2/2\mu^2)F_e^{(2)}(x) + \overline{F}_e(x)$. This implies the following lower bound for U:

$$U(x) \ge \frac{x}{\mu} + \frac{\mu_2}{2\mu^2} + \overline{F}_e(x) - \frac{1}{\mu} \int_x^\infty \overline{F}_e(t)\, dt. \tag{25}$$

This is the same as Brown [3], on noting that Brown [3] allows F to have an atom at zero, which we have excluded here only for convenience.

Observe that from (23) and (25) we derive the following two-sided bound for $U(x)$:

$$\frac{\mu_2}{2\mu^2} F_e^{(2)}(x) \le U(x) - \frac{x}{\mu} - \overline{F}_e(x) \le \frac{\mu_2}{2\mu^2} F(x).$$

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Next, since the IMRL class is a subfamily of the NWUE class, one might wonder whether the lower bound in (25) is a refinement over the corresponding bound for U in Corollary 2.3, and it is interesting to note that this is *not* true. In fact, for (25) to be an improvement over the lower bound given in (16), we need that

$$\frac{\mu_2}{2\mu^2} F_e^{(2)}(x) \geq F(x).$$

Using, for example, a Pareto distribution, one verifies easily that the last inequality is valid only for *some* x.

5. LOWER BOUNDS FOR TAILS OF COMPOUND GEOMETRIC DISTRIBUTIONS

Let $X_1, X_2, \ldots$, be i.i.d. variables with cumulative distribution function F and let N be a geometric random variable (r.v.) such that $P(N = n) = (1 - \phi)\phi^n$, $n = 0,1,2,\ldots, 0 < \phi < 1$. Then the tail of the distribution for the random sum $X = \sum_{i=0}^{N} X_i$ is given by

$$\psi(x) = P(X > x) = (1 - \phi) \sum_{k=1}^{\infty} \phi^k \overline{F^{*k}}(x). \tag{26}$$

Assume now that $\{N(x), x \geq 0\}$ is a renewal process with interarrival distribution F; Lin [11] proved the following relation between $N(x)$ and the compound geometric tail in (26):

$$\psi(x) = E[\phi^{N(x)}], \qquad x \geq 0$$

(see also Cai and Garrido [5]). Applying Jensen's inequality, we deduce that (see [15])

$$\psi(x) \geq \phi^{E[N(x)]} = \phi^{U(x)} \tag{27}$$

or, equivalently, solving for $U(x)$,

$$U(x) \geq \frac{\ln[\psi(x)]}{\ln \phi}.$$

Thus, bounds for $U(x)$ can be used to deduce bounds for $\psi(x)$ and vice versa. For instance, a bound for ψ is obtained immediately from Lorden's bound, $U(x) \leq x/\mu + \mu_2/\mu^2$, and (27); namely

$$\ln[\psi(x)] \geq \left(\frac{x}{\mu} + \frac{\mu_2}{\mu^2}\right) \ln \phi$$

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or

$$\psi(x) \geq \phi^{x/\mu+\mu_2/\mu^2}.$$

Further, using the bound $U_1(x)$ (cf. Theorem 2.1) rather than Lorden's gives a different bound for $\psi(x)$, which has the advantage that it is exact for $x = 0$ and it is better than the above bound for small values of x. Following the discussion after Theorem 2.1, we note in particular that the bound $\psi(x) \geq \phi^{U_1(x)}$ is uniformly better than the bound in the last displayed expression if the underlying distribution F is NBUE.

6. CONCLUDING REMARKS

Interest in bounds for the renewal function exists also in the more general case where F is not concentrated on the nonnegative half-axis, so that it represents the step distribution of a random walk and $U(x)$ denotes the expected number of visits in the interval $(-\infty, x]$ (see [10] for further details on this). An advantage of the method presented here is that it can be applied, without extra effort, for the case where F does not vanish on the negative half-axis. The passage from the one-sided case to the more general two-sided case is provided by the renewal equation

$$U(x) = 1_{[0,\infty)}(x) + \int_{(-\infty,\infty)} U(x-t)\, dF(t),$$

which should be used instead of (8). The analog of Lorden's bound for this more general case is given in [9], and concerning the performance of the bound, similar remarks apply to those for the one-sided case.

Finally, as mentioned after Theorem 2.1, the approach taken to prove that theorem can be used recursively. Thus, if one inserts the bound $U_1(x)$ into the renewal equation (8), one obtains the bound

$$U(x) \leq \frac{x}{\mu} + \frac{\mu_2 F^{*2}(x)}{\mu^2} + \overline{F_e}(x) + (\overline{F_e} * F)(x),$$

and, continuing in this way, we obtain after k successive iterations that

$$U(x) \leq \frac{x}{\mu} + \frac{\mu_2 F^{*k}(x)}{\mu^2} + \overline{F_e}(x) + [\overline{F_e} * (F + F^{*2} + \cdots + F^{*(k-1)})](x).$$

Letting $k \to \infty$ yields the bound

$$U(x) \leq \frac{x}{\mu} + (\overline{F_e} * U)(x),$$

which, in view of (14), shows that this procedure converges iteratively to the right solution for $U(x)$.

https://doi.org/10.1017/S0269964806060141 Published online by Cambridge University Press

Acknowledgments

This research was supported by the Greek Secretariat for Research and Technology (grant No. 01EP18).

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