PUNCH PROBLEM FOR PLANAR ANISOTROPIC ELASTIC HALF-PLANE

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ABSTRACT

The two dimensional punch problem for planar anisotropic elastic half-plane is revisited using the Lekhnitskii's formulation with aid of the Fourier transform and boundary integral equation. Four different conditions of contact problem for the rigid punch are analyzed in this study. From the combination of surface Green's function of half-plane and Hooke's law of anisotropic material, a set of Fredholm integral equations are obtained for mixed boundary value problems. After solving the integral equation according to specified contact condition, the explicit distributions of surface traction under the punch are obtained in closed-form. From the surface traction and Green's function of anisotropic half-plane, the full-field solutions of stresses are constructed. Numerical calculations of surface traction under the rigid punch are presented base on the analysis and are discussed in detail.

Keywords : Rigid punch, Planar anisotropic, Integral equation.

1. INTRODUCTION

Anisotropic elasticity has been an active research subjects in recent years as a result of the application of composite materials. Because of the inherent heterogeneous and anisotropic nature of composite materials, the theoretical analysis of such problems is more complicated than an analysis of isotopic materials. Therefore, many research ventures devoted themselves to studying the elastostatic problem of anisotropic material. There are essentially two formalisms for two-dimensional deformation of an anisotropic elastic material. The most well-known systematic study is the Lekhnitskii [1] formulation, the extension of the Airy stress function using in isotropic material system. And the other is the Stroh [2] formulation, resembling the general solution obtained by the Lekhnitskii formulation. Since then, the two formulations have become powerful mathematical tools for anisotropic elastic problems. Lekhnitskii obtained the two-dimensional solutions of an anisotropic half-space subjected to line force on the surface based on complex variable methods. Ting [3] introduced the Stroh formulations and solved a series of basic anisotropic problems; including infinite plane, half-plane, bimaterial, wedge and circular inclusion, *etc*. Ma and Lin [4] used Lekhnitskii formulation and Fourier transformation to obtain the analytical solution of an anisotropic layered half-plane.

Both contact and crack problems are important examples for the mixed boundary values problems, in which they have many engineering applications. The contact problems organize the chief part of tribology and the theory of elasticity. The problem for indentation of rigid punch, above all, plays an important role in contact problems. In addition, the punch problems can be referred as the special case of contact problem that one of the contact bodies is rigid. The first paper investigated the subject of contact mechanics that dated back to Hertz's work in 1882. Since then, the punch problem had been investigated by many researchers because of the application in engineering mechanics. While solving the mixed boundary value problems-such as the indentation problem of rigid punch-we have to deal properly with integral equations. Muskhelishvili [5] provided a methodology for solving singular integral equations with the theory of elasticity. Green and Zerna [6] used a potential function approach to solve the indentation of an anisotropic half-plane by a frictionless punch. From then on, the two dimensional punch problems were all the rage, thus causing many academicians to embark on problems of isotropic elastic material. Gladwell [7] and Johnson [8] summarized and discussed a variety of contact problems for isotropic elastic materials. In Johnson's study, the mixed boundary value problems for indentation of rigid punch were solved by a pair of integral equations.

With the analytical methods for punch problem of isotropic elastic material in place, the punch problem of anisotropic elastic material was analyzed in the literature. Klintworth and Strong [9] followed in the foot-

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DOI : 10.1017/jmech.2011.26
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https://doi.org/10.1017/jmech.2011.26 Published online by Cambridge University Press

steps of Johnson and Lekhnitskii and presented the stresses distribution for a planar anisotropic half-plane subject to indentation of the flattened end punch. Their work obtained the iterative solution of the general punch problem by superimposing independent solutions for normal and tangential tractions. On the other hand, some authors used the Stroh formalism to investigate the indentation of plane punch for an anisotropic elastic material. Fan and Keer [10] and Hwu and Fan [11,12] discussed the rigid punch problem of an anisotropic half-plane by combining Stroh's formalism and method of analytical continuation. The analytical solutions for both no slip and frictional punches have been obtained in their literature. Chao *et at.* [13] also use the method of analytical continuation to solve the indentation of heated punch on an anisotropic half-plane. Ciavarella [14] obtained the analytical solution of the pressure distribution for flat or conical indenters with rounded corners. Fu and Chandra [15] used potential theory and complex variable analysis to obtain the analytical solution for pressure profile under the frictionless punch. The problem of indentation of a flat punch to an isotropic layered half-plane media has been derived by Porter and Hills [16]. The contact force was found with aid of influence functions and numerical technique in their work. The Fredholm integral equations have been used by Wang *et al*. [17] to visit the punch problem of piezoelectric layered media as well. Comez [18] discussed the contact pressure and contact area for both frictional and frictionless stamp. In both of their works, the integral equations are solved by numerical method.

In this paper, the two dimensional indentation of rigid punch to a planar anisotropic half-plane is investigated. First, the surface Green's function of anisotropic half-plane subjected to a concentrated loading applied at the surface is obtained by Lekhnitskii formulation and Fourier transform technique. Then we set to work connecting the surface displacements and surface distributed loadings by Hooke's law of anisotropic material. Finally a set of integral equations is derived by applying the identities of anisotropic material constants. For the frictionless or frictional rigid punch, the analytical solutions can be obtained directly because the integral equations are uncoupled in the study. However, the integral equations are coupled as the rigid punch with no slip condition. For the punch problem with coupled integral equations, most authors derived the solutions by either iterative or numerical method. According to different conditions of contact region under the rigid punch, the integral equations are translated to Fredholm integral equations of first and second kind either uncoupled or coupled integral equations in this study. Once the analytical solutions for these integral equations are derived, the surface tractions in contact regions under the punch are expressed. With the combination of parseval theorem of Fourier transform and Green's function of anisotropic half-plane, the full-field distributions of stresses induced by rigid punch are obtained. Numerical results of the surface traction under the punch are discussed in detail.

2. ELASTIC CONSTANTS OF PLANAR ANISOTROPIC MATERIALS

The generalized Hook's law of a homogeneous, linear-elastic, and anisotropic body can be described by 21 independent elastic coefficients which relate strains and stresses in Cartesian *x*-*y*-*z* coordinate. It can be written in contract notation as

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\ & & a_{33} & a_{34} & a_{35} & a_{36} \\ & & & a_{44} & a_{45} & a_{46} \\ & sym. & & & a_{55} & a_{56} \\ & & & & & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}, \quad (1)$$

where σ_{ij} and ε_{ij} denote the stress tensor and infinitesimal strain tensor, respectively, $a_{ij} = a_{ji}$ are the components of compliance. If the components of compliance have the following properties

$$a_{14} = a_{15} = a_{24} = a_{25} = a_{46} = a_{56} = 0 , \quad (2)$$

then the in-plane and anti-plane deformations are uncoupled for such anisotropic material. The numbers of independent component of compliance reduces to 13 for anisotropic material with the plane of symmetry at $z = 0$, named monoclinic material. The Hooke's law of monoclinic material is expressed in a matrix form as

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & a_{16} \\ & a_{22} & a_{23} & 0 & 0 & a_{26} \\ & & a_{33} & 0 & 0 & a_{36} \\ & & & a_{44} & a_{45} & 0 \\ & sym. & & & a_{55} & 0 \\ & & & & & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{bmatrix}, \quad (3)$$

Obviously, monoclinic materials not only satisfy the conditions of Eq. (2) but also supply the compliance with $a_{34} = a_{35} = 0$. Hence, the monoclinic materials are referred to as planar anisotropic materials. For a two dimensional anisotropic in-plane problem, the stress-strain relations are only dependent on six independent materials constants. For plane stress case, the Hooke's law is expressed in a matrix form as follows

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}, \quad (4)$$

and $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0$. For plane strain case,

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix}, \quad (5)$$

and $\varepsilon_{zz} = \varepsilon_{xz} = \varepsilon_{yz} = 0$.

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The relation between s_{ij} and a_{ij} is

$$s_{ij} = a_{ij} - a_{i3}\, a_{j3} / a_{33} \ , \quad i, j = 1,\ 2,\ 6 \ . \tag{6}$$

In many applications, the anisotropic materials are out of orthotropic materials which have three reciprocally perpendicular planes of symmetry. There are only 4 independent material constants for two dimensional problem of orthotropic material. If the choice of the coordinate doesn't match up with the symmetry plane of orthotropic materials, the behavior of anisotropic took place for the chosen coordinate system. For this reason, transformation of the elastic compliance to a different coordinate system is necessary. For the practical case where the transformation is a rotation of z-axis with an counterclockwise angle θ, the new elastic compliance $\mathbf{s}'$ can be related to original compliance $\mathbf{s}$ as follows

$$\mathbf{s}' = (\mathbf{T}^{-1})^T \mathbf{s} \mathbf{T}^{-1} \ , \tag{7}$$

where

$$\mathbf{T} = \begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin 2\theta \\ \sin^2\theta & \cos^2\theta & -\sin 2\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} . \tag{8}$$

3. GREEN'S FUNCTION OF HALF-PLANE

For a two-dimensional plane problem, the stress of the body can be expressed in term of the Airy stress function $\Phi(x, y)$ as follows

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} \ , \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} \ , \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} \ . \tag{9}$$

Substituing Eq. (9) into the compatibility equation, the governing equation is expressed in terms of $\Phi(x, y)$. In absence of body force, the governing equation is

$$s_{22}\frac{\partial^4 \Phi}{\partial x^4} - 2s_{26}\frac{\partial^4 \Phi}{\partial x^3 \partial y} + (2s_{12} + s_{66})\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} - 2s_{16}\frac{\partial^4 \Phi}{\partial x \partial y^3} + s_{11}\frac{\partial^4 \Phi}{\partial y^4} = 0 \ , \tag{10}$$

for plane strain and

$$a_{22}\frac{\partial^4 \Phi}{\partial x^4} - 2a_{26}\frac{\partial^4 \Phi}{\partial x^3 \partial y} + (2a_{12} + a_{66})\frac{\partial^4 \Phi}{\partial x^2 \partial y^2} - 2a_{16}\frac{\partial^4 \Phi}{\partial x \partial y^3} + a_{11}\frac{\partial^4 \Phi}{\partial y^4} = 0 \ , \tag{11}$$

for plane stress. In fact, both Eqs. (10) and (11) are the same from the mathematical point of view. For plane strain problem, the general solution of the stress function Φ given in Lekhnitskii formulation is

$$\Phi = 2\,\mathrm{Re}\left[\Phi_1(z_1) + \Phi_2(z_2)\right] \ , \tag{12}$$

where $z_j = x + p_j y$. The p_j is the roots of the characteristic algebraic equation as follows

$$s_{11}\, p^4 - 2s_{16}\, p^3 + (2s_{12} + s_{66}) p^2 - 2s_{26}\, p + s_{22} = 0 \ . \tag{13}$$

For the general case, the four roots in Eq. (13) are expressed in two pairs of conjugate roots as

$$p_k = \alpha_k + i\beta_k \ , \quad p_{k+2} = \overline{p}_k \ , \quad k = 1,\ 2 \ . \tag{14}$$

where $i = \sqrt{-1}$, α_k and β_k are all real and $\beta_k > 0$.

From the integral transform method, the governing Eq. (10) in the Fourier transform domain is

$$s_{22}\omega^4 + 2s_{26}\, i\omega^3 \frac{d\tilde{\Phi}}{dy} - (2s_{12} + s_{66})\omega^2 \frac{d^2\tilde{\Phi}}{dy^2} - 2s_{16}\, i\omega \frac{d^3\tilde{\Phi}}{dy^3} + s_{11}\frac{d^4\tilde{\Phi}}{dy^4} = 0 \ . \tag{15}$$

The Fourier transform pairs are defined in the form

$$\tilde{\Phi}(\omega, y) = \int_{-\infty}^{\infty} \Phi(x, y)\, e^{-i\omega x} dx \ ,$$
$$\Phi(x, y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{\Phi}(\omega, y)\, e^{i\omega x} d\omega \ , \tag{16}$$

where ω is parameter in the transform domain and x is a spatial coordinate. The general solution of $\tilde{\Phi}$ in Eq. (15) is

$$\tilde{\Phi}(\omega, y) = \sum_{j=1}^{4} c_j\, e^{ip_j \omega y} \ , \tag{17}$$

where c_j are unknown constant to be determined by boundary condition. The general solutions of stresses and displacements in the transform domain are summarized in the matrix form as follows

$$\begin{bmatrix} \tilde{\sigma}_{yy} \\ \tilde{\sigma}_{xy} \\ \tilde{u}_x \\ \tilde{u}_y \end{bmatrix} = i\omega \begin{bmatrix} i\omega\, e^{ip_1\,\omega y} & i\omega\, e^{ip_2\,\omega y} & i\omega\, e^{ip_3\,\omega y} & i\omega\, e^{ip_4\,\omega y} \\ -i\omega\, p_1\, e^{ip_1\omega y} & -i\omega\, p_2\, e^{ip_2\,\omega y} & -i\omega\, p_3\, e^{ip_3\,\omega y} & -i\omega\, p_4\, e^{ip_4\,\omega y} \\ \xi(p_1)\, e^{ip_1\,\omega y} & \xi(p_2)\, e^{ip_2\,\omega y} & \xi(p_3)\, e^{ip_3\,\omega y} & \xi(p_4)\, e^{ip_4\,\omega y} \\ \eta(p_1)\, e^{ip_1\,\omega y} & \eta(p_2)\, e^{ip_2\,\omega y} & \eta(p_3)\, e^{ip_3\,\omega y} & \eta(p_4)\, e^{ip_4\,\omega y} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} , \tag{18}$$

where $\tilde{u}_x$ and $\tilde{u}_y$ are displacement components in the x and y direction, respectively, and

$$\xi(p) = s_{11}\, p^2 + s_{12} - s_{16}\, p \ , \quad \eta(p) = (s_{12}\, p^2 + s_{22} - s_{26}\, p)/\, p \ . \tag{19}$$

The coefficients of c_j in Eq. (18) can be determined by boundary conditions.

Consider the surface of planar anisotropic half-plane subjected to vertical and horizontal concentrated point loadings f_y and f_x at the original of coordinate as shown

https://doi.org/10.1017/jmech.2011.26 Published online by Cambridge University Press

in Fig. 1. By applying the boundary conditions to the general solution of Eq. (18), the solution of stresses and displacements in the transform domain is determined as follows

$$\begin{bmatrix} \tilde{\sigma}_{yy} \\ \tilde{\sigma}_{xy} \\ \tilde{u}_x \\ \tilde{u}_y \end{bmatrix} = i\omega \begin{bmatrix} i\omega e^{ip_1\,\omega y} & i\omega e^{ip_2\,\omega y} \\ -i\omega p_1\, e^{ip_1\,\omega y} & -i\omega p_2\, e^{ip_2\,\omega y} \\ \xi(p_1)\, e^{ip_1\,\omega y} & \xi(p_2)\, e^{ip_2\,\omega y} \\ \eta(p_1)\, e^{ip_1\,\omega y} & \eta(p_2)\, e^{ip_2\,\omega y} \end{bmatrix} \begin{bmatrix} \dfrac{f_x + p_2 f_y}{(p_2 - p_1)\,\omega^2} \\ -\dfrac{f_x + p_1 f_y}{(p_2 - p_1)\,\omega^2} \end{bmatrix}. \tag{20}$$

Taking the inverse Fourier transform for Eq. (20), the full-field distributions of stresses in the planar anisotropic half-plane are

$$\begin{bmatrix} \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} = \frac{1}{\pi} \mathrm{Re} \left\{ \begin{bmatrix} \dfrac{-i}{x + p_1 y} & \dfrac{-i}{x + p_2 y} \\ \dfrac{ip_1}{x + p_1 y} & \dfrac{ip_2}{x + p_2 y} \end{bmatrix} \begin{bmatrix} \dfrac{f_x + p_2 f_y}{p_2 - p_1} \\ -\dfrac{f_x + p_1 f_y}{p_2 - p_1} \end{bmatrix} \right\}, \tag{21}$$

where Re stands for the real part.

Since we have took the real part of Eq. (21), the stress components are expressed as follows

$$\begin{cases} \sigma_{yy} = -\dfrac{f_y y^2}{\pi} g(x, y) - \dfrac{f_x y^2}{\pi} h(x, y), \\ \sigma_{xy} = -\dfrac{f_y xy}{\pi} g(x, y) - \dfrac{f_x xy}{\pi} h(x, y), \end{cases} \tag{22}$$

where

$$\begin{cases} g(x, y) = \dfrac{(\beta_1 \beta_2^2 + \beta_1^2 \beta_2 + \alpha_1^2 \beta_2 + \alpha_2^2 \beta_1) y + (\alpha_1 \beta_2 + \alpha_2 \beta_1) x}{\left[(x + \alpha_1 y)^2 + (\beta_1 y)^2\right] \left[(x + \alpha_2 y)^2 + (\beta_2 y)^2\right]}, \\ h(x, y) = \dfrac{(\alpha_1 \beta_2 + \alpha_2 \beta_1) y + (\beta_2 + \beta_1) x}{\left[(x + \alpha_1 y)^2 + (\beta_1 y)^2\right] \left[(x + \alpha_2 y)^2 + (\beta_2 y)^2\right]}. \end{cases} \tag{23}$$

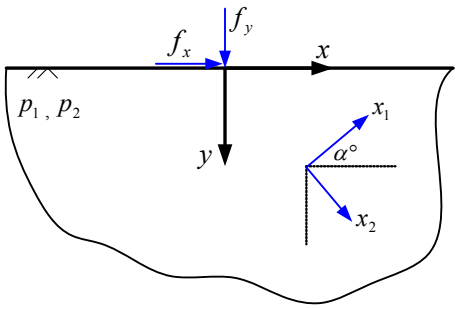


Fig. 1 Anisotropic half-plane subjected to concentrated force

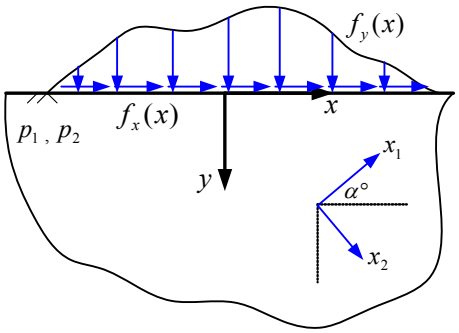


Fig. 2 Anisotropic half-plane subjected to arbitrary distributed force

In addition, the stress component σ_{xx} is

$$\sigma_{xx} = -\frac{f_y x^2}{\pi} g(x, y) - \frac{f_x x^2}{\pi} h(x, y). \tag{24}$$

Equations (22) and (24) are the Green's functions of planar anisotropic half-plane subjected to surface loadings. When the surface of half-plane is subjected to arbitrary distribution loading as shown in Fig. 2, the full-field solutions of stress for the half-plane can be obtained with aid of Eqs. (22) and (24) as follows.

$$\begin{cases} \sigma_{yy} = -\dfrac{y^2}{\pi} \int_{-b}^{a} f_y(s)\, g(x-s, y)\, ds - \dfrac{y^2}{\pi} \int_{-b}^{a} f_x(s) h(x-s, y)\, ds, \\ \sigma_{xy} = -\dfrac{y}{\pi} \int_{-b}^{a} f_y(s)(x-s)\, g(x-s, y)\, ds - \dfrac{y}{\pi} \int_{-b}^{a} f_x(s)(x-s) h(x-s, y)\, ds, \\ \sigma_{xx} = -\dfrac{1}{\pi} \int_{-b}^{a} f_y(s)(x-s)^2\, g(x-s, y)\, ds - \dfrac{1}{\pi} \int_{-b}^{a} f_x(s)(x-s)^2 h(x-s, y)\, ds. \end{cases} \tag{25}$$

So far, the full-field distributions of stress for planar anisotropic half-plane subjected to arbitrary surface loadings have been expressed as an integral form. Having the explicit form of stress, we can construct the relations between the surface loading and displacements by Hooke's law, *i.e.*, Eq. (5).

4. BOUNDARY INTEGRAL FORMULATION AND FREDHOLM INTEGRAL EQUATION

With the Green's function of stresses presented in Eq. (25) for arbitrary distributed loading, the solutions of strain field can be determined from Eq. (5). Integrate Eq. (5) and let $y = 0$, the surface displacements denoted as $\bar{u}_x$ and $\bar{u}_y$ can be represented as the integral forms in term of arbitrary surface traction $f_x(x)$ and $f_y(x)$.

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$$
\begin{aligned}
\overline{u}_x = & \frac{-s_{11}\,\mathrm{Im}(p_1\,p_2)}{\pi}\int_{-b}^{a} f_y(s)\ln(x-s)\,ds \\
& -\frac{\left(s_{12}-s_{11}\,\mathrm{Re}(p_1\,p_2)\right)}{2}\left[\int_{-b}^{x} f_y(s)\,ds-\int_{x}^{a} f_y(s)\,ds\right] \\
& +\frac{-s_{11}\,\mathrm{Im}(p_1+p_2)}{\pi}\int_{-b}^{a} f_x(s)\ln(x-s)\,ds \\
& -\frac{\left(s_{16}-s_{11}\,\mathrm{Re}(p_1+p_2)\right)}{2}\left[\int_{-b}^{x} f_x(s)\,ds-\int_{x}^{a} f_x(s)\,ds\right]+u_1\ ,
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\overline{u}_y = & \frac{-s_{22}\,\mathrm{Im}\left(p_1\,p_2(\overline{p}_1+\overline{p}_2)\right)}{\pi(p_1\,\overline{p}_1)(p_2\,\overline{p}_2)}\int_{-b}^{a} f_y(s)\ln(x-s)\,ds \\
& +\frac{1}{2}\left[s_{26}-\frac{s_{22}\,\mathrm{Re}\left(p_1\,p_2(\overline{p}_1+\overline{p}_2)\right)}{(p_1\,\overline{p}_1)(p_2\,\overline{p}_2)}\right]\left[\int_{-b}^{x} f_y(s)\,ds-\int_{x}^{a} f_y(s)\,ds\right] \\
& +\frac{-s_{22}\,\mathrm{Im}(p_1\,p_2)}{\pi(p_1\,\overline{p}_1)(p_2\,\overline{p}_2)}\int_{-b}^{a} f_x(s)\ln(x-s)\,ds \\
& +\frac{1}{2}\left[s_{12}-\frac{s_{22}\,\mathrm{Re}(p_1\,p_2)}{(p_1\,\overline{p}_1)(p_2\,\overline{p}_2)}\right]\left[\int_{-b}^{x} f_x(s)\,ds-\int_{x}^{a} f_x(s)\,ds\right]+u_2\ ,
\end{aligned}
\tag{27}
$$

where u_1 and u_2 are rigid body translation. The split in the integrations, $\int_{-b}^{x} dx$ and $\int_{x}^{a} dx$, in Eqs. (26) and (27) give rise to new difficulty in solving the problem. In order to obtain the simplest form and to remove the rigid body translation u_1 and u_2, differentiate the surface displacements in Eqs. (26) and (27) with x is performed. Therefore the gradient of surface displacement, $\overline{u}'_x$ and $\overline{u}'_y$, can be obtained by differentiating Eqs. (26) and (27) and expressed as follows.

$$
\begin{aligned}
\overline{u}'_x = & \frac{-s_{11}\,\mathrm{Im}(p_1\,p_2)}{\pi}\int_{-b}^{a}\frac{f_y(s)}{x-s}ds \\
& -\left(s_{12}-s_{11}\,\mathrm{Re}(p_1\,p_2)\right)f_y(x) \\
& +\frac{-s_{11}\,\mathrm{Im}(p_1+p_2)}{\pi}\int_{-b}^{a}\frac{f_x(s)}{x-s}ds \\
& -\left(s_{16}-s_{11}\,\mathrm{Re}(p_1+p_2)\right)f_x(x)\ ,
\end{aligned}
\tag{28}
$$

$$
\begin{aligned}
\overline{u}'_y = & \frac{-s_{22}\,\mathrm{Im}\left(p_1\,p_2(\overline{p}_1+\overline{p}_2)\right)}{\pi\,p_1\,\overline{p}_1\,p_2\,\overline{p}_2}\int_{-b}^{a}\frac{f_y(s)}{x-s}ds \\
& +\left[s_{26}-\frac{s_{22}\,\mathrm{Re}\left(p_1\,p_2(\overline{p}_1+\overline{p}_2)\right)}{p_1\,\overline{p}_1\,p_2\,\overline{p}_2}\right]f_y(x) \\
& +\frac{-s_{22}\,\mathrm{Im}(p_1\,p_2)}{\pi\,p_1\,\overline{p}_1\,p_2\,\overline{p}_2}\int_{-b}^{a}\frac{f_x(s)}{(x-s)}ds \\
& +\left[s_{12}-\frac{s_{22}\,\mathrm{Re}(\overline{p}_1\,\overline{p}_2)}{p_1\,\overline{p}_1\,p_2\,\overline{p}_2}\right]f_x(x)\ .
\end{aligned}
\tag{29}
$$

Equations (28) and (29) include material constants s_{ij} and eigenvalues p_i. The eigenvalues p_i and coefficients s_{ij} have the by following relations

$$
\begin{cases}
\mathrm{Re}\left[p_1+p_2\right]=\dfrac{s_{16}}{s_{11}}\ , \\
\mathrm{Re}\left[p_1\,p_2(\overline{p}_1+\overline{p}_2)\right]=\dfrac{s_{26}}{s_{11}}\ , \\
p_1\,\overline{p}_1\,p_2\,\overline{p}_2=\dfrac{s_{22}}{s_{11}}\ .
\end{cases}
\tag{30}
$$

Using the identity in Eq. (30) for Eqs. (28) and (29), these equations can be simplified as follows.

$$
-\overline{u}'_x = A\hat{f}_y(s)+Bf_y(x)+E\hat{f}_x(s)\ , \tag{31}
$$

$$
-\overline{u}'_y = D\hat{f}_y(s)+A\hat{f}_x(s)-Bf_x(x)\ , \tag{32}
$$

where

$$
\begin{aligned}
& A=\frac{s_{11}}{\pi}\mathrm{Im}[p_1\,p_2]\ ,\quad B=s_{12}-s_{11}\,\mathrm{Re}[p_1\,p_2]\ , \\
& E=\frac{s_{11}}{\pi}\mathrm{Im}[p_1+p_2]\ , \\
& D=\frac{s_{11}}{\pi}\mathrm{Im}\left[p_1\,p_2(\overline{p}_1+\overline{p}_2)\right]\ ,\quad \hat{f}_y(s)=\int_{-b}^{a}\frac{f_y(s)}{x-s}ds\ , \\
& \hat{f}_x(s)=\int_{-b}^{a}\frac{f_x(s)}{x-s}ds\ .
\end{aligned}
$$

Equations (31) and (32) express the relation between surface tractions and surface displacement gradients for planar anisotropic half-plane. We are now ready to solve the punch problem for anisotropic half-plane. With the conditions of contact under the punch on the surface, the contact force under the rigid punch will be analyzed.

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5. CONTACT FORCE FOR THE RIGID PUNCH PROBLEM DUE TO VARIOUS CONTACT CONDITIONS

For the usual boundary value problems of half-plane, the surface tractions are prescribed on loading region $(-b, a)$. But for contact problems of rigid punch, the displacements are prescribed on loading region rather than surface tractions. Thus, contact problems of rigid punch are regarded as mixed boundary value problems. In order to obtain the contact force under the punch, it is necessary to specify the contact conditions under the punch. There are four types of contact conditions that prescribed on region $(-b, a)$ under the punch will be discussed in this section.

Case (1): All surface tractions $f_x(x), f_y(x)$ are prescribed on loading region.

Case (2): Surface displacements and surface tractions, *i.e.*, $f_x(x)$ and $\overline{u}_y(x)$, or $f_y(x)$ and $\overline{u}_x(x)$, are prescribed.

Case (3): Surface displacement $\overline{u}_y(x)$ is prescribed and the vertical and horizontal surface tractions have the relation $f_x(x) = \mu f_y(x)$, where μ is constant coefficient of friction.

Case (4): All surface displacements, $\overline{u}_x(x)$ and $\overline{u}_y(x)$, are prescribed.

Now, the four classes will be discussed in the following presentation.

Case (1)

This is the simplest boundary values problem. For any specified surface tractions $f_x(x)$ and $f_y(x)$, the full-field solution of anisotropic half-plane can be obtained from Eq. (25) directly.

Case (2)

Consider that the surface displacement $\overline{u}_y(x)$ and surface traction under rigid punch $f_x(x)$ are prescribed. Eq. (32) can be simplified as

$$\int_{-b}^{a} \frac{f_y(s)}{x-s} ds = G(x) , \tag{33}$$

where

$$G(x) = \frac{1}{D}\left(Bf_x(x) - A\int_{-b}^{a} \frac{f_x(s)}{x-s} ds - \overline{u}_y'(x) \right) . \tag{34}$$

Equation (33) is the Fredholm integral equation of first kind, the solution can be found as

$$f_y(x) = \frac{1}{\pi^2\sqrt{(b+x)(a-x)}} \int_{-b}^{a} \frac{\sqrt{(s+b)(s-a)}G(s)}{x-s} ds + \frac{C}{\pi^2\sqrt{(b+x)(a-x)}} , \tag{35}$$

where

$C = \pi\int_{-b}^{a} f_y(x)\, dx = \pi F_y$, and F_y is the resultant force applied on the punch.

For the simplest case, flat end and frictionless punch as shown in Fig. 3, *i.e.*, $f_x(x) = 0$, $\overline{u}_y'(x) = 0$, we have $G(x) = 0$. If the coordinate is taken at the center of the punch, the solution of Eq. (35) reduce to

$$f_y(x) = \frac{F_y}{\pi\sqrt{a^2 - x^2}} . \tag{36}$$

The result in Eq. (36) implies that surface traction $f_y(x)$ is independent of material properties and is the same as the solution of isotropic case. By substituting surface traction of Eq. (36) to Eq. (25), the full-field stress distributions in the half-plane can be obtained.

If the profile of the indenter is of circular as shown in Fig. 4. The surface displacement under the punch can be expressed as

$$\overline{u}_y = \frac{a^2 - x^2}{2R} . \tag{37}$$

From Eq. (34), the function $G(x)$ can be found as

$$G(x) = \frac{x}{DR} . \tag{38}$$

Substitute Eq. (38) into Eq. (35) and the general solution of pressure distribution would be obtained as follow.

$$f_y(x) = \frac{1}{\sqrt{a^2 - x^2}}\left[\frac{1}{\pi DR}\left(DRF_y + \frac{1}{2}a^2 - x^2 \right) \right] . \tag{39}$$

With aid of the conditions that f_y vanished at $x = \pm a$, the pressure distribution can be simplified furthermore as

$$f_y(x) = \frac{2F_y}{\pi a^2}\sqrt{a^2 - x^2} . \tag{40}$$

According Eqs. (36) and (40), we found that the contact force under the frictionless punch is independent to the material property of half-plane whether the indenter is flat or circular. For the surface displacement $\overline{u}_x(x)$ and surface traction $f_y(x)$ are prescribed on the contact region, the solution of $f_x(x)$ can be solved by the similar way.

Case (3)

Consider that the surface displacement $\overline{u}_y(x)$ is prescribed and the traction under the rigid punch have the relation $f_x(x) = \mu f_y(x)$. In this case, Eq. (32) can be summarized as

$$f_y(x) + \frac{\lambda}{\pi}\int_{-b}^{a} \frac{f_y(s)}{x-s} ds = G(x) , \tag{41}$$

where

$$\lambda = -\pi\frac{D + \mu A}{B\mu} \quad \text{and} \quad G(x) = \frac{\overline{u}_y'(x)}{B\mu} .$$

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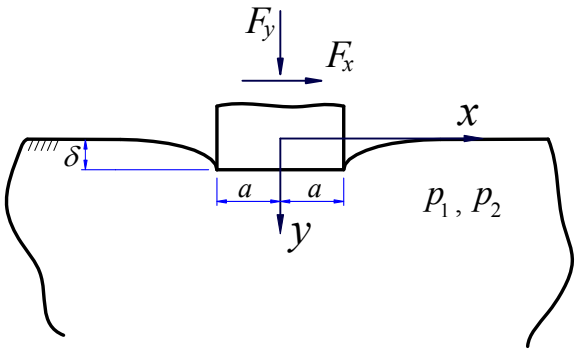


Fig. 3 Anisotropic half-plane subjected to rigid punch with flatted end

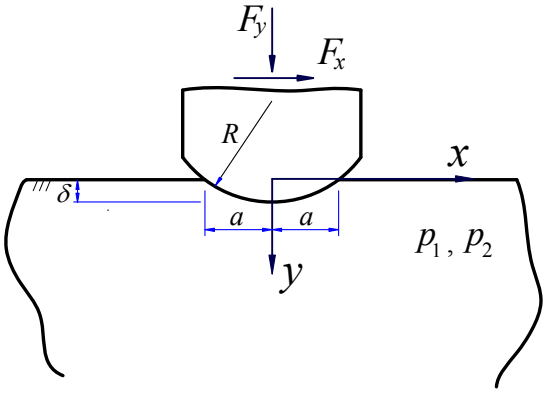


Fig. 4 Anisotropic half-plane subjected to rigid punch with rounded end

Equation (41) is the Fredholm integral equation of second kind. By letting $b = a$ and the solution of integral equation can be expressed as follows.

$$f_y(x) = \frac{G(x)}{1+\lambda^2} - \frac{\lambda}{\pi(1+\lambda^2)\sqrt{a^2-x^2}}\left(\frac{a+x}{a-x}\right)^{\gamma}\int_{-a}^{a}\sqrt{a^2-s^2}\left(\frac{a-s}{a+s}\right)^{\gamma}\frac{G(s)}{x-s}ds - \frac{\lambda\, a\, C}{\pi\sqrt{1+\lambda^2}\sqrt{a^2-x^2}}\left(\frac{a+x}{a-x}\right)^{\gamma}, \tag{42}$$

where

$$C = \frac{1}{a}\int_{-a}^{a} f_y(x)\,dx = \frac{F_y}{a} \quad \text{and} \quad \gamma = \frac{1}{\pi}\cot^{-1}\lambda .$$

Because of λ is real and so is γ. For surface traction solution of Eq. (42), the integral is difficult to obtain analytical explicit results unless $G(x)$ and γ are particularly simple. For the simplest case, the flat end punch so that $G(x) = 0$, the surface traction expression (42) is reduced to

$$f_y(x) = -\frac{\cos(\pi\gamma)F_y}{\pi\sqrt{a^2-x^2}}\left(\frac{a+x}{a-x}\right)^{\gamma}, \tag{43}$$

and the associated surface traction $f_x(x)$ is

$$f_x(x) = -\frac{\mu\cos(\pi\gamma)F_y}{\pi\sqrt{a^2-x^2}}\left(\frac{a+x}{a-x}\right)^{\gamma}. \tag{44}$$

The solution in the study is identical with the result of Hwu and Fan [11,12]. And the analytical check via the aid of numerical calculation was shown in Appendix *A*. When $\mu \to 0$, the distribution of contact force given in Eq. (43) is the same as the surface traction of frictionless punch given in Class (2).

Case (4)

Consider that surface displacements $\overline{u}_x(x)$, $\overline{u}_y(x)$ are prescribed at contact region under rigid punch. From Eqs. (31) and (32), it is found that two integral equations are coupled for this case. To obtain the standard type of the first kind or the second kind Fredholm integral equation, the two integral equations must be combined. Introduce the constants λ_x and $i\lambda_y$, multiply them into (31) and (32), respectively and we will have the following equations.

$$-\lambda_x\,\overline{u}_x' = \lambda_x\,A\hat{f}_y(s) + \lambda_x\,Bf_y(x) + \lambda_x\,E\hat{f}_x(s), \tag{45}$$

$$-i\lambda_y\,\overline{u}_y' = i\lambda_y\,D\hat{f}_y(s) + i\lambda_y\,A\hat{f}_x(s) - i\lambda_y\,Bf_x(x). \tag{46}$$

Adding Eq. (45) to Eq. (46), the following result is obtained.

$$-(\lambda_x\,\overline{u}_x' + +i\lambda_y\,\overline{u}_y') = (\lambda_x\,A + i\lambda_y D)\hat{f}_y(s) + (\lambda_x\,E + i\lambda_y\,A)\hat{f}_x(s) + \lambda_x\,Bf_y(x) - i\lambda_y\,Bf_x(x). \tag{47}$$

Compare Eq. (47) with the standard Fredholm integral equation of second kind as follow

$$F(x) + \frac{\lambda}{\pi}\int_{-a}^{a}\frac{F(s)}{x-s}\,ds = G(x). \tag{48}$$

It is found that λ_x and λ_y should have the relation

$$\lambda_x = \frac{i\pi D}{\lambda B - \pi A}\lambda_y . \tag{49}$$

The eigenvalue λ of the Fredholm integral equation in (48) can be determined as

$$\lambda = \pm\frac{\pi\sqrt{A^2 - CD}}{B}. \tag{50}$$

For convenience, we can choose

$$\lambda = \frac{\pi\sqrt{A^2-CD}}{B}, \quad \lambda_x = i\,\pi\,D, \quad \lambda_y = \lambda B - \pi A . \tag{51}$$

Comparing Eqs. (47) and (48), we let

$$\begin{cases} G(x) = -\pi D\,\overline{u}_x'(x) + (\pi A - \lambda B)\,\overline{u}_y'(x), \\ F(x) = \pi BDf_y(x) + (\pi A - B\lambda)\,Bf_x(x). \end{cases} \tag{52}$$

λ is named the eigenvalue of the Fredholm integral equation in Eq. (48) due to the fact that the property of

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λ decides the characteristic of the solution for Eq. (48). Because of $A^2 - CD < 0$ and B is real, the engenvalue λ is pure imagery. Thus, we let $\lambda = i\lambda_1$, where λ_1 is real. In additional, the material constant has the property

$$1 > \frac{B}{\pi\sqrt{CD - A^2}} > 0 \,, \tag{53}$$

so we have $\lambda_1 > 1$.

Take Eq. (52) into the general solutions of Eq. (48) and we obtain the contact force under the rigid punch which is presented as follow.

$$\begin{aligned} &\pi BDf_y(x) + (\pi A - iB\lambda_1)\, Bf_x(x) \\ &= \left[\begin{aligned} &\frac{-i\lambda_1}{\pi(1-\lambda_1^2)\sqrt{a^2-x^2}} \left(\frac{a+x}{a-x}\right)^{i\eta} \int_{-a}^{a} \sqrt{a^2-s^2} \left(\frac{a-s}{a+s}\right)^{i\eta} \frac{G(s)}{x-s}\, ds \\ &+ \frac{G(x)}{1-\lambda_1^2} - \frac{a\ C}{\pi\sqrt{1-\frac{1}{\lambda_1^2}}\sqrt{a^2-x^2}} \left(\frac{a+x}{a-x}\right)^{i\eta} \end{aligned} \right], \end{aligned} \tag{54}$$

where $C = \frac{1}{a}\int_{-a}^{a} F(x)\, dx = \frac{\pi BDF_y + (\pi A - iB\lambda_1)F_x}{a}$ and $\eta = -\frac{i}{\pi}\cot^{-1}(i\lambda_1)$.

In Eq. (54), the term $\left(\frac{a+x}{a-x}\right)^{i\eta}$ can be rewritten as

$$\left(\frac{a+x}{a-x}\right)^{i\eta} = \cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) + i\sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right). \tag{55}$$

Consider the flat end punch so that $G(x) = 0$. Take image-part and real-part for Eq. (54) and the surface tractions $f_x(x)$ and $f_y(x)$ under the punch are expressed as

$$f_x(x) = \frac{\left[\pi A \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) - B\lambda_1 \cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right)\right] F_x + \pi D \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_y}{\pi B\sqrt{\lambda_1^2 - 1}\sqrt{a^2-x^2}}, \tag{56}$$

$$f_y(x) = -\frac{\left[\pi A \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) + B\lambda_1 \cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right)\right] F_y + \pi E \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_x}{\pi B\sqrt{\lambda_1^2 - 1}\sqrt{a^2-x^2}}. \tag{57}$$

The solution in the study is identical with the result of Hwu and Fan [11]. The analytical check via the aid of numerical calculation was shown in Appendix *B*.

For orthotropic materials, Eqs. (56) and (57) can be reduced to the following form.

$$f_x(x) = \frac{\lambda_1}{\pi\sqrt{\lambda_1^2 - 1}\sqrt{a^2-x^2}} \left[\cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_x - \sqrt[4]{\frac{s_{22}}{s_{11}}} \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_y \right], \tag{58}$$

$$f_y(x) = \frac{-\lambda_1}{\pi\sqrt{\lambda_1^2 - 1}\sqrt{a^2-x^2}} \left[\sqrt[4]{\frac{s_{11}}{s_{22}}} \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_x + \cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_y \right], \tag{59}$$

in which $\lambda_1 = \frac{\pi\sqrt{ED}}{B}$. For isotropic material, the material constants $s_{11} = s_{22}$, $\lambda_1 = \frac{1+\kappa}{\kappa - 1}$ and the contact force under the punch can be simplified as follow.

$$f_x(x) = \frac{(1+k)}{2\pi\sqrt{k}\sqrt{a^2-x^2}} \left[\cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_x - \sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_y \right], \tag{60}$$

$$f_y(x) = \frac{-(1+k)}{2\pi\sqrt{k}\sqrt{a^2-x^2}} \left[\sin\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_x + \cos\left(\eta \ln\left(\frac{a+x}{a-x}\right)\right) F_y \right], \tag{61}$$

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where $\kappa = 3 - 4\nu$ for plane strain and $\kappa = \frac{3-\nu}{1+\nu}$ for plane stress.

It is noted that the integral Eqs. (31) and (32) are coupled in case (4). The coupled integral equations are difficult to be solved directly and many authors used numerical or iterative method. In this work, we derive the analytical solution of coupled integral equations by the relations of material coefficients, *i.e.*, Eq. (30), in conjunction with the examining the structure of the couple equations.

6. NUMERICAL RESULTS

In this section, the contact force under rigid punch is investigated by numerical calculation. The material of T300/934 graphite/epoxy is employed for numerical calculations. The elastic behavior of such material is an orthotropic material in the material axes $x_1 - x_2$. The elastic compliance constants are

$$\begin{bmatrix} s_{11} & s_{12} & s_{16} \\ s_{12} & s_{22} & s_{26} \\ s_{16} & s_{26} & s_{66} \end{bmatrix} = \begin{bmatrix} 7.63 & -1.68 & 0 \\ -1.68 & 97.1 & 0 \\ 0 & 0 & 144.5 \end{bmatrix} \times 10^{-3} \frac{1}{\text{GPa}} . \tag{62}$$

If the material axes $x_1 - x_2$ are rotated through an angle α counterclockwise from the geometric coordinate $x - y$ as shown in Figs. 1 and 2, the material are transformed to planar anisotropic material.

Figure 5 shows the contact force $f_y(x)$ for the indentation of flat-ended punch into an orthotropic material with coefficients of friction μ = 0, 1, 2, 3 and 4, respectively. The singularity located near the edges of contact region is presented in the figure. From this figure it is shown that the contact force is unsymmetrical due to the friction force.

Figure 6 shows the result for the indentation of flat-ended punch with coefficients of friction μ = 4. The material constants are different by rotating the material axes with respect to geometric coordinate for α = 0, 30, 85, 90 degree. The figure show that the surface tractions varied by different material constant.

Figures 7 and 8 are the contact force under the punch with no slip for different material constants. For no slip condition, the normal force is different slightly by different material constants.

Figure 9 present the contact force for the flat-ended punch indentation into orthotropic material with contact conditions of frictionless, friction and no slip. The figure shows that the contact force is similar for frictionless and no slip condition but is distorted for friction condition.

7. CONCLUSIONS

The distributions of contact force under the punch for the indentation of a rigid punch into an anisotropic half-plane are analyzed in the study. From the surface Green's function of anisotropic half-plane and the conditions of contact under the punch, the problems are derived as a set of integral equations. For the rigid punch with frictionless, friction and no slip, the integral equations can be simplified as standard Fredholm integral equations and the contact force can be determined. If the profile of the rigid punch is flat-ended, the analytical closed-form solution can be derived. The distributions of contact force under the flat-ended rigid punch are presented by numerical calculation. By examples of calculation we can realized the characteristics of contact force for different conditions of contact such as frictionless, friction and no slip. On the other hand, the material constants, such as isotropic, orthotropic and anisotropic material, will affect the contact force as well. Since the surface distribution under the rigid punch is known, the full-field stress and displacement solution can be easily determined.

ACKNOWLEDGEMENTS

This research was partially supported by the National Science Council in Taiwan through Grant NSC 95-2221-E-218-011.

APPENDIX A

Because of the different formulations are used in this study. There are some difficulties to do fully analytical check. Thus, analytic with aid of numerical check via the existing expressions have been summary as follows.

The solution of frictional punch with flatted end in Hwu and Fan [11,12] is

$$f_y(x) = \frac{F_y}{2\pi} \frac{\beta + \bar{\beta}}{\beta} \frac{(a-x)^{\delta-1}}{(a+x)^{\delta}} = \frac{F_y \sin \pi\,\delta}{\pi (a+x)^{\delta} (a-x)^{1-\delta}} \tag{A1}$$

where $\delta = \frac{1}{2\pi} \arg(-\bar{\beta}/\beta)$, $0 \le \delta < 1$, and $\beta = m_{22}^* + \mu(m_{12}^* + m_{32}^*)$. m_{12}^*, m_{22}^*, and m_{32}^* are the {12}, {22}, {32} components of $\mathbf{M}^{-1}$. M is the impedance tensor defined as $\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1} = \mathbf{L}(\mathbf{I} - i\mathbf{S})^{-1}$.

The result in author's study is

$$f_y(x) = \frac{\cos(\pi\,\gamma)\, F_y}{\pi \sqrt{a^2 - x^2}} \left(\frac{a+x}{a-x} \right)^{\gamma} , \tag{A2}$$

The Eq. (A2) can be rewritten as

$$f_y(x) = \frac{\cos(\pi\,\gamma) F_y}{\pi (a-x)^{1/2+\gamma} (a+x)^{1/2-\gamma}} \tag{A3}$$

Compare Eqs. (A1) and (A3) and we find that the two equations are not the same unless $\gamma + \delta = 1/2$.

Use the material constants in Eq. (62) and the transformation of material constants in Eqs. (7) and (8) with the coefficient of friction μ = 0.2 and the numerical compare have been listed as Table A1.

Table A1 The values of δ and γ with respect to different material constants.

The result in Table 1 proved that the Eqs. (A1) and (A3) are identical.

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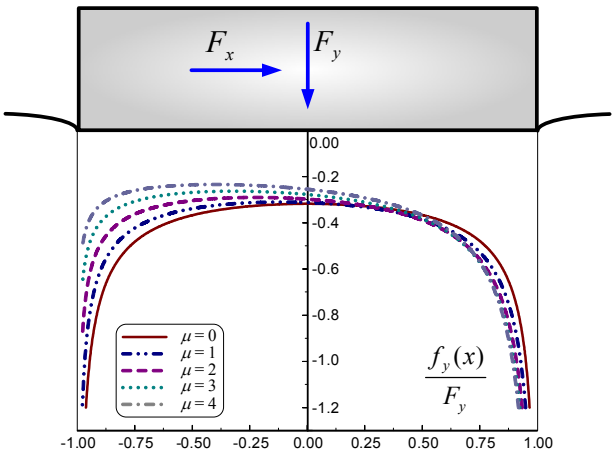


Fig. 5 Contact force $f_y(x)$ for the indentation of a flat-ended punch into an orthotropic half-plane

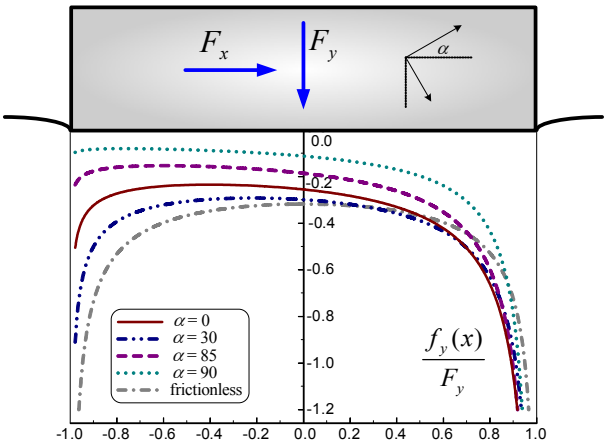


Fig. 6 Contact force $f_y(x)$ for the indentation of a flat-ended punch into an anisotropic half-plane

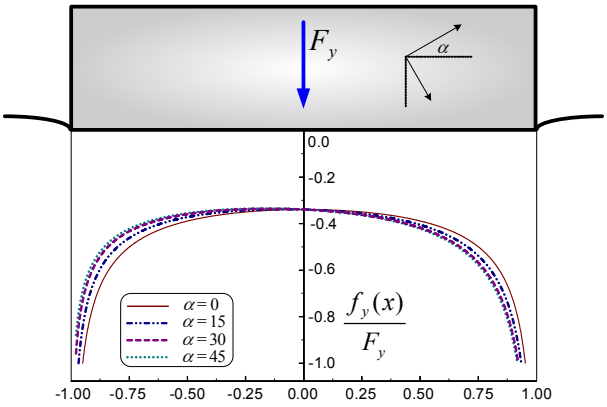


Fig. 7 Contact force $f_y(x)$ for the indentation of a punch into an anisotropic half-plane with no slip

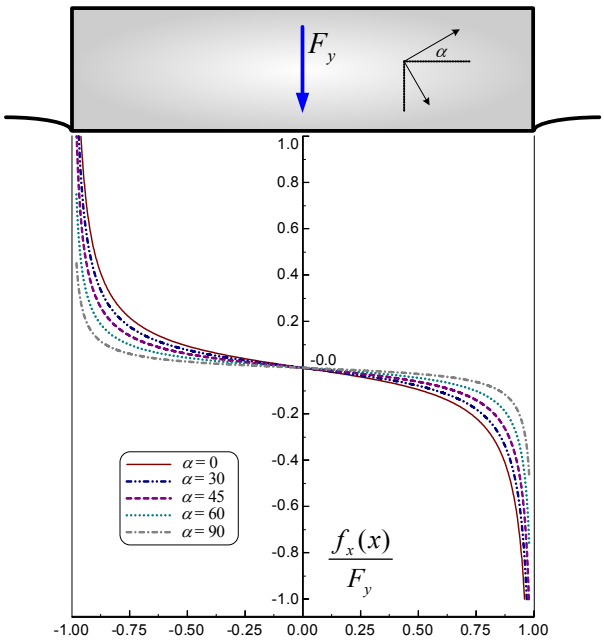


Fig. 8 Contact force $f_x(x)$ for the indentation of a punch into an anisotropic half-plane with no slip

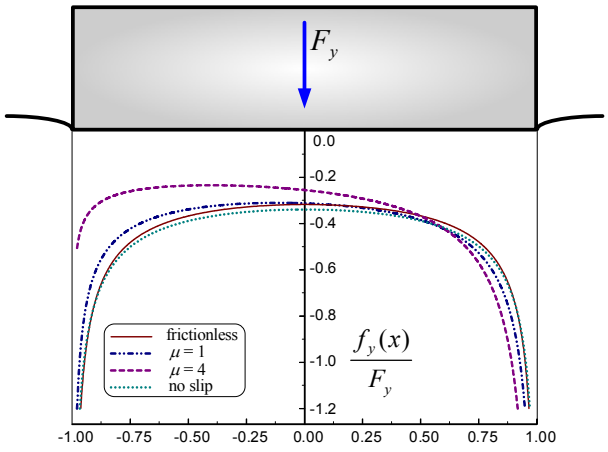


Fig. 9 Contact force $f_y(x)$ for the indentation of a punch into an orthotropic half-plane with different conditions of contact

Table A1 The values of δ and γ with respect to different material constants

θ	0°	30°	45°	60°	75°	90°
δ	0.48822	0.48665	0.48346	0.47748	0.46778	0.46376
γ	0.01178	0.01335	0.01654	0.02252	0.03222	0.03624

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APPENDIX B

The solution of no slip punch with flatted end in Hwu and Fan [11] is

$$\begin{bmatrix} f_x(x) \\ f_y(x) \end{bmatrix} = \frac{1}{\pi\sqrt{a^2-x^2}} \left\{ \mathbf{I} + \frac{1-c_R}{\beta^2}(\mathbf{S}^T)^2 + \frac{c_I}{\beta}\mathbf{S}^T \right\} \begin{bmatrix} F_x \\ F_y \end{bmatrix}, \tag{B1}$$

where $\beta^2 = -\frac{1}{2}\mathrm{tr}(\mathbf{S}^2)$, $c_R + ic_I = \cosh \pi\beta \cdot e^{-i\varepsilon \ln\left(\frac{a+x}{a-x}\right)}$ and $\varepsilon = \frac{1}{2\pi}\ln\frac{1+\beta}{1-\beta}$.

The c_R and c_I can be expressed as

$$c_R = \frac{1}{\sqrt{1-\beta^2}}\cos\ (\varepsilon \ln X), \quad c_I = \frac{-1}{\sqrt{1-\beta^2}}\sin\ (\varepsilon \ln X), \tag{B2}$$

where $X = \ln\frac{a+x}{a-x}$.

The result in the author's study is

$$\begin{bmatrix} f_x(x) \\ f_y(x) \end{bmatrix} = \frac{1}{\pi\sqrt{a^2-x^2}} [\Lambda] \begin{bmatrix} F_x \\ F_y \end{bmatrix}, \tag{B3}$$

where

$$[\Lambda] = \frac{-1}{B\sqrt{\lambda_1^2-1}} \begin{bmatrix} \pi A \sin(\eta \ln X) - B\lambda_1 \cos(\eta \ln X) & \pi D \sin(\eta \ln X) \\ \pi E \sin(\eta \ln X) & \pi A \sin(\eta \ln X) - B\lambda_1 \cos(\eta \ln X) \end{bmatrix}. \tag{B4}$$

We use the material constants in Eq. (62) and the transformation of material constants in Eqs. (7) and (8) to do the numerical compare.

(i) For $\theta = 0°$,

Hwu and Fan's [11] result is

$$\mathbf{I} + \frac{1-c_R}{\beta^2}(\mathbf{S}^T)^2 + \frac{c_I}{\beta}\mathbf{S}^T = \begin{bmatrix} 1.0674\cos(\varepsilon X) & -2.0157\sin(\varepsilon X) \\ 0.5652\sin(\varepsilon X) & 1.0674\cos(\varepsilon X) \end{bmatrix},$$

where $\varepsilon = 0.11621$.

The result of the author's study is

$$[\Lambda] = \begin{bmatrix} 1.0674\cos(\eta X) & -2.0157\sin(\eta X) \\ 0.5652\sin(\eta X) & 1.0674\cos(\eta X) \end{bmatrix}.$$

where $\eta = 0.11621$.

(ii) For $\theta = 30°$,

Hwu and Fan's [11] result is

$$\mathbf{I} + \frac{1-c_R}{\beta^2}(\mathbf{S}^T)^2 + \frac{c_I}{\beta}\mathbf{S}^T = \begin{bmatrix} 1.0674\cos(\varepsilon X) - 0.6281\sin(\varepsilon X) & -1.6531\sin(\varepsilon X) \\ 0.9278\sin(\varepsilon X) & 0.6281\sin(\varepsilon X) + 1.06739\cos(\varepsilon X) \end{bmatrix},$$

where $\varepsilon = 0.11621$.

The result of the author's study is

$$[\Lambda] = \begin{bmatrix} 1.0674\cos(\eta X) - 0.6281\sin(\eta X) & -1.6531\sin(\eta X) \\ 0.9278\sin(\eta X) & 0.6281\sin(\eta X) + 1.0674\cos(\eta X) \end{bmatrix}.$$

where $\eta = 0.11621$.

(iii) For $\theta = 45°$,

Hwu and Fan's [11] result is

$$\mathbf{I} + \frac{1-c_R}{\beta^2}(\mathbf{S}^T)^2 + \frac{c_I}{\beta}\mathbf{S}^T = \begin{bmatrix} 1.0674\cos(\varepsilon X) - 0.7253\sin(\varepsilon X) & -1.2905\sin(\varepsilon X) \\ 1.2905\sin(\varepsilon X) & 1.0674\cos(\varepsilon X) + 0.7253\sin(\varepsilon X) \end{bmatrix},$$

where $\varepsilon = 0.11621$.

The result of the author's study is

$$[\Lambda] = \begin{bmatrix} 1.0674\cos(\eta X) - 0.7253\sin(\eta X) & -1.2905\sin(\eta X) \\ 1.2905\sin(\eta X) & 1.0674\cos(\eta X) + 0.7253\sin(\eta X) \end{bmatrix}.$$

where $\eta = 0.11621$.

The results in case (i), (ii) and (iii) proved that the Eqs. (B1) and (B3) are identical.

https://doi.org/10.1017/jmech.2011.26 Published online by Cambridge University Press

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(Manuscript received March 12, 2010,
accepted for publication April 26, 2010.)

https://doi.org/10.1017/jmech.2011.26 Published online by Cambridge University Press