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Syzygy sequences of the N-center problem

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Abstract. The purpose of this paper is to consider the N-center problem with collinear centers, to identify its syzygy sequences that can be realized by minimizers of the Lagrangian action functional and to count the number of such syzygy sequences. In particular, we show that the number of such realizable syzygy sequences of length ℓ greater than or equal to two for the 3-center problem is at least $F_{\ell+2} - 2$, where $\{F_n\}$ is the Fibonacci sequence. Moreover, with fixed length ℓ, the density of such realizable syzygy sequences of length ℓ for the N-center problem approaches one as N increases to infinity. Using reflection symmetry, the minimizers that we found can be extended to periodic solutions.

1. *Introduction: syzygy sequences and realizability*

Syzygy, as an astronomical term, is often referred to as a collinear configuration of the three-body problem. Here we adopt it for collinear configuration of any system of celestial bodies.

The N-center problem is a simplified version of the $(N + 1)$-body problem; it concerns the motion of a point mass moving in space under the Newtonian gravitational attraction due to N fixed bodies. These fixed bodies are called *centers* of the system. In this paper, we consider the planar N-center problem with collinear centers $\mathbf{c}_1, \ldots, \mathbf{c}_N$. A syzygy is a configuration with the moving body falling on the line through those fixed centers.

Let $m_k > 0$ be the mass of the center $\mathbf{c}_k$ and let the mass of the free particle be one. The equation of motion for the planar N-center problem is

$$\ddot{z} = -\sum_{i=1}^{N} \frac{m_i(z - \mathbf{c}_i)}{|z - \mathbf{c}_i|^3}. \tag{1}$$

Here $z(t) \in \mathbb{R}^2 \cong \mathbb{C}$ represents the position of the free particle at time t. For simplicity, we assume that our collinear centers are sitting on the real axis $\mathbb{R}$ with coordinates

$$\mathbf{c}_i = (c_i, 0), \quad c_1 < c_2 < \cdots < c_{N-1} < c_N. \tag{2}$$

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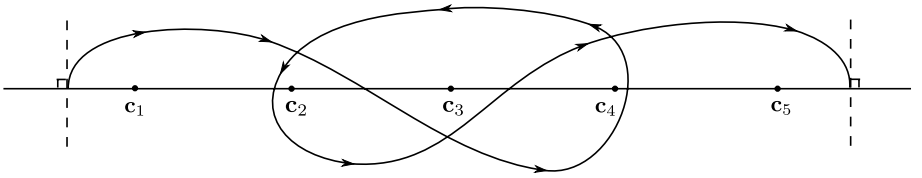


FIGURE 1. An example of a realizable syzygy sequence.

Let $c_0 = -\infty$, $c_{N+1} = \infty$. Consider $N+1$ intervals on the real axis $\mathbb{R}$ given by

$$I_i = (c_{i-1}, c_i), \quad i = 1, 2, \ldots, N+1.$$

A *syzygy sequence* of *length* ℓ (≥ 2) is a finite sequence $\mathbf{s} = (s_1, \ldots, s_\ell)$ satisfying

$$\mathbf{s} = (s_1, \ldots, s_\ell) \in \{1, \ldots, N+1\}^\ell, \quad s_i \neq s_{i+1}, \quad i = 1, \ldots, \ell - 1.$$

We say that a syzygy sequence $\mathbf{s} = (s_1, \ldots, s_\ell)$ is *realizable* if there exists a solution z for (1) satisfying

$$\text{there exists } t_1 < t_2 < \cdots < t_\ell \quad \text{such that}$$
$$z(t_i) \in I_{s_i} \quad i = 1, \ldots, \ell \quad \text{and} \quad z(t) \notin \mathbb{R} \text{ for any } t \in \bigcup_{i=1}^{\ell-1} (t_i, t_{i+1}).$$

In this case, the solution z is said to *realize* the syzygy sequence $\mathbf{s}$. Figure 1 depicts a path realizing the syzygy sequence (1, 3, 5, 2, 4, 6). The solution z is smooth on $[t_1, t_\ell]$ as it does not collide with any center during this time interval.

Due to the time-reversal symmetry of (1), a syzygy sequence $(s_1, \ldots, s_\ell)$ is realizable if and only if its reflection $(s_\ell, \ldots, s_1)$ is realizable. For this reason, syzygy sequences $(s_1, \ldots, s_\ell)$ and $(s_\ell, \ldots, s_1)$ are considered *equivalent*. Throughout this paper, the 'number of (realizable) syzygy sequences' means the 'number of equivalence classes'.

It is well known that the N-center problem with $N \geq 3$ and positive energy has no analytic integral [**1**, **13**, **14**]. A natural question about such a system is to determine topological classes of paths or loops that can be realized by classical solutions. In [**18**] Soave–Terracini consider the N-center problem with small negative energy and prove the existence of solutions within many topological classes of paths or loops. Their approach is based on the Maupertuis' principle and broken geodesic arguments. For the particular case with N collinear centers, their results imply the realizability of many syzygy sequences. Apart from these works, our interests in syzygy sequences are also partially motivated by recent advances on syzygies of the three-body problem [**8**, **15**–**17**].

The purpose of this paper is to identify syzygy sequences that can be realized by minimizers of the Lagrangian action functional A_T and to count the number of such syzygy sequences. In particular, we show that, up to equivalence, the number of realizable syzygy sequences of length $\ell \geq 2$ for the 3-center problem is at least $F_{\ell+2} - 2$, where $\{F_n\}$ is the Fibonacci sequence. Moreover, with fixed ℓ the density of realizable syzygy sequences of length ℓ for the N-center problem approaches one as $N \to \infty$. Furthermore, these solution curves that realize syzygy sequences are minimizers of A_T on some suitable function spaces.

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Given a syzygy sequence $\mathbf{s} = (s_1, \ldots, s_\ell)$ of length ℓ, due to the minimizing property and boundary constraints, a minimizer that realizes this syzygy sequence must hit the real axis (where all the centers are located) orthogonally at the initial and ending moments: t_1, t_ℓ. By reflection symmetry, we acquire a $2(t_\ell - t_1)$-periodic solution which realizes the syzygy sequence $(s_1, \ldots, s_{\ell-1}, s_\ell, s_{\ell-1}, \ldots, s_1)$. This solution realizes a free homotopy class in the N-punctured plane. A more detailed study along this direction can be found in [**22**].

In contrast to previous applications of variational methods to the N-body and N-center problems (see [**4**, **7**, **10**, **11**, **18**, **19**] and references therein), we focus on the syzygy problem and apply a minimization method to the fixed-time Lagrangian action functional on spaces with purely topological constraints. The same minimization approach has been adopted by Castelli in his thesis [**2**, §5.4], for more general symmetrical arrangements of centers. The topological constraint prevents the action minimizer from staying too far away from the centers and, in conjunction with local deformation, collisions can be avoided following arguments similar to those in [**2**, **18**]. The resulting classification of realizable syzygy sequences provides an interesting comparison with the 2-center problem, which is an integrable case (see [**21**]) whose realizable syzygy sequences can be completely classified [**9**]. Our paper can be also regarded as a continuation of the second author's other paper [**22**] which applies the same approach to the problem, but without assuming collinearity of centers. Some lemmas and propositions in this paper have appeared in various forms in the above-mentioned references as well as in [**22**].

The variational formulation of the problem, together with some elementary properties of action minimizers, will be given in §2. Section 3 contains our main results on realizable syzygy sequences. In §4, we fix the length of syzygy sequences and use some combinatorial arguments to provide lower bound estimates for the number of syzygy sequences that can be realized by minimizers of the action functional A_T.

2. *Variational set-up*

Equation (1) is the Euler–Lagrange equation of the the *action functional* defined by

$$A_{T_0,T_1}(z) := \int_{T_0}^{T_1} \frac{1}{2}|\dot{z}|^2 + V(z)\, dt, \tag{3}$$

where

$$V(z) = \sum_{j=1}^{N} \frac{m_j}{|z - \mathbf{c}_j|}$$

is the potential energy. It is well known the action functional A_{T_0,T_1} is of class C^1 in the space of collision-free Sobolev paths $H^1([T_0, T_1], \mathcal{X})$, where $\mathcal{X} := \mathbb{C} \setminus \{\mathbf{c}_j : j = 1, \ldots, \ell\}$, and its critical points are classical solutions of equation (1). For simplicity, the action functional $A_{0,T}$ will be denoted by A_T.

Definition 2.1. Given a syzygy sequence $\mathbf{s}$ of length ℓ and $T_0 < T_1$, we say that a path $z \in H^1([T_0, T_1], \mathcal{X})$ is $\mathbf{s}$-*generic* or *generic with respect to* $\mathbf{s}$ if it satisfies the following conditions:

(i) there exist $t_1 < t_2 < \cdots < t_\ell$ with $t_1 = T_0$, $t_\ell = T_1$ and $z(t_j) \in I_{s_j}$, $j = 1, \ldots, \ell$; and
(ii) $\mathrm{Im}(z(t)) = (-1)^{j-1}|\mathrm{Im}(z(t))|$, for any $t \in (t_j, t_{j+1})$, $j = 1, \ldots, \ell - 1$.

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We denote the set of paths in $H^1([T_0, T_1], \mathcal{X})$ which are homotopic to some $\mathbf{s}$-generic path relative to $\{T_0, T_1\}$ by $\Omega_{\mathbf{s}}^*$, and denote its weak closure in $H^1([T_0, T_1], \mathbb{C})$ by $\Omega_{\mathbf{s}}$. The union of all $\Omega_{\mathbf{s}}$ will be denoted by Ω.

The definition of an $\mathbf{s}$-generic path can be extended to paths in $\Omega_{\mathbf{s}}$ if we replace condition (i) by

(i') there exist $t_1 \le t_2 \le \cdots \le t_\ell$ with $t_1 = T_0$, $t_\ell = T_1$ and $z(t_j) \in \bar{I}_{s_j}$, $j = 1, \ldots, \ell$.

For a generic path, we are not requiring the t_j to be the only moments at which a syzygy occurs. Of course, not all paths from $\Omega_{\mathbf{s}}$ are $\mathbf{s}$-generic. However, we have the following lemma which will be useful later.

LEMMA 2.2. *For any $\zeta \in \Omega_{\mathbf{s}}$, there is an $\mathbf{s}$-generic $z \in \Omega_{\mathbf{s}}$ with the same action value as ζ on $[T_0, T_1]$: $A_{T_0,T_1}(z) = A_{T_0,T_1}(\zeta)$.*

Proof. By Definition 2.1, for any $\zeta \in \Omega_{\mathbf{s}}$, there are $T_0 = t_1 \le t_2 \le \cdots \le t_\ell = T_1$ such that $\zeta(t_j) \in \bar{I}_{s_j}$, $j = 1, \ldots, \ell$. Define a new path $z \in \Omega_{\mathbf{s}}$ as

$$z(t) = \zeta(t) \quad \text{if } t = t_j \text{ for some } 1 \le j \le \ell,$$

$$z(t) = \begin{cases} \zeta(t) & \text{if } t \in (t_j, t_{j+1}),\ \zeta(t) \in \mathbb{R} \text{ or } \dfrac{\mathrm{Im}(\zeta(t))}{|\mathrm{Im}(\zeta(t))|} = (-1)^{j-1}, \\ \bar{\zeta}(t) & \text{if } t \in (t_j, t_{j+1}),\ \zeta(t) \notin \mathbb{R} \text{ and } \dfrac{\mathrm{Im}(\zeta(t))}{|\mathrm{Im}(\zeta(t))|} = (-1)^{j},\ 1 \le j \le \ell - 1. \end{cases}$$

Then z is generic with respect to $\mathbf{s}$. As all centers are located on the real axis, $A_{t_1,t_\ell}(z) = A_{t_1,t_\ell}(\zeta)$ since the potential is invariant under the complex conjugacy. □

In the next lemma, we show the existence of action minimizers in $\Omega_{\mathbf{s}}$. The proof is similar to the case of Newtonian N-body problems, for which the main idea of the proof is similar to that of Poincaré's inequality; a version has appeared in [**12**, §III]. The proof below is a close analogy to that of [**3**, Proposition 3].

LEMMA 2.3. *For any syzygy sequence $\mathbf{s}$ and $T_0 < T_1$, A_{T_0,T_1} attains its infimum on $\Omega_{\mathbf{s}}$.*

Proof. For simplicity, we assume that $T_0 = 0$ and let $T_1 = T$. In what follows, $L^p([0, T], \mathbb{C})$ and $H^1([0, T], \mathbb{C})$ will be denoted by L^p and H^1, respectively.

It is well known that the action functional $A_T = A_{0,T}$ is weakly lower semi-continuous on H^1. As $\Omega_{\mathbf{s}}$ is weakly closed, it is sufficient to show that A_T is coercive in $\Omega_{\mathbf{s}}$: that is, $A_T(z) \to \infty$ if $\|z\|_{H^1} \to \infty$.

It is given that $z \in \Omega_{\mathbf{s}}$. Let $i = \mathbf{s}_1$. Then $z(0) \in \bar{I}_i$ and $z(t^*) \in \bar{I}_j$ for some $t^* \in (0, T]$ and $j \ne i$. Assume, without loss of generality, that $i < j$. Then, $|z(0) - c_i| \le |z(0) - z(t^*)| \le \|\dot{z}\|_{L^1}$. Let $C = \max\{|\mathbf{c}_k| : k = 1, \ldots, N\}$. Then, for any $t \in [0, T]$,

$$|z(t)| \le |z(t) - z(0)| + |z(0) - c_i| + |c_i| \le 2\|\dot{z}\|_{L^1} + C \le 2\sqrt{T}\|\dot{z}\|_{L^2} + C.$$

Therefore

$$\|z\|_{L^2}^2 \le \left(2\sqrt{T}\|\dot{z}\|_{L^2} + C\right)^2 T$$

and so $\|\dot{z}\|_{L^2} \to \infty$ as $\|z\|_{H^1} \to \infty$. This implies coercivity of A_T since $A_T(z) \ge \frac{1}{2}\|\dot{z}\|_{L^2}^2$. □

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We remark here that the proof of Lemma 2.3 not only shows the existence of the action minimizer z, but also provides upper bounds for $\|z\|_{C^0}$ and $\|z\|_{H^1}$. This is in contrast with fixed-energy problems, and is a by-product of the Lagrangian minimizing method with fixed-time constraint.

Of course, a minimizer obtained by the above result could contain collisions. However, the next lemma tells us that this could happen at most a finite number of times.

Definition 2.4. Given a $z \in H^1([T_0, T_1], \mathbb{C})$, we say it is a *generalized solution* of equation (1) if it satisfies the following conditions:

(i) the set of collision moments $\Delta(z) := \{t \in [T_0, T_1] : z(t) \in \mathbb{C} \setminus \mathcal{X}\}$ is an isolated set;
(ii) $z(t)$ satisfies equation (1) for any $t \in [T_0, T_1] \setminus \Delta(z)$; and
(iii) the energy is preserved through collisions: $\frac{1}{2}|\dot{z}(t)|^2 - V(z(t)) = \text{constant}$, for any $t \in [T_0, T_1] \setminus \Delta(z)$.

LEMMA 2.5. *If z is a minimizer of A_{T_0,T_1} on $\Omega_{\mathbf{s}}$, then it is a generalized solution of (1).*

The proof of the above result is an easier version of a similar result for the N-body problem, for which a detailed proof can be found in [**5**, §3.3] and [**20**, §4.1].

3. *Properties of minimizers*

In this section, we will show that, for certain syzygy sequences, the action minimizers obtained by Lemma 2.3 are free of collision. Hence they are solutions realizing the corresponding syzygy sequences. By Lemma 2.5, we just need to show that isolated collisions cannot exist. The key idea is that for most isolated collisions, by making a proper local deformation near an isolated collision, we can obtain a new path with strictly smaller action and reach a contradiction.

For this, consider an arbitrary generalized solution $z \in H^1([\tau - \delta, \tau + \delta], \mathbb{C})$ with a single collision at the moment τ, where $z(\tau) = \mathbf{c}_j$ for some $\delta > 0$. Put $z(t)$ in polar coordinates with $\mathbf{c}_j$ as the origin

$$z(t) - \mathbf{c}_j = \rho(t)e^{i\theta(t)} \quad \text{for all } t \in [\tau - \delta, \tau) \cup (\tau, \tau + \delta].$$

The following asymptotic estimates are well known to experts. Detailed proofs can be found in [**22**, Propositions 2.1 and 2.2]. Below, the notation $f(t) \sim c\,|t - \tau|^\alpha$ means $f(t) = c\,|t - \tau|^\alpha + o(|t - \tau|^\alpha)$.

PROPOSITION 3.1. *There is a constant $\mu > 0$ such that:*

(a) $\rho(t) \sim \mu|t - \tau|^{2/3}$, $\dot{\rho}(t) \sim \frac{2}{3}\mu|t - \tau|^{-1/3}$ *as* $t \to \tau+$;
(b) $\rho(t) \sim \mu|t - \tau|^{2/3}$, $\dot{\rho}(t) \sim -\frac{2}{3}\mu|t - \tau|^{-1/3}$ *as* $t \to \tau-$.

PROPOSITION 3.2. *There exist θ_-, θ_+ such that*

$$\lim_{t \to \tau^\pm} \theta(t) = \theta_\pm, \qquad \lim_{t \to \tau^\pm} \dot{\theta}(t) = 0.$$

For the Kepler problem or the 1-center problem, we know that, for any two different points with the same positive distance from the origin (where the center is placed), the parabolic collision-ejection path connecting them has action value strictly larger than a direct and an indirect Keplerian arc joining them with the same transfer time. This result was attributed to Marchal in [**6**]. A proof can be found in [**11**] as well as in [**19**], [**18**]

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and [**22**], and, in the last three references, the result was proved for general homogeneous potentials.

Based on the result of the Kepler problem and the asymptotic estimates, we have the following local deformation property using the *blow-up* technique introduced by Terracini.

PROPOSITION 3.3. *Following the notation introduced above, if* $|\theta_+ - \theta_-| < 2\pi$*, then, for any* $\delta^* > 0$ *small enough, there is an* $\varepsilon = \varepsilon(\delta^*) > 0$ *and a collision-free path* $\zeta \in H^1([\tau - \delta, \tau + \delta], \mathcal{X})$ *which satisfies:*

(a) $\zeta(t) = z(t)$ *for any* $t \in [\tau - \delta, \tau + \delta] \setminus [\tau - \delta^*, \tau + \delta^*]$*;*

(b) $|\zeta(t) - \mathbf{c}_j| \le \varepsilon$ *for any* $t \in [\tau - \delta^*, \tau + \delta^*]$*;*

(c) $\mathrm{Arg}(\zeta(\tau \pm \delta) - \mathbf{c}_j) = \mathrm{Arg}(z(\tau \pm \delta) - \mathbf{c}_j)$*; and*

(d) $A_{\tau-\delta,\tau+\delta}(\zeta) < A_{\tau-\delta,\tau+\delta}(z)$.

Furthermore, ε *goes to zero as* δ^* *goes to zero.*

We omit the proof here; see [**22**, Proposition 3.2] for details. While Proposition 3.3 can be applied to isolated collisions happening at interior moments of $[T_0, T_1]$, we need similar results for boundary moments: T_0, T_1.

PROPOSITION 3.4. *Given* $|\theta_0 - \theta_+| < \pi$*, for any* $\delta^* > 0$ *small enough, there is an* $\varepsilon = \varepsilon(\delta^*) > 0$ *and a collision-free path* $\zeta \in H^1([\tau, \tau + \delta], \mathcal{X})$ *which satisfies:*

(a) $\zeta(t) = z(t)$ *for any* $t \in [\tau + \delta^*, \tau + \delta]$;

(b) $|\zeta(t) - \mathbf{c}_j| \le \varepsilon$ *for any* $t \in [\tau, \tau + \delta^*]$;

(c) $\mathrm{Arg}(\zeta(\tau) - \mathbf{c}_j) = \theta_0$ *and* $\mathrm{Arg}(\zeta(\tau + \delta) - \mathbf{c}_j) = \mathrm{Arg}(z(\tau + \delta) - \mathbf{c}_j)$*; and*

(d) $A_{\tau,\tau+\delta}(\zeta) < A_{\tau,\tau+\delta}(z)$.

Furthermore, ε *goes to zero as* δ^* *goes to zero.*

PROPOSITION 3.5. *Given* $|\theta_0 - \theta_-| < \pi$*, for any* $\delta^* > 0$ *small enough, there is an* $\varepsilon = \varepsilon(\delta^*) > 0$ *and a collision-free path* $\zeta \in H^1([\tau - \delta, \tau], \mathcal{X})$ *which satisfies:*

(a) $\zeta(t) = z(t)$ *for any* $t \in [\tau - \delta, \tau - \delta^*]$;

(b) $|\zeta(t) - \mathbf{c}_j| \le \varepsilon$ *for any* $t \in [\tau - \delta^*, \tau]$;

(c) $\mathrm{Arg}(\zeta(\tau) - \mathbf{c}_j) = \theta_1$ *and* $\mathrm{Arg}(\zeta(\tau - \delta) - \mathbf{c}_j) = \mathrm{Arg}(z(\tau - \delta) - \mathbf{c}_j)$*; and*

(d) $A_{\tau-\delta,\tau}(\zeta) < A_{\tau-\delta,\tau}(z)$.

Furthermore, ε *goes to zero as* δ^* *goes to zero.*

Proofs of the last two propositions are almost the same as the proof of Proposition 3.3 and we will not repeat details here.

Notice that Proposition 3.3 does not hold when $\theta_+ - \theta_- = \pm 2\pi$. In this case, the collision can be regarded as elastic. Since the energy is conserved, the absolute value of the velocity of the free particle will be the same after collision and the direction of it reverses. Using this fact, we can show that, in this case, it must be a *collision–reflection* solution, as stated in the following lemma. For a precise proof, one just needs to perform a Levi-Civita regularization near the isolated collision. Details can be found in [**18**, §4.4] or [**22**, §5].

LEMMA 3.6. *If* $\theta_- = \theta_+ (\mathrm{mod}\ 2\pi)$*, then the generalized solution* z *is a* ***collision–reflection*** *solution of (1): that is,*

$$z(\tau + t) = z(\tau - t) \quad \text{for any } t \in [0, \delta].$$

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With the above results, we are now ready to prove the main results of this section.

THEOREM 3.7. *If a syzygy sequence* $\mathbf{s}$ *of length* ℓ *with* $\ell \geq 2$ *satisfies*

$$|s_j - s_{j+1}| \geq 2 \quad \text{for all } j \in \{1, \ldots, \ell - 1\} \quad \text{and} \quad \max\{|s_j - s_k| : 1 \leq j < k \leq \ell\} \geq 3, \tag{4}$$

then it can be realized by a collision-free minimizer of the action functional A_T *on* $\Omega_{\mathbf{s}}$.

It is not hard to see that Theorem 3.7 is a direct corollary of Lemma 2.3 and the following proposition.

PROPOSITION 3.8. *Given a finite syzygy sequence* $\mathbf{s}$ *of length* ℓ *with* $\ell \geq 2$*, if it satisfies conditions (4) in Theorem 3.7, then any minimizer* z *of* A_T *on* $\Omega_{\mathbf{s}}$ *has the following properties:*

(a) $z(t) \in \mathcal{X}$ *for any* $t \in [0, T]$*;*
(b) *there exist* $0 = t_1 < t_2 < \cdots < t_\ell = T$ *such that* $z(t_j) \in I_{s_j}$ *for any* $j = 1, \ldots, \ell$*;*
(c) $\mathrm{Im}(z(t)) \neq 0$ *and* $\mathrm{Im}(z(t)) = (-1)^{j-1}|\mathrm{Im}(z(t))|$ *for any* $t \in (t_j, t_{j+1})$, $j = 1, \ldots, \ell - 1$;
(d) $z|_{[t_j, t_{j+1}]}$ *is free of self-intersections for any* $j = 1, \ldots, \ell - 1$; *and*
(e) z *is orthogonal to the real axis at* $z(0)$ *and* $z(T)$.

Remark 3.9. By property (e) in this proposition, the minimizer can be extended to a periodic solution which realizes the syzygy sequence $(s_1, \ldots, s_{\ell-1}, s_\ell, s_{\ell-1}, \ldots, s_1)$.

Proof. By Lemma 2.2, there is no loss of generality by assuming that z is $\mathbf{s}$-generic. For the moment, assume that property (a) holds. As $z \in \Omega_{\mathbf{s}}^*$, it is obvious that property (b) must hold.

Since z is generic, property (c) holds if we can show that $\mathrm{Im}(z(t)) \neq 0$ for any $t \in [0, T] \setminus \{t_1, t_2, \ldots, t_\ell\}$. We shall argue by contradiction. Assume that there exists $\tau \in (t_j, t_{j+1})$ with $z(\tau) \in \mathbb{R}$ for some $j \in \{1, \ldots, \ell - 1\}$.

First, $\mathrm{Im}(\dot{z}(\tau))$ cannot be zero, otherwise before and after the moment τ the particle would stay on the real axis until a collision happens in one of the directions, which violates property (a). Second, $\mathrm{Im}(\dot{z}(\tau)) \neq 0$ cannot happen either. Otherwise z would intersect the real axis transversally at $\tau \in (t_j, t_{j+1})$, which cannot happen to a generic path. This proves property (c).

Property (d) follows directly from property (c) and the fact that z is an action minimizer.

Property (e) is simply the natural boundary condition resulting from the first variation of the action functional.

Now we will start to prove property (a). Since z is an $\mathbf{s}$-generic path, there are $0 = t_1 \leq t_2 \leq \cdots \leq t_\ell = T$ such that

$$\begin{cases} \mathrm{Im}(z(t_j)) = 0 & \text{for any } j = 1, \ldots, \ell, \\ \mathrm{Im}(z(t)) = (-1)^{j-1}|\mathrm{Im}(z(t))| & \text{for any } t \in (t_j, t_{j+1}) \text{ and } j = 1, \ldots, \ell - 1. \end{cases} \tag{5}$$

Meanwhile, $|s_j - s_{j+1}| \geq 2$ implies

$$\mathrm{dist}(\bar{I}_{s_j}, \bar{I}_{s_{j+1}}) := \min\{|q - p| : q \in \bar{I}_{s_j}, p \in \bar{I}_{s_{j+1}}\} > 0.$$

As a result, $0 = t_1 < t_2 < \cdots < t_\ell = T$.

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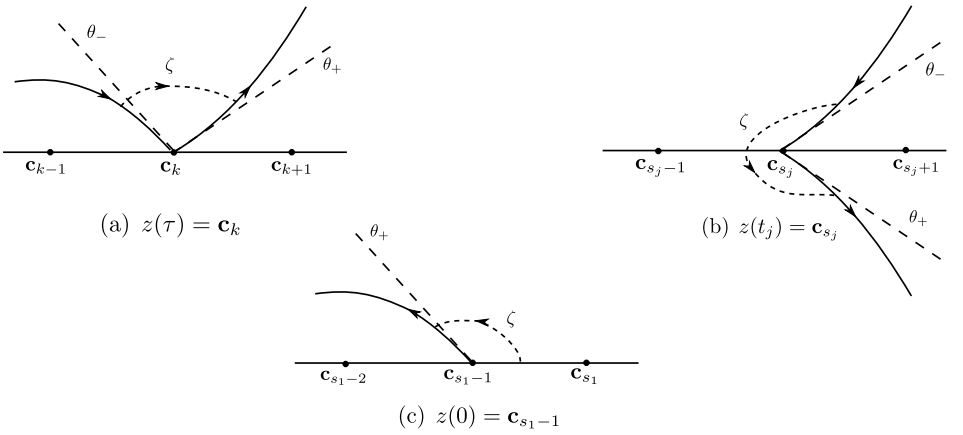


FIGURE 2. Local deformation.

By Lemma 2.5, the set of collision moments $\Delta(z)$ is isolated. In the following, we will show that it is, in fact, empty. This will be demonstrated in three steps.

Step 1. Show that a collision cannot happen at any moment in (t_j, t_{j+1}), $j = 1, \ldots, \ell - 1$.

Suppose there is a $\tau \in (t_j, t_{j+1})$ with $z(\tau) = \mathbf{c}_k$ for an odd j (the proof for j being even is similar). Choose a $\delta > 0$ small enough such that $[\tau - \delta, \tau + \delta] \subset (t_j, t_{j+1})$ and $[\tau - \delta, \tau + \delta] \cap \Delta(z) = \tau$.

Put $z|_{[\tau-\delta,\tau+\delta]}$ in polar coordinates: $z(t) - \mathbf{c}_k = \rho(t)e^{i\theta(t)}$. As j is odd, by (5), we may assume that $0 \le \theta(t) \le \pi$ for $t \ne \tau$ and, by Proposition 3.2, there are $\theta_\pm \in [0, \pi]$ with $\lim_{t\to\tau^\pm} \theta(t) = \theta_\pm$.

Obviously, $|\theta_+ - \theta_-| < 2\pi$. By Proposition 3.3, we can make a local deformation and get a new path in $\Omega_{\mathbf{s}}$ with action value strictly smaller than that of z (see Figure 2(a)), which is absurd. This finishes our proof of Step 1.

Step 2. A collision cannot occur at any t_j, $j = 2, \ldots, \ell - 1$.

We argue by contradiction. Assume that $t_j \in \Delta(z)$ for an even j (the case of j being odd is similar). Since $z(t_j) \in \bar{I}_{s_j}$, either $z(t_j) = \mathbf{c}_{s_j-1}$ or $\mathbf{c}_{s_j}$.

First, assume that $z(t_j) = \mathbf{c}_{s_j}$. Choose a $\delta > 0$ small enough, such that $[t_j - \delta, t_j + \delta] \subset (t_{j-1}, t_{j+1})$ and $[t_j - \delta, t_j + \delta] \cap \Delta(z) = \{\tau\}$. Put $z|_{[t_j-\delta,t_j+\delta]}$ into polar coordinates with the origin at $\mathbf{c}_{s_j}$, as before. As j is even, by (5), we can assume that

$$\theta(t) \in [0, \pi] \quad \text{if } t \in [t_j - \delta, t_j), \qquad \theta(t) \in [\pi, 2\pi] \quad \text{if } t \in (t_j, t_j + \delta].$$

The particular ranges of the argument are chosen to ensure that the deformed path goes around $\mathbf{c}_{s_j}$ counter-clockwise. As a result, $\theta_- = \lim_{t\to t_j^-} \theta(t) \in [0, \pi]$ and $\theta_+ = \lim_{t\to t_j^+} \theta(t) \in [\pi, 2\pi]$.

As long as $\theta_+ - \theta_- < 2\pi$, by Proposition 3.3, we can make a local deformation within $\Omega_{\mathbf{s}}$ and get a contradiction (see Figure 2(b)).

The only exception not excluded by local deformation is when $\theta_- = 0$ and $\theta_+ = 2\pi$. In this case, by Lemma 3.6, $z|_{[t_j-\delta,t_j+\delta]}$ is an collision–reflection solution. Furthermore,

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before and after the collision, the free particle must stay on the real axis between $\mathbf{c}_{s_j}$ and $\mathbf{c}_{s_j+1}$ until it collides with one of these two centers. To be precise, there is a $t^* > 0$ such that

$$z(t_j - t) = z(t_j + t) \in (c_{s_j}, c_{s_j+1}) \quad \text{for all } t \in (0, t^*) \quad \text{and}$$
$$z(t_j - t^*) = z(t_j + t^*) \in \{\mathbf{c}_{s_j}, \mathbf{c}_{s_j+1}\}.$$

Since $|s_{j+1} - s_j| \geq 2$ and $|s_j - s_{j-1}| \geq 2$, by the result from Step 1, this can happen only when the following equalities hold simultaneously

$$s_{j+1} - s_j = s_{j-1} - s_j = 2, \quad t^* = t_{j+1} - t_j = t_j - t_{j-1} \quad \text{and}$$
$$z(t_{j-1}) = z(t_{j+1}) = \mathbf{c}_{s_j+1}.$$

If the other possibility, $z(t_j) = \mathbf{c}_{s_j-1}$, occurs, then, by the same argument as above, the only case not excluded by the local deformation argument is when the following equalities hold simultaneously

$$s_{j+1} - s_j = s_{j-1} - s_j = -2, \quad z(t_{j+1}) = z(t_{j-1}) = \mathbf{c}_{s_j-2}.$$

The only difference is that, this time, in the polar coordinates, we need to choose

$$\theta(t) \in [0, \pi] \quad \text{if } t \in [t_j - \delta, t_j), \quad \theta(t) \in [-\pi, 0] \quad \text{if } t \in (t_j, t_j + \delta],$$

as we need the deformed path to go around the center $\mathbf{c}_{s_j-1}$ clockwise.

By the above arguments, whenever $t_j \in \Delta(z)$, so are t_{j-1} and t_{j+1} (whenever they are defined). By induction, $t_k \in \Delta(z)$ for any $k = 1, \ldots, \ell$. Furthermore, the particle has to stay on the real interval between two neighboring centers all the time. As a result, $|s_j - s_k|$ is either zero or two, which is a contradiction to (4).

Step 3. A collision cannot occur at $t = 0$ or T.

We will only give details for $t = 0$, while the other case is similar. Since $z(0) \in \bar{I}_{s_1}$, a collision can only happen with $\mathbf{c}_{s_1-1}$ or $\mathbf{c}_{s_1}$. Assume that $z(0) = \mathbf{c}_{s_1-1}$ (if $s_1 = 1$, then c_{s_1-1} is at negative infinity and this is impossible). Choose a $\delta < t_2$; by the above results, $[0, \delta] \cap \Delta(z) = \{0\}$. Choose polar coordinates with $\mathbf{c}_{s_1-1}$ as the origin as before; by (5), we may assume that $\theta(t) \in [0, \pi]$ for any $t \in (0, \delta]$. Then $\theta_+ = \lim_{t \to 0^+} \theta(t) \in [0, \pi]$.

If $\theta_+ \neq \pi$, we can apply Proposition 3.4 to get a contradiction by choosing a suitable local deformation (see Figure 2(c)).

If $\theta_+ = \pi$, then after the initial moment zero, the free particle must stay on the real axis between $\mathbf{c}_{s_1-2}$ and $\mathbf{c}_{s_1-1}$ until it first collides with one of them at some $t^* \in (0, T]$. By the result of Step 1, $t^* = t_2$. If $l > 2$, this contradicts the result in Step 2. If $l = 2$, then $t^* = t_2 = T$. However, by condition (4), $|s_2 - s_1| \geq 3$. This means that $z(T)$ can never equal to $\mathbf{c}_{s_1-2}$ or $\mathbf{c}_{s_1-1}$, which is again a contradiction.

We have now proved that $z(0) \neq \mathbf{c}_{s_1-1}$. The proof for $z(0) \neq \mathbf{c}_{s_1}$ is similar. □

The condition $|s_j - s_{j+1}| \geq 2$ for any j, in Theorem 3.7, can be seen as forcing the particle to cross at least two centers between every two consecutive syzygies. It was brought to our attention by Susanna Terracini that similar conditions and discussions have appeared in [**2**, §5.4]. The following result shows that even if this condition does not hold for some j, certain syzygy sequences are still realizable.

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THEOREM 3.10. *Suppose a syzygy sequence* $\mathbf{s}$ *of length* $\ell \geq 4$ *satisfies the following conditions:*

(a) $|s_1 - s_2| \geq 2$ *and* $|s_\ell - s_{\ell-1}| \geq 2$*;*

(b) *if* $s_{j+1} - s_j = 1$*, then either* $s_{j-1} < s_j < s_{j+1} < s_{j+2}$ *or* $s_{j+2} < s_{j-1} < s_j < s_{j+1}$*; and*

(c) *if* $s_j - s_{j+1} = 1$*, then either* $s_{j-1} > s_j > s_{j+1} > s_{j+2}$ *or* $s_{j+2} > s_{j-1} > s_j > s_{j+1}$.

Then $\mathbf{s}$ *can be realized by a collision-free minimizer of the action functional* A_T *on* $\Omega_{\mathbf{s}}$.

Again the above theorem follows directly from Lemma 2.3 and the next proposition.

PROPOSITION 3.11. *Given a syzygy sequence* $\mathbf{s}$ *of length* $\ell \geq 4$*, if it satisfies the same conditions as in Theorem 3.10, then the conclusions of Proposition 3.8 will still hold.*

Proof. By the same argument as used in the proof of Proposition 3.8, it is enough to show that any minimizer z of A_T on $\Omega_{\mathbf{s}}$ is free of collision. Without loss of generality, assume that z is an $\mathbf{s}$-generic path. Then z satisfies (5) for some $0 = t_1 \leq t_2 \leq \cdots \leq t_\ell = T$. Although (t_j, t_{j+1}) may be empty for some j at this moment, the same arguments as used in the proof of Proposition 3.8, Step 1, will still work and

$$z(t) \in \mathcal{X} \quad \text{for any } t \in (t_j, t_{j+1}) \quad \text{and} \quad j = 1, \ldots, \ell - 1. \tag{6}$$

Obviously, $t_j < t_{j+1}$, if $|s_{j+1} - s_j| \geq 2$. By condition (a) of Theorem 3.10, $t_1 < t_2$ and $t_{\ell-1} < t_\ell$. In the following, we will show that $t_j < t_{j+1}$, even when $|s_{j+1} - s_j| = 1$.

Assume that $s_{j+1} - s_j = 1$ (the other case is similar) and $\tau = t_j = t_{j+1}$ for an even j (the case of j being odd is similar). As a result, $z(t_j) = z(t_{j+1}) = \mathbf{c}_{s_j}$. Notice that, by condition (a) of Theorem 3.10, $j \notin \{1, \ell - 1, \ell\}$ and therefore s_{j-1}, s_{j+2} are well defined.

Choose a $\delta > 0$ small enough, such that $[\tau - \delta, \tau + \delta] \subset [t_{j-1}, t_{j+2}]$ and $[\tau - \delta, \tau + \delta] \cap \Delta(z) = \{\tau\}$. Put $z|_{[\tau-\delta,\tau+\delta]}$ into polar coordinates with the origin at $\mathbf{c}_{s_j}$ as before. By (5), we may assume that

$$\theta(t) \in [0, \pi] \quad \text{if } t \in [\tau - \delta, \tau), \quad \theta(t) \in [2\pi, 3\pi] \quad \text{if } t \in (\tau, \tau + \delta].$$

The ranges of θ are chosen to ensure that the deformed path will go around $\mathbf{c}_{s_j}$ counterclockwise. Then $\theta_- = \lim_{t \to \tau^-} \theta(t) \in [0, \pi]$ and $\theta_+ = \lim_{t \to \tau^+} \theta(t) \in [2\pi, 3\pi]$.

In the following, we need to discuss two different cases.

Case 1. $s_{j-1} < s_j < s_{j+1} < s_{j+2}$.

First, we claim $\theta_+ = \theta_- + 2\pi$ cannot happen. Otherwise conditions of Lemma 3.6 would be satisfied and z would be a collision–reflection solution near τ: that is, there would be a $t^* > 0$ such that

$$z(\tau - t) = z(\tau + t) \quad \text{for all } t \in [0, t^*].$$

By (6), $z(t) \in \mathcal{X}$ satisfies equation (1) for any $t \in (t_{j-1}, t_{j+2}) \setminus \{\tau\}$. By the existence and uniqueness theorem of ordinary differential equations, this t^* can be chosen so that $[t_{j-1}, t_{j+2}] \subset [\tau - t^*, \tau + t^*]$. Using the reflection symmetry of z on $[\tau - t^*, \tau + t^*]$ and the fact that

$$z(t_{j-1}) \in \bar{I}_{s_{j-1}}, \quad z(t_{j+2}) \in \bar{I}_{s_{j+2}} \quad \text{and} \quad \bar{I}_{s_{j-1}} \cap \bar{I}_{s_{j+2}} = \emptyset,$$

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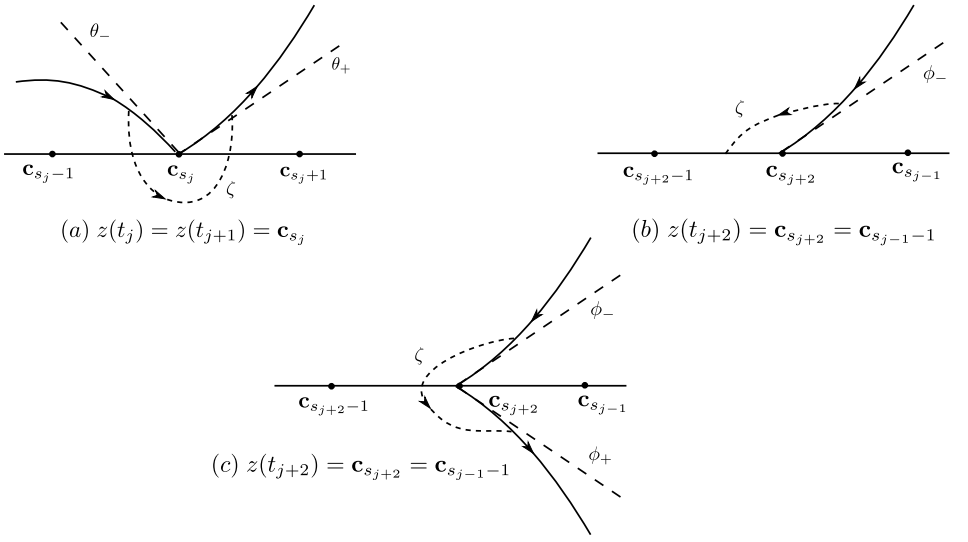


FIGURE 3. Local deformation.

we know that $\mathrm{Im}(\dot{z}) \neq 0$ on $(t_{j-1}, t_{j+2}) \setminus \{\tau\}$, because otherwise z would move along $I_{s_j} \cup I_{s_{j+1}}$ until a collision occurs at some moment in $(t_{j-1}, t_{j+2}) \setminus \{\tau\}$. Furthermore, the reflection symmetry implies the existence of $\tau_1 \in (t_{j-1}, t_{j+2}) \setminus \{\tau\}$ such that $z(\tau_1) \in I_{s_{j-1}} \cup I_{s_{j+2}}$. We must have $\mathrm{Im}(\dot{z}(\tau_1)) \neq 0$, but then z would intersect the real axis transversally at τ_1, which cannot happen to a generic path.

Second, we claim that there is no loss of generality by assuming that $\theta_+ < \theta_- + 2\pi$. Otherwise, because $z(t_{j-1}) < z(\tau) < z(t_{j+2})$ (they are all real numbers) and $\mathrm{Im}(z(t)) \geq 0$ for any $t \in [t_{j-1}, t_{j+2}]$, there must be $\tau_1 \in (t_{j-1}, \tau)$ and $\tau_2 \in (\tau, t_{j+2})$ with $z(\tau_1) = z(\tau_2)$. By reversing the time parameter on $[\tau_1, \tau_2]$, we get a new path in $\Omega_{\mathbf{s}}$, which is still an action minimizer, but with $\theta_+ < \theta_- + 2\pi$.

As a result, $\theta_+ \in (\theta_-, \theta_- + 2\pi)$ and we can use Proposition 3.3 to make a local deformation near the isolated collision $z(\tau)$ to get a contradiction (see Figure 3(a)).

Case 2: $s_{j+2} < s_{j-1} < s_j < s_{j+1}$.

By the same arguments as used in case 1, it is enough to show that $\theta_+ \neq \theta_- + 2\pi$. Now assume that $\theta_+ = \theta_- + 2\pi$. Then $z(t)$ is a collision–reflection solution near the moment τ for some $t^* > 0$, as in case 1. Furthermore t^* can be chosen so that $(t_{j-1}, t_{j+2}) \subset (\tau - t^*, \tau + t^*)$.

Since $z(t_{j-1}) \in \bar{I}_{s_{j-1}}$, $z(t_{j+2}) \in \bar{I}_{s_{j+2}}$ and $s_{j+2} < s_{j-1}$, this can happen only when

$$z(t_{j-1}) = \mathbf{c}_{s_{j-1}-1} = z(t_{j+2}) = \mathbf{c}_{s_{j+2}} \quad \text{and} \quad s_{j+2} = s_{j-1} - 1.$$

Suppose, otherwise, with $s_{j+2} < s_{j-1} - 1$. Then $\bar{I}_{s_{j+2}} \cap \bar{I}_{s_{j-1}} = \emptyset$, and we get a contradiction, as in case 1.

Now $z(t_{j+2}) = \mathbf{c}_{s_{j+2}}$ is an isolated collision. For $\delta > 0$ small enough, we set $z(t) - \mathbf{c}_{s_{j+2}} = r(t)e^{i\phi(t)}$ for any $t \in [t_{j+2} - \delta, t_{j+2} + \delta] \cap [0, T]$. As j is even, so is $j + 2$. We may assume that

$$\phi(t) \in [0, \pi] \quad \text{if } t \in [t_{j+2} - \delta, t_{j+2});$$

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Then

$$\phi_- = \lim_{t \to t_{j+2}^-} \phi(t) \in [0, \pi].$$

Provided $j+2 < \ell$ and $t_{j+2} < t_{j+3}$, we can also assume that

$$\phi(t) \in [\pi, 2\pi] \quad \text{if } t \in (t_{j+2}, t_{j+2} + \delta].$$

Below, we will see that $j+2 < \ell$ actually implies $t_{j+2} < t_{j+3}$.

First, notice that $\phi_- \neq 0$. Suppose otherwise. Then, before the collision at $t = t_{j+2}$, the particle must stay on the real axis between the two centers $\mathbf{c}_{s_{j+2}}$ and $\mathbf{c}_{s_{j+2}+1}$. Since $z(t_{j+1}) = \mathbf{c}_{s_j}$ and $c_{s_j} > c_{s_{j-1}} = c_{s_{j+2}+1}$, there must be a collision moment $\tau_1 \in (t_{j+1}, t_{j+2})$, which is a contradiction to (6).

If $j+2 = \ell$, as $\phi_- \in (0, \pi]$, by Proposition 3.5, we can make a local deformation of z near $z(t_{j+2})$ to move it away from $\mathbf{c}_{s_{j+2}}$, but still contained in $(c_{s_{j+2}-1}, c_{s_{j+2}})$, so as to reduce the action value and get a contradiction (see Figure 3(b)).

If $j+2 < \ell$, notice that $s_{j+3} - s_{j+2} = 1$ cannot happen, otherwise conditions (a) and (b) of Theorem 3.10 would imply that $j+3 < \ell$ and

$$s_{j+1} < s_{j+2} < s_{j+3} < s_{j+4} \quad \text{or} \quad s_{j+4} < s_{j+1} < s_{j+2} < s_{j+3},$$

which violates $s_{j+2} < s_{j+1}$. Hence either $s_{j+3} - s_{j+2} \geq 2$ or $s_{j+3} - s_{j+2} \leq -1$ must hold. Since $z(t_{j+2}) = \mathbf{c}_{s_{j+2}}$ is the right-hand end of the interval $\bar{I}_{s_{j+2}}$, in either case, we must have $t_{j+2} < t_{j+3}$.

As $j+2$ is even, $\mathrm{Im}(z(t)) \leq 0$ for any $t \in [t_{j+2}, t_{j+3}]$. For $\delta > 0$ small enough, we may assume that

$$\phi(t) \in [\pi, 2\pi] \quad \text{if } t \in (t_{j+2}, t_{j+2} + \delta].$$

Then $\phi_+ = \lim_{t \to t_{j+2}^+} \phi(t) \in [\pi, 2\pi]$. We have $\phi_+ - \phi_- < 2\pi$ since $\phi_- \in (0, \pi]$. Then, by Proposition 3.3, we can make a local deformation of z near $z(t_{j+2})$ so as to reduce the action value and get a contradiction (see Figure 3(c)). This finishes case 2.

To summarize, we have proved that $0 = t_1 < t_2 < \cdots < t_\ell = T$. The rest of the proof is similar to that of Proposition 3.8. We omit it here. □

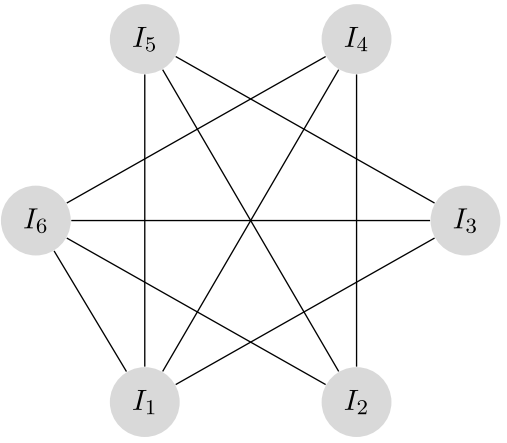
4. *Counting realizable syzygy sequences*

Consider the simple graph G_{N+1} with vertices $I_1, \ldots, I_{N+1}$ and edges from I_i to I_j for any i, j with $|i-j| > 1$. For example, G_6 is the graph below.

The adjacency matrix $\mathfrak{A}_{N+1}$ of the graph G_{N+1} is the $(N+1) \times (N+1)$ binary matrix

$$\mathfrak{A}_{N+1} = \begin{pmatrix} 0 & 0 & 1 & \cdots & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 0 \end{pmatrix}. \tag{7}$$

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It is the square matrix consisting of ones except for its tridiagonal which consists of zeros.

Paths in the graph G_{N+1} from I_i to I_j consisting of ℓ edges (repetition allowed) are called *walks of length* ℓ. Since G_{N+1} is simple, any walk of length ℓ can be uniquely expressed as an $(\ell+1)$-tuple $(I_{s_1}, \dots, I_{s_{\ell+1}})$. Clearly, its subscripts $(s_1, \dots, s_{\ell+1})$ is a syzygy sequence of length $\ell+1$. This correspondence is not surjective as $|s_i - s_{i+1}| = 1$ is allowed for the syzygy sequence $(s_1, \dots, s_{\ell+1})$ but not for the walk $(I_{s_1}, \dots, I_{s_{\ell+1}})$ of the graph G_{N+1}. As for syzygy sequences, walks $(I_{s_1}, \dots, I_{s_{\ell+1}})$ and $(I_{s_{\ell+1}}, \dots, I_{s_1})$ are considered *equivalent*.

LEMMA 4.1. *Consider the N-center problem (1), (2). Let* $\mathfrak{A}_{N+1}$ *be as in (7). It is given that* $T > 0$. *The number of syzygy sequences of length* $\ell \geq 2$ *which can be realized by collision-free minimizers of the action functional* A_T *on* Ω *is at least*

$$\frac{1}{2}\sum_{i,j=1}^{N+1} (\mathfrak{A}_{N+1}^{\ell-1})_{ij} - N + 1.$$

Proof. A well-known property of adjacency matrices states that the entry $(\mathfrak{A}_{N+1}^{\ell-1})_{ij}$ of $\mathfrak{A}_{N+1}^{\ell-1}$ in row i and column j is the number of walks in G_{N+1} of length $\ell-1$ from vertex I_i to I_j. This can be easily obtained by induction on ℓ. When $i = j$, each walk is counted twice. So half of the sum of entries in $\mathfrak{A}_{N+1}^{\ell-1}$ is the total number (up to equivalence) of walks of length $\ell-1$. By Theorem 3.7, we know that walks in G_{N+1} of length $\ell-1$ correspond to syzygy sequences of length ℓ that are realized by collision-free minimizers of A_T, except for walks of the form

$$(I_i, I_j, I_i, I_j, \dots, I_i, I_j) \quad \text{or} \quad (I_i, I_j, I_i, I_j, \dots, I_i, I_j, I_i), \quad |i-j| = 2. \tag{8}$$

The lemma follows since there are, up to equivalence, $N-1$ sequences of the form (8). □

Below, we provide a lower bound for the number of realizable syzygy sequences. We begin with the case $N = 3$, which has a surprisingly simple bound in terms of the

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Fibonacci sequence

$$(F_0, F_1, F_2, F_3, \ldots) = (0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots):$$
$$\text{i.e. } F_0 = 0,\ F_1 = 1,\ F_{n+2} = F_n + F_{n+1} \text{ for } n \geq 0.$$

THEOREM 4.2. *Let $\{F_n\}$ be the Fibonacci sequence and let $N = 3$. For any $\ell \geq 2$, there are at least $F_{\ell+2} - 2$ syzygy sequences of length ℓ for the 3-center problem (1), (2) which can be realized by collision-free minimizers of the action functional A_T on the space Ω.*

Proof. The first few powers of $\mathfrak{A}_{N+1} = \mathfrak{A}_4$ are

$$\mathfrak{A}_4 = \begin{pmatrix} 0&0&1&1\\ 0&0&0&1\\ 1&0&0&0\\ 1&1&0&0 \end{pmatrix}, \quad \mathfrak{A}_4^2 = \begin{pmatrix} 2&1&0&0\\ 1&1&0&0\\ 0&0&1&1\\ 0&0&1&2 \end{pmatrix}, \quad \mathfrak{A}_4^3 = \begin{pmatrix} 0&0&2&3\\ 0&0&1&2\\ 2&1&0&0\\ 3&3&0&0 \end{pmatrix}, \quad \mathfrak{A}_4^4 = \begin{pmatrix} 5&3&0&0\\ 3&2&0&0\\ 0&0&2&3\\ 0&0&3&5 \end{pmatrix}.$$

By induction, we find

$$\mathfrak{A}_4^{2k-1} = \begin{pmatrix} 0 & 0 & F_{2k-1} & F_{2k}\\ 0 & 0 & F_{2k-2} & F_{2k-1}\\ F_{2k-1} & F_{2k-2} & 0 & 0\\ F_{2k} & F_{2k-1} & 0 & 0 \end{pmatrix}, \quad \mathfrak{A}_4^{2k} = \begin{pmatrix} F_{2k+1} & F_{2k} & 0 & 0\\ F_{2k} & F_{2k-1} & 0 & 0\\ 0 & 0 & F_{2k-1} & F_{2k}\\ 0 & 0 & F_{2k} & F_{2k+1} \end{pmatrix}, \quad k \in \mathbb{N},$$

Regardless of whether $\ell \in \mathbb{N}$ is even or odd, the sum of elements in $\mathfrak{A}_4^\ell$ is

$$2(F_{\ell+1} + 2F_\ell + F_{\ell-1}) = 2F_{\ell+3}.$$

Then, by Lemma 4.1, there are at least $F_{\ell+3} - 2$ realizable syzygy sequences of length $\ell + 1$. □

Remark 4.3. For $N = 3$, the number of (equivalent) syzygy sequences of length ℓ is $2 \cdot 3^{\ell-1}$. Therefore, the density of realizable syzygy sequences of length ℓ is at least

$$\frac{F_{\ell+2} - 2}{2 \cdot 3^{\ell-1}}.$$

Since the ratio of consecutive Fibonacci numbers F_{k+1}/F_k approaches the golden ratio $(1 + \sqrt{5})/2$ as $k \to \infty$, the ratio above converges to zero as $\ell \to \infty$.

In sharp contrast with this remark, the next theorem shows that 'almost all' syzygy sequences are realizable when the number of centers is large.

THEOREM 4.4. *Consider the N-center problem (1), (2). For fixed $\ell \geq 2$, the density of realizable syzygy sequences of length ℓ approaches one as N increases to infinity.*

Proof. Apparently, there are $(N + 1)N^{\ell-1}/2$ syzygy sequences. The sum of elements in $\mathfrak{A}_{N+1}$ is $N(N - 1)/2$, so, by Lemma 4.1, there are at least $(N - 1)(N - 2)/2$ realizable syzygy sequences of length $\ell = 2$. Therefore the theorem holds for $\ell = 2$.

Observe that, by writing $\mathfrak{A}_{N+1}^\ell$ as $\mathfrak{A}_{N+1}^{\ell-1}\mathfrak{A}_{N+1}$, the entry of $\mathfrak{A}_{N+1}^\ell$ in row i and column j is precisely

$$(\mathfrak{A}_{N+1}^\ell)_{ij} = \sum_{\substack{k=1\\ |k-j|>1}}^{N+1} (\mathfrak{A}_{N+1}^{\ell-1})_{ik}. \tag{9}$$

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The summation on the right-hand side contains $N-1$ or $N-2$ terms, depending on whether $j \in \{1, N+1\}$ or not. Since the number of ones in each row of $\mathfrak{A}_{N+1}$ is $N-1$ or $N-2$, by (9), each entry of $\mathfrak{A}_{N+1}^2$ is at least $N-5$. By (9) again, each entry of $\mathfrak{A}_{N+1}^3$ is at least $(N-5)(N-2)$. Inductively, for $m \geq 0$, each entry of $\mathfrak{A}_{N+1}^{2+m}$ is bounded from below by $(N-5)(N-2)^m$. Therefore, for $\ell \geq 3$, the sum of elements in $\mathfrak{A}_{N+1}^{\ell-1}$ is at least

$$(N+1)^2(N-5)(N-2)^{\ell-3}.$$

By Lemma 4.1, there are at least

$$\tfrac{1}{2}(N+1)^2(N-5)(N-2)^{\ell-3} - N + 1 \tag{10}$$

realizable syzygy sequences of length ℓ. Therefore the density of realizable syzygy sequences of length ℓ is bounded from below by

$$\frac{(N+1)^2(N-5)(N-2)^{\ell-3} - 2(N-1)}{(N+1)N^{\ell-1}},$$

which approaches one as N increases to infinity. □

In the proof of Theorem 4.4, we have obtained a lower bound (10) in terms of N, ℓ for the number of realizable syzygy sequences of length ℓ. This bound is rather crude and does not offer any information for cases $N = 4, 5$. Here is a simple refinement for this lower bound.

THEOREM 4.5. *Consider the N-center problem (1), (2). Let $\mathfrak{A}_{N+1}$ be as in (7). For $\ell \geq m+2$, $m \in \mathbb{N}$, the number of syzygy sequences of length ℓ which can be realized by collision-free minimizers of the action functional A_T on the space Ω is at least*

$$\frac{1}{2}(N+1)(N-2)^{\ell-m-1} \sum_{i=1}^{N+1} \min_k \left\{(\mathfrak{A}_{N+1}^m)_{ik}\right\} - N + 1.$$

Proof. By (9), for each $m \in \mathbb{N}$ and $j \in \{1, 2, \ldots, N+1\}$,

$$(\mathfrak{A}_{N+1}^{m+1})_{ij} \geq (N-2) \min_k \left\{(\mathfrak{A}_{N+1}^m)_{ik}\right\}.$$

The right-hand side is independent of j, so

$$(\mathfrak{A}_{N+1}^{m+2})_{ij} \geq (N-2) \min_k \left\{(\mathfrak{A}_{N+1}^{m+1})_{ik}\right\} \geq (N-2)^2 \min_k \left\{(\mathfrak{A}_{N+1}^m)_{ik}\right\}.$$

Inductively, we find

$$(\mathfrak{A}_{N+1}^{m+s})_{ij} \geq (N-2)^s \min_k \left\{(\mathfrak{A}_{N+1}^m)_{ik}\right\} \quad \text{for all } s \in \mathbb{N}.$$

Let $\ell = m + s + 1$ and sum both sides over $i, j \in \{1, 2, \ldots, N+1\}$. Then

$$\sum_{i,j=1}^{N+1} (\mathfrak{A}_{N+1}^{\ell-1})_{ij} \geq (N+1)(N-2)^{\ell-m-1} \sum_{i=1}^{N+1} \min_k \left\{(\mathfrak{A}_{N+1}^m)_{ik}\right\} \quad \text{for all } \ell \geq m+2.$$

Now the theorem follows, from Lemma 4.1. □

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For example, let $(N, m) = (4, 5)$ and $(5, 5)$. $\mathfrak{A}^m_{N+1}$ are, respectively,

$$\mathfrak{A}^5_5 = \begin{pmatrix} 18 & 10 & 22 & 26 & 34 \\ 10 & 4 & 14 & 20 & 26 \\ 22 & 14 & 16 & 14 & 22 \\ 26 & 20 & 14 & 4 & 10 \\ 34 & 26 & 22 & 10 & 18 \end{pmatrix}, \quad \mathfrak{A}^5_6 = \begin{pmatrix} 88 & 63 & 78 & 89 & 88 & 108 \\ 63 & 42 & 55 & 69 & 73 & 88 \\ 78 & 55 & 60 & 66 & 69 & 89 \\ 89 & 69 & 66 & 60 & 55 & 78 \\ 88 & 73 & 69 & 55 & 42 & 63 \\ 108 & 88 & 89 & 78 & 63 & 88 \end{pmatrix}.$$

Let $\ell \geq 7$. Applying Theorem 4.5 for $N = 4$, there are at least $105 \cdot 2^{\ell-6} - 3$ syzygy sequences of length ℓ which are collision-free minimizers of A_T; for $N = 5$ there are at least $960 \cdot 3^{\ell-6} - 4$ such realizable syzygy sequences of length ℓ.

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