THE TRANSVERSE FLEXURE OF THIN ELASTIC PLATES SUPPORTED AT SEVERAL POINTS

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ABSTRACT. This paper depends upon the method developed by Kolossoff and Muskhelishvili for problems of plane elasticity and later extended to plate problems by Lechnitzky. Exact solutions in closed forms are obtained for the problem of a thin circular plate supported at several interior or boundary points and normally loaded over the area of an eccentric circle, the load being symmetrical with respect to the centre of the circle and the boundary of the plate being free. Explicit formulae for the deflexion, the bending and twisting moments and shearing stresses are given at any point of the plate. As limiting cases plates in the form of the infinite plane and half plane are also considered.

Introduction. A method of handling thin circular plates, subject to normal loadings symmetrically distributed with respect to the centre of the plate, and supported at several points along the boundary was developed by Nadai (11) and quoted by Timoshenko (13). The same method was used by De Beer (5) to obtain the deflexion at any point of a symmetrically loaded circular plate supported in a number of points, regularly distributed around the boundary. Following similar lines and using solutions previously obtained (1, 2) for circular plates simply supported along the entire boundary, the author (3) found infinite series solutions for circular plates subject to certain loadings over the area of an eccentric circle, when the plate is supported at several points along the boundary, the supports being symmetrically situated with respect to the common diameter of plate and circle. The loadings considered in (1), (2) and (3) include uniform as well as linearly varying loadings.

Lurie (8) applied Muskhelishvili's method for solving the biharmonic equation to the discussion of several problems of the bending of circular plates. Friedmann (6) considered the problem of the bending of a thin isotropic plate when acted upon by any number of normal concentrated forces at interior or boundary points. As special cases of his formulae he found the solutions of certain particular cases which were considered earlier by Lurie (8) and Nadai (11). Applying Muskhelishvili's complex variable method Chankvetadze (4) found a solution in closed form for a thin circular plate normally and uniformly loaded over the whole plate and supported at a number of points of its periphery. The same method was used by Washizu (14) to discuss the bending of a circular plate under several concentrated loads on the boundary, the faces of the plate being free from normal forces. The boundary condition taken by Washizu is of a form different from that used by Friedmann and Chankvetadze and may be obtained from it by differentiation. Yi-Yuan Yu (15) studied the bending of isotropic homogeneous thin plates, the boundaries of which were subjected to isolated couples and forces. Three particular problems are worked out in detail, namely, those of circular plates subjected respectively to two bending moments, to two twisting

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moments both applied at the ends of a diameter, and to four forces applied at the ends of two perpendicular diameters.

In this paper solutions in finite terms are obtained for the problem of a thin circular plate supported at several interior points and normally loaded over the area of an eccentric circle, the load being symmetrical with respect to the centre of the circle and the boundary of the plate being free. The solutions obtained are based on the classical Poisson–Kirchhoff theory of small deflexions of thin plates and are exact within the assumptions underlying the theory. No special treatment is required when all supports or some of them lie at the boundary, and the solutions in this case can be obtained from those in the case of interior supports by a limiting process of letting all supports or some of them tend to the boundary. The solutions given include as special cases those of Friedmann and Chankvetadze as well as the solution obtained by Yu for concentrated edge forces. As limiting cases solutions are deduced for thin elastic plates in the form of the infinite plane and the half plane supported at any number of points and normally loaded over a circle.

1. *Preliminary results and statement of the problem.* The transverse displacement w, measured positively downwards, at any point $z = x+iy = re^{i\theta}$ of the mid-plane of a normally loaded thin elastic plate of thickness h and flexural rigidity D bounded by a closed curve C satisfies the biharmonic equation

$$D\nabla^4 w = p, \tag{1·1}$$

where p is the normal load intensity at the point z. The general solution of (1·1) may be represented in terms of two analytic functions $\phi(z)$, $\chi(z)$ of the complex variable z and a particular integral $W(z,\bar{z})$:

$$w = \bar{z}\phi(z) + z\overline{\phi}(\bar{z}) + \chi(z) + \overline{\chi}(\bar{z}) + W(z,\bar{z}), \tag{1·2}$$

where bars are used to denote conjugate complex quantities. In an unloaded region $p = 0$ and $W(z,\bar{z})$ may be taken as zero.

The bending moments per unit length M_x, M_y, twisting moment per unit length M_{xy}, and shearing forces per unit length Q_x, Q_y may conveniently be given (7), (12) by

$$\left.\begin{aligned} M_x + M_y &= -4D(1+\nu)\,\partial^2 w/\partial z\,\partial\bar{z} &&= -4D(1+\nu)[2\Re\phi'(z) + \partial^2 W/\partial z\,\partial\bar{z}],\\ M_x - M_y + 2iM_{xy} &= -4D(1-\nu)\,\partial^2 w/\partial z^2 &&= -4D(1-\nu)[\bar{z}\phi''(z) + \psi'(z) + \partial^2 W/\partial z^2],\\ Q_x - iQ_y &= -8D\,\partial^3 w/\partial z^2\,\partial\bar{z} &&= -8D[\phi''(z) + \partial^3 W/\partial z^2\,\partial\bar{z}], \end{aligned}\right\} \tag{1·3}$$

where

$$\psi(z) = \chi'(z), \tag{1·4}$$

ν is Poisson's ratio, and accents denote derivatives with respect to z. If s and n are arcs of orthogonal curves in the mid-plane of the plate, then

$$\left.\begin{aligned} M_n + M_s &= M_x + M_y,\\ M_n - M_s + 2iM_{ns} &= e^{2i\alpha}(M_x - M_y + 2iM_{xy}),\\ Q_n - iQ_s &= e^{i\alpha}(Q_x - iQ_y), \end{aligned}\right\} \tag{1·5}$$

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where α is the angle which the direction n at the point considered makes with the x axis. In the special case of polar coordinates $\alpha = \theta$, $e^{i\alpha} = z/r$, and substituting from (1·3) in (1·5) we get

$$\left.\begin{aligned} M_r + M_\theta &= -4D(1+\nu)[2\Re\phi'(z) + \partial^2 W/\partial z\,\partial\bar{z}], \\ M_r - M_\theta + 2iM_{r\theta} &= -4D(1-\nu)[z\phi''(z) + z^2(\psi' + W'')/r^2], \\ Q_r - iQ_\theta &= -8Dz[\phi''(z) + \partial^3 W/\partial z^2\,\partial\bar{z}]/r. \end{aligned}\right\} \tag{1·6}$$

Let the plate be normally loaded over the area bounded by a closed curve C_0 inside C and supported at any number of points. Let 1 and 2 refer to the two regions inside C_0 and between C_0 and C respectively. Since the region 2 is unloaded we may take $W_2(z,\bar{z}) = 0$, while $W_1(z,\bar{z})$ depends on the type of loading considered and can be easily obtained if the transverse load intensity is known.

Let the moment M_n and shear resultant per unit length V_n along the boundary C of the plate be given as functions of the arc length s along C by

$$M_n = m(s), \quad V_n = Q_n - \partial M_{ns}/\partial_s = v(s), \tag{1·7}$$

where n denotes the normal exterior to C. It can be shown (7) that the boundary condition has the final form

$$\kappa\phi_2(z) + z\overline{\phi}_2'(\bar{z}) + \overline{\psi}_2(\bar{z}) = f \quad \text{on } C, \tag{1·8}$$

where

$$\kappa = (3+\nu)/(\nu-1), \tag{1·9}$$

and

$$f = f_1 + if_2 = \frac{1}{2D(1-\nu)}\int_0^s \left[m(s) + \int_0^s v(s)\,ds\right](dx + i\,dy). \tag{1·10}$$

The necessary and sufficient conditions for the physical continuity required across C_0 can be taken as

$$[w]_1^2 = [\partial w/\partial z]_1^2 = [\partial^2 w/\partial z\,\partial\bar{z}]_1^2 = [\partial^3 w/\partial z^2\,\partial\bar{z}]_1^2 = 0. \tag{1·11}$$

The problem to be solved now becomes one of using the boundary condition (1·8) and the continuity conditions (1·11) to determine the functions $\phi_1(z), \psi_1(z), \chi_1(z)$; $\phi_2(z), \psi_2(z), \chi_2(z)$. Once these functions are determined, the deflexion, moments and shearing forces at any point of the plate may be computed by means of equations (1·2) and (1·3) or (1·6).

For a circular plate with the origin as centre and radius c, we put $z = c\zeta$, and the boundary C in the z plane is mapped on the boundary γ of the unit circle in the ζ plane. The boundary condition (1·8) is now transformed to

$$\kappa\Phi_2(\sigma) + \sigma\bar{\Phi}_2'(\bar{\sigma}) + \bar{\Psi}_2(\bar{\sigma}) = f \quad \text{on } \gamma, \tag{1·12}$$

where $\Phi_2(\zeta) = \phi_2(z)$, $\Psi_2(\zeta) = \psi_2(z)$ and the function f has also been transformed and is now a function of $\sigma = e^{i\theta}$, namely the value of ζ on the boundary γ of the unit circle.

When all supports lie at interior points of the plate so that the boundary of the plate is free we have

$$m(s) = v(s) = 0. \tag{1·13}$$

Noticing that the terms $C_1 + iC_1'z$ in $\phi(z)$ and $C_2 + C_2'z$ in $\chi(z)$, where C_1' is a real constant and C_1, C_2, C_2' are arbitrary complex constants, make no contributions to the moments and shearing forces at any point, we see that in the case of a free boundary the function

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f in (1·12) may be taken as $A+iB\sigma$, B being real. The two terms $iC_1'z$ in $\phi(z)$ and $C_2'z$ in $\chi(z)$ may be dropped, since the first contributes nothing to w and the contribution of the second term is similar to that of the term C_1 in $\phi(z)$ and may be absorbed into it. Hence apart from constant terms in $\phi_2(z)$ and $\psi_2(z)$ the function f in (1·8) and (1·12) may be taken as zero in the case of a free boundary. Thus in our problem the boundary condition may be written in the form

$$\kappa\Phi_2(\sigma)+\sigma\bar{\Phi}_2'(\bar{\sigma})+\bar{\Psi}_2'(\bar{\sigma})=0 \quad \text{on } \gamma. \tag{1·14}$$

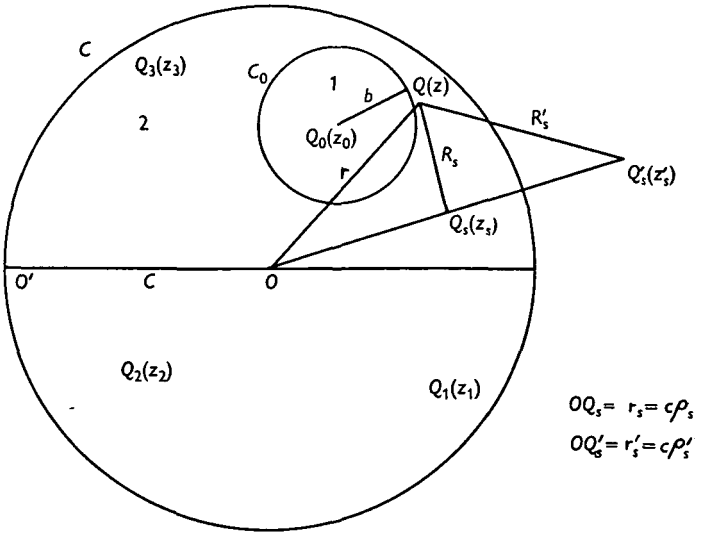


Fig. 1

2. *Circular plate supported at several points under normal loading over an eccentric circle.* Consider a thin circular plate with centre O, radius c and boundary C. Let C_0 denote the boundary of an eccentric circle of centre $Q_0(z_0)$ and radius b. Let

$$z = x+iy = re^{i\theta} = c\zeta$$

be the complex variable of any point Q of the plate referred to O as origin. We suppose that the plate is normally loaded over the area of the eccentric circle and supported at the m interior points Q_s $(z_s = r_s e^{i\theta_s})$ where $s = 1, 2, \ldots, m$. Let Q_s' $(z_s' = r_s' e^{i\theta_s})$ be the inverse of Q_s with respect to C, so that

$$\bar{z}_s z_s' = c^2, \quad \bar{\zeta}_s \zeta_s' = 1. \tag{2·1}$$

Let us now put (see Fig. 1)

$$Z_s = z - z_s, \quad Z_s' = z_s' - z; \quad |Z_s| = R_s, \quad |Z_s'| = R_s'. \tag{2·2}$$

For all points on C_0 we have $$Z_0\bar{Z}_0 = b^2. \tag{2·3}$$

Let the normal load intensity p on the plate be given by

$$p_1 = p_0 R_0^{n-2} \quad (n \geqslant 2), \quad p_2 = 0. \tag{2·4}$$

The total load P_0 on the plate is therefore given by

$$P_0 = 2\pi p_0 b^n/n. \tag{2·5}$$

For $n = 2$ we have uniform loading over the eccentric circle.

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We denote the concentrated reaction, measured positively downwards at the point of support Q_s by P_s. The conditions of equilibrium of the plate give

$$\sum_{s=0}^{m} P_s = 0, \quad \sum_{s=0}^{m} P_s z_s = \sum_{s=0}^{m} P_s \zeta_s = 0. \tag{2·6}$$

The particular integrals $W_1(z, \bar{z})$ and $W_2(z, \bar{z})$ may be taken as

$$W_1(z, \bar{z}) = (16\alpha/n^2 n'^2)\,(Z_0 \bar{Z}_0)^{\frac{1}{2}n'}, \quad W_2(z, \bar{z}) = 0, \tag{2·7}$$

where

$$n' = n+2 \quad \text{and} \quad \alpha = p_0/16D. \tag{2·8}$$

Substituting from (1·2) and (2·7) in the continuity conditions (1·11) and using (2·3) we get

$$[\bar{z}\phi(z) + z\overline{\phi}(\bar{z}) + \chi(z) + \overline{\chi}(\bar{z})]_1^2 = 16\alpha b^{n'}/n^2 n'^2, \tag{2·9a}$$

$$[\bar{z}\phi'(z) + \overline{\phi}(\bar{z}) + \psi(z)]_1^2 = 8\alpha b^{n'}/n^2 n' Z_0, \tag{2·9b}$$

$$[\phi'(z) + \overline{\phi}'(\bar{z})]_1^2 = 4\alpha b^n/n^2, \tag{2·9c}$$

$$[\phi''(z)]_1^2 = 2\alpha b^n/n Z_0. \tag{2·9d}$$

As tentative solutions for these continuity conditions we assume that

$$[\phi]_1^2 = Z_0(A + B \log Z_0/b), \tag{2·10a}$$

$$[\chi + \bar{z}_0\phi]_1^2 = C + D \log Z_0/b, \tag{2·10b}$$

where A, B, C and D are real constants to be determined. Substituting from (2·10a) in (2·9c), (2·9d) and equating the coefficients of powers of Z_0 on both sides we get

$$A = kP_0(1-n)/n, \quad B = kP_0, \tag{2·11}$$

where

$$k = 1/16\pi D. \tag{2·12}$$

Hence we have

$$[\phi]_1^2 = kP_0 Z_0(1/n - 1 + \log Z_0/b). \tag{2·13}$$

Substituting from (2·13) and (2·10b) in (2·9a), (2·9b) and equating the coefficients of powers of Z_0 on both sides we get

$$C = kP_0 b'^2(1+n')/n', \quad D = kP_0 b'^2, \tag{2·14}$$

where

$$b'^2 = b^2\delta_n, \quad \delta_n = n/n'. \tag{2·15}$$

Substituting from (2·14) and (2·13) in (2·10b) we find

$$[\chi]_1^2 = kP_0\left[\bar{z}_0 Z_0\left(1 - \frac{1}{n} - \log\frac{Z_0}{b}\right) + b'^2\left(1 + \frac{1}{n'} + \log\frac{Z_0}{b}\right)\right]. \tag{2·16}$$

The singular part of the deflexion function w near the concentrated force P_s is given by

$$w_{\text{sin.}} = (P_s/8\pi D)\, R_s^2 \log R_s. \tag{2·17}$$

The contributions of the expression (2·17) to the functions $\phi(z)$ and $\chi(z)$ are

$$\phi_{\text{sin.}}(z) = kP_s Z_s \log Z_s, \quad \chi_{\text{sin.}}(z) = -kP_s \bar{z}_s Z_s \log Z_s. \tag{2·18}$$

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Making use of (2·13), (2·16) and the singular parts of $\phi(z)$ and $\chi(z)$ given by (2·18) we now assume that

$$\phi_2(z) = k\sum_{s=0}^{m} P_s Z_s \log Z_s + \phi_0(z), \tag{2·19a}$$

$$\chi_2(z) = -k\left[\sum_{s=0}^{m} P_s \bar{z}_s Z_s \log Z_s - P_0 b'^2 \log Z_0\right] + \chi_0(z), \tag{2·19b}$$

where $\phi_0(z)$ and $\chi_0(z)$ are functions of z regular all over the plate. Substituting from (2·19a), (2·19b) in (2·13), (2·16) we get

$$\phi_1(z) = k\left[\sum_{s=1}^{m} P_s Z_s \log Z_s + P_0 Z_0\left(1 - \frac{1}{n} + \log b\right)\right] + \phi_0(z), \tag{2·20a}$$

$$\chi_1(z) = -k\left[\sum_{s=1}^{m} P_s \bar{z}_s Z_s \log Z_s + P_0\left\{\bar{z}_0 Z_0\left(1 - \frac{1}{n} + \log b\right) + b'^2\left(1 + \frac{1}{n'} - \log b\right)\right\}\right] + \chi_0(z). \tag{2·20b}$$

Substituting from (2·19), (2·20) and (2·7) in (1·2) we get

$$w_1 = w_0 + 2k\left[\sum_{s=1}^{m} P_s R_s^2 \log R_s + P_0\left\{\left(1 - \frac{1}{n}\right)(R_0^2 - b^2) + \frac{4(R_0^{n'} - b^{n'})}{nn'^2 b^n} + (R_0^2 + b'^2)\log b\right\}\right], \tag{2·21a}$$

$$w_2 = w_0 + 2k\left[\sum_{s=0}^{m} P_s R_s^2 \log R_s + P_0 b'^2 \log R_0\right], \tag{2·21b}$$

where

$$w_0 = \bar{z}\phi_0(z) + z\overline{\phi_0}(\bar{z}) + \chi_0(z) + \overline{\chi_0}(\bar{z}). \tag{2·22}$$

It is easily verified from (2·21a), (2·21b) that $w_1 = w_2$ on $R_0 = b$. There remains now the determination of the regular functions $\phi_0(z)$ and $\chi_0(z)$ from the boundary condition (1·14). Substituting from (2·19) in (1·14) remembering that $\Phi_2(\zeta) = \phi_2(z)$ and $\Psi_2(\zeta) = \chi_2'(z)$, we find that the boundary condition takes the form

$$\kappa\Phi_0(\sigma) + \sigma\bar{\Phi}_0'(\bar{\sigma}) + \bar{\Psi}_0(\bar{\sigma}) = F \quad \text{on } \gamma, \tag{2·23}$$

where

$$\Phi_0(\zeta) = \phi_0(z), \quad \Psi_0(\zeta) = \psi_0(z) = \chi_0'(z),$$

$$F = -ck\left[\sum_{s=0}^{m} P_s(\sigma - \zeta_s)\left\{\kappa \log(\sigma - \zeta_s) + \log\frac{1 - \bar{\zeta}_s\sigma}{\sigma}\right\} + \frac{P_0 q'^2 \sigma}{1 - \bar{\zeta}_0\sigma}\right], \tag{2·24}$$

and $q = b/c$, $q' = b'/c$. Hence we have

$$\bar{F} = -ck\left[\sum_{s=0}^{m} P_s \frac{1 - \bar{\zeta}_s\sigma}{\sigma}\left\{\kappa \log\frac{1 - \bar{\zeta}_s\sigma}{\sigma} + \log(\sigma - \zeta_s)\right\} + \frac{P_0 q'^2}{\sigma - \zeta_0}\right]. \tag{2·25}$$

Each of the two analytic functions $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ can always be represented by a power series consisting of only positive powers of ζ. Furthermore, because of the indeterminateness of the functions, we may write

$$\left.\begin{aligned}\Phi_0(\zeta) &= \alpha_1\zeta + \alpha_2\zeta^2 + \ldots,\\ \Psi_0(\zeta) &= \beta_0 + \beta_1\zeta + \beta_2\zeta^2 + \ldots,\end{aligned}\right\} \tag{2·26}$$

where α_1 is a real constant and $\alpha_2, \alpha_3, \ldots; \beta_0, \beta_1, \beta_2, \ldots$ are complex constants. To solve for $\Phi_0(\zeta)$ and $\Psi_0(\zeta)$ from the boundary condition (2·23) we multiply the equation and

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its conjugate throughout by $\frac{1}{2\pi i}\frac{d\sigma}{\sigma-\zeta}$ and integrate around γ, ζ being any point inside γ. The procedure is entirely the same as that used by Muskhelishvili (9) in obtaining the general solution for a circular plate loaded in its plane. The following results for the plate problem are easily found:

$$\Phi_0(\zeta) = \frac{1}{2\pi i\kappa}\int_\gamma \frac{F\,d\sigma}{\sigma-\zeta} - \frac{\alpha_1\zeta}{\kappa}, \tag{2·27a}$$

$$\Psi_0(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{\overline{F}\,d\sigma}{\sigma-\zeta} + \frac{\alpha_1 - \Phi_0'(\zeta)}{\zeta}, \tag{2·27b}$$

where

$$\alpha_1 = \frac{1}{\kappa+1}\left[\frac{d}{d\bar{\zeta}}\left(\frac{1}{2\pi i}\int_\gamma \frac{F\,d\sigma}{\sigma-\zeta}\right)\right]_{\zeta=0}. \tag{2·27c}$$

We have now to evaluate the contour integrals occurring in (2·27). Applying formulae (70·3), (70·4′) of (10), p. 270, and paying attention to the logarithmic singularities inside γ we get

$$\left.\begin{aligned}
\frac{1}{2\pi i}\int_\gamma (\sigma-\zeta_s)\log(\sigma-\zeta_s)\frac{d\sigma}{\sigma-\zeta} &= 1-\zeta_s+(\zeta-\zeta_s)\log(\zeta-1),\\
\frac{1}{2\pi i}\int_\gamma (\sigma-\zeta_s)\log(1-\bar{\zeta}_s\sigma)\frac{d\sigma}{\sigma-\zeta} &= (\zeta-\zeta_s)\log(1-\bar{\zeta}_s\zeta),\\
\frac{1}{2\pi i}\int_\gamma (\sigma-\zeta_s)\log\sigma\frac{d\sigma}{\sigma-\zeta} &= 1+(\zeta-\zeta_s)\log(\zeta-1),\\
\frac{1}{2\pi i}\int_\gamma \frac{\sigma\,d\sigma}{(1-\bar{\zeta}_0\sigma)(\sigma-\zeta)} &= \frac{\zeta}{1-\bar{\zeta}_0\zeta};
\end{aligned}\right\} \tag{2·28}$$

$$\left.\begin{aligned}
\frac{1}{2\pi i}\int_\gamma (1-\bar{\zeta}_s\sigma)\log(1-\bar{\zeta}_s\sigma)\frac{d\sigma}{\sigma(\sigma-\zeta)} &= \frac{1-\bar{\zeta}_s\zeta}{\zeta}\log(1-\bar{\zeta}_s\zeta),\\
\frac{1}{2\pi i}\int_\gamma (1-\bar{\zeta}_s\sigma)\log\sigma\frac{d\sigma}{\sigma(\sigma-\zeta)} &= \frac{1}{2\pi i}\int_\gamma (1-\bar{\zeta}_s\sigma)\log(\sigma-\zeta_s)\frac{d\sigma}{\sigma(\sigma-\zeta)}\\
&= \frac{1}{\zeta}\log(1-\zeta)-\bar{\zeta}_s\log(\zeta-1),\\
\frac{1}{2\pi i}\int_\gamma \frac{d\sigma}{(\sigma-\zeta_0)(\sigma-\zeta)} &= 0.
\end{aligned}\right\} \tag{2·29}$$

From (2·24) and (2·28) we find, using the equilibrium conditions, that

$$\frac{1}{2\pi i}\int_\gamma \frac{F\,d\sigma}{\sigma-\zeta} = -ck\left[\sum_{s=0}^{m} P_s u_s \log v_s + \frac{P_0 q'^2\zeta}{v_0}\right], \tag{2·30}$$

where

$$u_s = \zeta-\zeta_s, \quad v_s = 1-\bar{\zeta}_s\zeta. \tag{2·31}$$

Similarly, we have from (2·25), (2·29) and (2·6)

$$\frac{1}{2\pi i}\int_\gamma \frac{\overline{F}\,d\sigma}{\sigma-\zeta} = -(ck\kappa/\zeta)\sum_{s=0}^{m} P_s v_s \log v_s. \tag{2·32}$$

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Substituting from (2·30) in (2·27*c*) we get

$$\alpha_1 = -\frac{ck}{\kappa+1}\left(\sum_{s=0}^{m} P_s\rho_s^2 + P_0 q'^2\right), \tag{2·33}$$

where $\rho_s = |\zeta_s| = r_s/c$. Substituting from (2·30), (2·32), (2·33) in (2·27*a*), (2·27*b*) and remembering that complex constants C_1 and C_2 may be added to Φ_0 and χ_0 without affecting the moments and shearing forces we get

$$\Phi_0(\zeta) = -\frac{ck}{\kappa}\left[\sum_{s=0}^{m} P_s u_s \log v_s + \frac{\zeta}{\kappa+1}\sum_{s=0}^{m} P_s(1-\rho_s^2) + P_0 q'^2\zeta\left(\frac{1}{v_0} - \frac{1}{\kappa+1}\right)\right] + C_1, \tag{2·34}$$

$$\Psi_0(\zeta) = \frac{ck}{\kappa}\left[\sum_{s=0}^{m} P_s\left\{\left(\frac{1-\kappa^2}{\zeta}v_s + \bar{\zeta}_s\right)\log v_s - \frac{1-\rho_s^2}{v_s}\bar{\zeta}_s\right\} + P_0 q'^2\bar{\zeta}_0\left(\frac{1}{v_0} + \frac{1}{v_0^2}\right)\right], \tag{2·35}$$

and hence

$$\chi_0(\zeta) = \frac{c^2k}{\kappa}\left[\sum_{s=0}^{m} P_s\bar{\zeta}_s u_s \log v_s + (1-\kappa^2)\, I_m(\zeta) + P_0 q'^2\left(\frac{1}{v_0} - \log v_0\right)\right] + C_2, \tag{2·36}$$

where

$$I_m(\zeta) = \sum_{s=0}^{m} P_s\int_0^1 (1-\bar{\zeta}_s\zeta t)\log(1-\bar{\zeta}_s\zeta t)\, dt/t. \tag{2·37}$$

Substituting from (2·34) and (2·36) in (2·22) we get

$$w_0 = -\frac{2c^2k}{\kappa}\left[\sum_{s=0}^{m} P_s U_s^2 \log V_s + P_0 q'^2 \log V_0 + (\kappa^2-1)\,\Re I_m(\zeta) - (1-\rho^2)\left\{\frac{1}{\kappa+1}\sum_{s=0}^{m} P_s(1-\rho_s^2) + P_0 q'^2\left(\Re\frac{1}{v_0} - \frac{1}{\kappa+1}\right)\right\}\right] + \gamma_1 x + \gamma_2 y + \gamma_3, \tag{2·38}$$

where

$$\rho = r/c, \quad U_s = |u_s| = R_s/c, \quad V_s = |v_s| = R_s'/r_s', \tag{2·39}$$

and γ_1, γ_2, γ_3 are real constants. Substituting from (2·38) in (2·21) we get

$$w_1 = \frac{2c^2k}{\kappa}\left[\sum_{s=1}^{m} P_s U_s^2(\kappa\log U_s - \log V_s) + P_0(U_0^2+q'^2)(\kappa\log q - \log V_0) + (1-\kappa^2)\,\Re I_m(\zeta) + \frac{\kappa P_0}{c^2}\left\{\left(1-\frac{1}{n}\right)(R_0^2-b^2) + \frac{4(R_0^{n'}-b^{n'})}{nn'^2b^n}\right\} + \frac{1-\rho^2}{\kappa+1}\left\{\sum_{s=0}^{m} P_s(1-\rho_s^2) + P_0 q'^2\left(\Re\frac{\kappa+1}{v_0} - 1\right)\right\}\right] + \gamma_1 x + \gamma_2 y + \gamma_3, \tag{2·40a}$$

$$w_2 = \frac{2c^2k}{\kappa}\left[\sum_{s=0}^{m} P_s U_s^2(\kappa\log U_s - \log V_s) + P_0 q'^2(\kappa\log U_0 - \log V_0) + (1-\kappa^2)\Re I_m(\zeta) + \frac{1-\rho^2}{\kappa+1}\left\{\sum_{s=0}^{m} P_s(1-\rho_s^2) + P_0 q'^2\left(\Re\frac{\kappa+1}{v_0} - 1\right)\right\}\right] + \gamma_1 x + \gamma_2 y + \gamma_3. \tag{2·40b}$$

When all supports, or some of them, lie at the boundary then the functions ϕ_1, ψ_1, χ_1; ϕ_2, ψ_2, χ_2 and deflexions w_1 and w_2 can be obtained from the previous expressions by letting all supports, or some of them, tend to the boundary. If, for example, the sth support lies at the boundary we put $\rho_s = 1$, $v_s = -u_s/\zeta_s$ and $V_s = U_s$. In the case of a circular plate supported at any number of points of its periphery and normally

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loaded over the area of an eccentric circle we easily get from (2·19), (2·20), (2·34) and (2·35), omitting terms which do not affect the deflexion, moments and shearing forces,

$$\phi_1 = ck\left[\frac{\kappa-1}{\kappa}\sum_{s=1}^{m} P_s u_s \log u_s + P_0 u_0\left(1-\frac{1}{n}+\log q-\frac{1}{\kappa}\log v_0\right)+\frac{P_0\zeta}{\kappa}\left(\frac{\rho_0^2+q'^2-1}{\kappa+1}-\frac{q'^2}{v_0}\right)\right]+C, \tag{2·41a}$$

$$\psi_1 = ck\left[\frac{1-\kappa^2}{\kappa\zeta}\sum_{s=0}^{m} P_s \log v_s+(\kappa-1)\sum_{s=1}^{m} P_s\bar{\zeta}_s \log u_s + P_0\bar{\zeta}_0\left\{\kappa\log v_0-\log q+\frac{1}{n}+\frac{\rho_0^2+q'^2-1}{\kappa v_0}+\frac{q'^2}{\kappa v_0^2}\right\}\right]; \tag{2·41b}$$

$$\phi_2 = ck\left[\frac{\kappa-1}{\kappa}\sum_{s=1}^{m} P_s u_s \log u_s + P_0 u_0\left(\log u_0-\frac{1}{\kappa}\log v_0\right)+\frac{P_0\zeta}{\kappa}\left(\frac{\rho_0^2+q'^2-1}{\kappa+1}-\frac{q'^2}{v_0}\right)\right]+C, \tag{2·42a}$$

$$\psi_2 = ck\left[\frac{1-\kappa^2}{\kappa\zeta}\sum_{s=0}^{m} P_s \log v_s+(\kappa-1)\sum_{s=1}^{m} P_s\bar{\zeta}_s \log u_s+\frac{P_0 q'^2}{u_0} + P_0\bar{\zeta}_0\left\{\kappa\log v_0-\log u_0+\frac{\rho_0^2+q'^2-1}{\kappa v_0}+\frac{q'^2}{\kappa v_0^2}\right\}\right]. \tag{2·42b}$$

Putting $P_0 = 0$ in (2·40a) or (2·40b) we get

$$w = \frac{2c^2k}{\kappa}\left[\sum_{s=1}^{m} P_s U_s^2(\kappa\log U-\log V_s)+(1-\kappa^2)\,\Re I_m(\zeta)+\frac{1-\rho^2}{\kappa+1}\sum_{s=1}^{m} P_s(1-\rho_s^2)\right]+\gamma_1 x+\gamma_2 y+\gamma_3, \tag{2·43}$$

which is the solution for a circular plate acted upon by the normal concentrated forces P_s at the points Q_s ($s = 1, 2, \ldots, m$). Substituting for κ from (1·9) in (2·43) we easily see that this solution agrees with Friedmann's result ((6), p. 259).

Putting $P_0 = 0$ in (2·41) or (2·42), so that the plate is under concentrated edge forces only, we find that the results obtained reduce to Yu's expressions ((15), equations (28), (29)) for $\phi(\zeta)$ and $\psi(\zeta)$, noting that the second term on the right side of equation (28) in Yu's paper yields a term of the type $iC\zeta$ (C being a real constant), which may be omitted without affecting either the deflexion or the moments and shearing forces.

The solution of Chankvetadze's problem (4) is now obtained by putting $z_0 = \zeta_0 = 0$, $b = c$, $n = 2$ and $q'^2 = \frac{1}{2}$ in (2·41).

Let M_r^1, M_θ^1, $M_{r\theta}^1$, Q_r^1, Q_θ^1 be the moments and shearing forces in region 1 and M_r^2, M_θ^2, $M_{r\theta}^2$, Q_r^2, Q_θ^2 be those in region 2. Denoting the contributions of $\phi_0(z)$ and $\psi_0(z)$ to the moments and shearing forces by M_r^0, M_θ^0, $M_{r\theta}^0$, Q_r^0, Q_θ^0 and substituting from (2·7), (2·19) and (2·20) in (1·6) we get

$$\left.\begin{aligned}
M_r^1+M_\theta^1 &= M_r^0+M_\theta^0-\frac{1+\nu}{2\pi}\left[\sum_{s=1}^{m} P_s\log U_s+P_0\left\{\log q+\frac{1}{n}\left(\frac{R_0^n}{b^n}-1\right)\right\}\right],\\
M_r^1-M_\theta^1 &= M_r^0-M_\theta^0-\frac{1-\nu}{8\pi\rho^2}\left[\sum_{s=1}^{m} P_s\left\{\left(\frac{\rho^2-\rho_s^2}{U_s}\right)^2-\rho_s^2\right\}+(\rho_0^2-U_0^2)P_0\right.\\
&\qquad\left.+\frac{2P_0}{n'}\left(\frac{R_0}{b}\right)^n\left\{\left(\frac{\rho^2-\rho_0^2+U_0^2}{U_0}\right)^2-2\rho^2\right\}\right],\\
M_{r\theta}^1 &= M_{r\theta}^0-\frac{1-\nu}{16\pi i\rho^2}\left[\sum_{s=1}^{m}\frac{\rho^2-\rho_s^2}{U_s^2}(\bar{\zeta}\zeta_s-\zeta\bar{\zeta}_s)+P_0(\bar{\zeta}\zeta_0-\zeta\bar{\zeta}_0)\left\{\frac{2}{n'}\left(1+\frac{\rho^2-\rho_0^2}{U_0^2}\right)\left(\frac{U_0}{q}\right)^n-1\right\}\right];
\end{aligned}\right\} \tag{2·44a}$$

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$$\left.\begin{aligned} Q_r^1 &= Q_r^0 - \frac{1}{4\pi r}\left[\sum_{s=1}^{m} P_s \frac{\rho^2-\rho_s^2}{U_s^2} + P_0\left\{\left(1+\frac{\rho^2-\rho_0^2}{U_0^2}\right)\left(\frac{U_0}{q}\right)^n - 1\right\}\right], \\ Q_\theta^1 &= Q_\theta^0 - \frac{i}{4\pi r}\left[\sum_{s=1}^{m} P_s \frac{\bar\zeta\zeta_s - \zeta\bar\zeta_s}{U_s^2} + P_0 \frac{\bar\zeta\zeta_0 - \zeta\bar\zeta_0}{U_0^2}\left(\frac{U_0}{q}\right)^n\right]; \end{aligned}\right\} \tag{2·44b}$$

$$\left.\begin{aligned} M_r^2 + M_\theta^2 &= M_r^0 + M_\theta^0 - \frac{1+\nu}{2\pi}\sum_{s=0}^{m} P_s \log U_s, \\ M_r^2 - M_\theta^2 &= M_r^0 - M_\theta^0 - \frac{1-\nu}{8\pi\rho^2}\left[\sum_{s=0}^{m} P_s\left\{\left(\frac{\rho^2-\rho_s^2}{U_s}\right)^2 - \rho_s^2\right\} - 2P_0 q'^2\, \Re\frac{\zeta^2}{u_0^2}\right], \\ M_{r\theta}^2 &= M_{r\theta}^0 - \frac{1-\nu}{16\pi i\rho^2}\left[\sum_{s=0}^{m} P_s \frac{\rho^2-\rho_s^2}{U_s^2}(\bar\zeta\zeta_s - \zeta\bar\zeta_s) - 2iP_0 q'^2\, \Im\frac{\zeta^2}{u_0^2}\right]; \end{aligned}\right\} \tag{2·45a}$$

$$\left.\begin{aligned} Q_r^2 &= Q_r^0 - \frac{1}{4\pi r}\sum_{s=0}^{m} P_s(\rho^2-\rho_s^2)/U_s^2, \\ Q_\theta^2 &= Q_\theta^0 - \frac{i}{4\pi r}\sum_{s=0}^{m} P_s(\bar\zeta\zeta_s - \zeta\bar\zeta_s)/U_s^2. \end{aligned}\right\} \tag{2·45b}$$

Substituting from (2·34), (2·35) in (1·6) with $W = 0$ we get

$$\left.\begin{aligned} M_r^0 + M_\theta^0 &= \frac{\nu+1}{2\pi\kappa}\left[\sum_{s=0}^{m} P_s\left\{\log V_s + \frac{\rho_s^2}{1+\nu} - \frac{(1-\rho_s^2)(1-\rho^2\rho_s^2)}{2V_s^2}\right\} + P_0 q'^2\left\{\Re\frac{1}{v_0^2} - \frac{1}{\kappa+1}\right\}\right], \\ M_r^0 - M_\theta^0 &= \frac{\nu-1}{8\pi\kappa\rho^2}\left[\sum_{s=0}^{m} P_s\left\{2(\kappa^2-1)\log V_s + \rho_s^2 + \frac{(1-\rho^2\rho_s^2)^2 - 2\rho_s^2(1-\rho^2)^2}{V_s^2}\right.\right. \\ &\qquad \left.\left. - \frac{(1-\rho^2)(1-\rho_s^2)(1-\rho^2\rho_s^2)^2}{V_s^4}\right\} + 2P_0 q'^2\left\{1 + (2\rho^2-3)\,\Re\frac{1}{v_0^2} + 2(1-\rho^2)\,\Re\frac{1}{v_0^3}\right\}\right], \\ M_{r\theta}^0 &= \frac{\nu-1}{16\pi i\kappa\rho^2}\left[\sum_{s=0}^{m} P_s\left\{\left(\frac{\rho_s^2-\rho^2}{V_s^2} + \frac{(1-\rho^2)(1-\rho_s^2)(1-\rho^2\rho_s^2)}{V_s^4}\right)(\bar\zeta\zeta_s - \zeta\bar\zeta_s) + (\kappa^2-1)\log\frac{v_s}{\bar v_s}\right\}\right. \\ &\qquad \left. + 2iP_0 q'^2\left\{(2\rho^2-3)\,\Im\frac{1}{v_0^2} + 2(1-\rho^2)\,\Im\frac{1}{v_0^3}\right\}\right]; \end{aligned}\right\} \tag{2·46a}$$

$$\left.\begin{aligned} Q_r^0 &= -\frac{1}{4\pi\kappa r}\left[\sum_{s=0}^{m}\frac{P_s}{V_s^2}\left\{(1-2\rho^2+\rho^2\rho_s^2)\rho_s^2 + \frac{(1-\rho_s^2)(1-\rho^2\rho_s^2)^2}{V_s^2}\right\} - 4P_0 q'^2\,\Re\frac{\zeta\bar\zeta_0}{v_0^3}\right], \\ Q_\theta^0 &= \frac{i}{4\pi\kappa r}\left[\sum_{s=0}^{m}\frac{P_s}{V_s^2}\left\{1 + \frac{(1-\rho_s^2)(1-\rho^2\rho_s^2)}{V_s^2}\right\}(\bar\zeta\zeta_s - \zeta\bar\zeta_s) + 4iP_0 q'^2\,\Im\frac{\zeta\bar\zeta}{v_0^3}\right]. \end{aligned}\right\} \tag{2·46b}$$

It can be easily verified from (2·45) and (2·46) that the boundary conditions $M_r^2 = 0$, $cQ_r^2 = \partial M_{r\theta}^2/\partial\theta$ are satisfied on C.

The solution of the problem of an infinitely large thin elastic plate normally loaded over the area of a circle and supported at any number of points can be easily deduced from the previous results by letting c tend to infinity. From (2·34), (2·36) we see that in this case $\phi_0(z) = C_1$, $\chi_0(z) = C_2$, $w_0 = \gamma_1 x + \gamma_2 y + \gamma_3$ and then (2·19), (2·20) and (2·21) give ϕ_1, χ_1, w_1; ϕ_2, χ, w_2 at any point of the plate while the moments and shearing forces are given by (2·44) and (2·45).

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3. *Symmetrical bending of a circular plate supported at points regularly distributed along the circumference of a concentric circle.* Let the circular plate, with centre O and radius c, be symmetrically loaded over the area of a concentric circle of radius b ($p_1 = p_0 r^{n-2}, p_2 = 0, n \geqslant 2$), and supported at the m points $z_s = ae^{is\alpha}$ ($s = 1, 2, \ldots, m$; $\alpha = 2\pi/m$). In this case we have

$$P_1 = P_2 = \ldots = P_m = -P_0/m. \tag{3·1}$$

Substituting from (3·1) in (2·21), (2·38) and (2·37) we get

$$w_1 = w_0 + 2kP_0\left[(r^2+b'^2)\log q - \frac{1}{m}\sum_{s=1}^{m} R_s^2 \log(R_s/c) + \left(1-\frac{1}{n}\right)(r^2-b^2) + \frac{4(r^{n'}-b^{n'})}{nn'^2 b^n}\right], \tag{3·2a}$$

$$w_2 = w_0 + 2kP_0\left[(r^2+b'^2)\log\rho - \frac{1}{m}\sum_{s=1}^{m} R_s^2 \log(R_s/c)\right], \quad \rho = r/c, \quad q = b/c, \tag{3·2b}$$

where

$$w_0 = \frac{2kP_0}{m\kappa}\left[\sum_{s=1}^{m} R_s^2 \log\frac{R_s'}{r_s'} + (\kappa^2-1)c^2\,\Re L(\zeta) + \frac{m(1-\rho^2)(a^2+\kappa b'^2)}{(\kappa+1)}\right] + \gamma_1 x + \gamma_2 y + \gamma_3, \tag{3·2c}$$

$$L(\zeta) = \sum_{s=1}^{m}\int_0^1 (1-\bar{\zeta}_s\zeta u)\log(1-\bar{\zeta}_s\zeta u)\,du/u. \tag{3·3}$$

We now consider separately two cases:

Case I. $a \geqslant b$. In this case the supports lie in the unloaded area and the conditions $w_2 = 0$ at $z = z_s$ ($s = 1, 2, \ldots, m$) give

$$\gamma_1 = \gamma_2 = 0 \tag{3·4a}$$

and

$$\gamma_3 = \frac{2kP_0}{\kappa}\left[a^2\{A_m(t) + (1-\kappa^2)B_m(t)\} + \kappa(a^2-b'^2)\log t - \frac{(1-t^2)(a^2+\kappa b'^2)}{\kappa+1}\right], \tag{3·4b}$$

where $t = a/c$,

$$mA_m(t) = \sum_{s=1}^{m}(1-\cos s\alpha)[\kappa\log\{2(1-\cos\alpha)\} - \log(1-2t^2\cos\alpha+t^4)], \tag{3·5a}$$

$$mt^2 B_m = L(t) = \sum_{s=1}^{m}(1-t^2e^{-is\alpha})\log(1-t^2e^{-is\alpha}) - \frac{1}{m}\phi(t^{2m})$$

$$= \log(1-t^{2m}) - t^2\sum_{s=1}^{m} e^{-is\alpha}\log(1-t^2e^{-is\alpha}) - \frac{1}{m}\phi(t^{2m}), \tag{3·5b}$$

and

$$\phi(x) = -\int_0^x \frac{\log(1-u)}{u}\,du = \sum_{n=1}^{\infty}\frac{x^n}{n^2}, \quad \phi(1) = \tfrac{1}{6}\pi^2. \tag{3·5c}$$

The deflexion at the centre of the plate is now given by putting $z = 0$ in (3·2a). Remembering that $k = 1/16\pi D$ and substituting from (3·4) we get

$$w_c = \frac{P_0 a^2}{8\pi\kappa D}\left[A_m(t) + (1-\kappa^2)B_m(t) + \kappa C + \frac{t^2+\kappa q'^2}{\kappa+1}\right], \tag{3·6a}$$

where

$$C = \frac{b'^2}{a^2}\left[\log\frac{b}{a} - \frac{(n+3)}{(n+2)}\right]. \tag{3·6b}$$

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For a concentrated load at the centre $C = 0$, $q' = 0$ and (3·6a) gives

$$w_c = \frac{P_0 a^2}{8\pi\kappa D}\left[A_m(t) + (1-\kappa^2)\,B_m(t) + \frac{t^2}{\kappa+1}\right]. \tag{3·7}$$

For a circular plate symmetrically loaded over a concentric circle and supported at points regularly distributed along the boundary we let t tend to 1. In this case we have

$$mA_m(1) = 4(\kappa-1)\sum_{s=1}^{m-1}\sin^2\tfrac{1}{2}s\alpha\log(2\sin\tfrac{1}{2}s\alpha), \tag{3·8a}$$

$$mB_m(1) = 2\sum_{s=1}^{m-1}\sin^2\tfrac{1}{2}s\alpha\log(2\sin\tfrac{1}{2}s\alpha) - \frac{1}{2}\sum_{s=1}^{m-1}(\pi - s\alpha)\sin s\alpha - \frac{\pi^2}{6m}; \tag{3·8b}$$

and (3·6) gives

$$w_c = \frac{P_0 c^2}{8\pi\kappa D}\left[\frac{\kappa^2-1}{2m}\left\{\sum_{s=1}^{m-1}(\pi - s\alpha)\sin s\alpha + \frac{\pi^2}{3m}\right\} - \frac{2(\kappa-1)^2}{m}\sum_{s=1}^{m-1}\sin^2\tfrac{1}{2}s\alpha\log(2\sin\tfrac{1}{2}s\alpha) + \kappa q'^2\left(\log q - \frac{n+3}{n'}\right) + \frac{1+\kappa q'^2}{\kappa+1}\right], \tag{3·9}$$

which reduces to

$$w_c = \frac{P_0 c^2}{8\pi\kappa D}\left[\frac{\kappa^2-1}{2m}\left\{\sum_{1}^{m-1}(\pi - s\alpha)\sin s\alpha + \frac{\pi^2}{3m}\right\} - \frac{2(\kappa-1)^2}{m}\sum_{1}^{m-1}\sin^2\frac{s\alpha}{2}\log\left(2\sin\frac{s\alpha}{2}\right) + \frac{1}{\kappa+1}\right] \tag{3·10}$$

for a concentrated load at the centre. When $m = 3$ we obtain

$$w_c = \frac{P_0 c^2}{16\pi D}\left[\frac{8}{(1-\nu)(3+\nu)}\left\{2\log 3 - \frac{\pi}{27}(1+\nu)(\pi + 3\sqrt{3})\right\} + \frac{2}{\kappa(\kappa+1)}\right]. \tag{3·11}$$

Putting $\nu = \frac{1}{4}, \kappa = -\frac{13}{3}$ in (3·11) we get

$$w_c = 0{\cdot}067P_0c^2/D = 0{\cdot}754P_0c^2/Eh^3, \tag{3·12}$$

which agrees with Nadai's result quoted by Timoshenko (13), p. 271.

Case II. $a \leqslant b$. In this case the supports lie in the loaded region and the conditions $w_1 = 0$ at $z = z_s$ give $\gamma_1 = \gamma_2 = 0$ and

$$\gamma_3 = \frac{2kP_0}{\kappa}\left[a^2\{A_m(t) + (1-\kappa^2)\,B_m(t)\} + 2a^2\kappa\log t - \kappa(a^2+b'^2)\log q + \kappa\left(1-\frac{1}{n}\right)(b^2-a^2) + \frac{4\kappa(b^{n'}-a^{n'})}{nn'^2b^n} - \frac{(1-t^2)(a^2+\kappa b'^2)}{\kappa+1}\right]. \tag{3·13}$$

The deflexion at the centre is now given by

$$w_c = \frac{P_0 a^2}{8\pi\kappa D}\left[A_m(t) + (1-\kappa^2)\,B_m(t) + \kappa C' + \frac{t^2+\kappa q'^2}{\kappa+1}\right], \tag{3·14}$$

where

$$C' = \frac{1}{n} - 1 - \frac{4}{nn'^2}\left(\frac{a}{b}\right)^n + \log\frac{a}{b}. \tag{3·15}$$

When the circular plate is completely and symmetrically loaded we put $b = c$ and we get

$$w_c = \frac{P_0 a^2}{8\pi\kappa D}\left[A_m(t) + (1-\kappa^2)\,B_m(t) + \kappa\left(\frac{1}{n} - 1 - \frac{4t^n}{nn'^2} + \log t\right) + \frac{t^2+\kappa\delta_n}{1+\kappa}\right]. \tag{3·16}$$

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Putting $t = 1$ in (3·16) we obtain the following expression for the deflexion at the centre of a completely loaded plate supported at points regularly distributed around its periphery:

$$w_c = \frac{P_0 c^2}{8\pi\kappa D}\left[\frac{\kappa^2-1}{2m}\left\{\sum_1^{m-1}(\pi - s\alpha)\sin s\alpha + \frac{\pi^2}{3m}\right\}\right.$$
$$\left. - \frac{2(\kappa-1)^2}{m}\sum_1^{m-1}\sin^2\frac{s\alpha}{2}\log\left(2\sin\frac{s\alpha}{2}\right) - \frac{\kappa n(n+3)}{n'^2} + \frac{\kappa\delta_n + 1}{\kappa+1}\right], \quad (3\cdot17)$$

which agrees with (3·9) on putting $q = 1$. For $m = 3$ and a uniform load over the whole plate ($n = 2$, $n' = 4$, $\delta_2 = \frac{1}{2}$) equation (3·17) gives

$$w_c = \frac{P_0 c^2}{16\pi D}\left[\frac{8}{(1-\nu)(3+\nu)}\left\{2\log 3 - \frac{\pi}{27}(1+\nu)(\pi + 3\sqrt{3})\right\} + \frac{\kappa+2}{\kappa(\kappa+1)} - \frac{5}{4}\right], \quad (3\cdot18)$$

which does not agree with Chankvetadze's formulae ((4), eq. (32)), the correction of which was made in a previous paper ((3), eq. 65), but agrees with eq. (64) of (3), obtained by a totally different method.

4. *Bending of a half-plane normally loaded over a circle and supported at any number of points.* Consider now how the various functions ϕ_1, χ_1; ϕ_2, χ_2 given by (2·20), (2·19) change under change of origin from the centre of the plate O to the point O' ($z = -c$) (see Fig. 1). Assuming that z' denotes the complex variable of any point P of the plate referred to O' as origin we have $z = z' - c$, $z_s = z'_s - c$ so that* $Z_s = z - z_s$ remains unchanged and the new functions become

$$\phi_1 = k\left[\sum_{s=1}^{m} P_s Z_s \log Z_s + P_0 Z_0\left(1 - \frac{1}{n} + \log b\right)\right] + \phi_0(z'-c), \quad (4\cdot1a)$$

$$\chi_1 = -k\left[\sum_{s=1}^{m} P_s \bar{z}'_s Z_s \log Z_s + P_0\left\{\bar{z}'_0 Z_0\left(1 - \frac{1}{n} + \log b\right) + b'^2\left(1 + \frac{1}{n'} - \log b\right)\right\}\right]$$
$$+ \chi_0(z'-c) - c\phi_0(z'-c); \quad (4\cdot1b)$$

$$\phi_2 = k\sum_{s=0}^{m} P_s Z_s \log Z_s + \phi_0(z'-c), \quad (4\cdot2a)$$

$$\chi_2 = -k\left[\sum_{s=0}^{m} P_s \bar{z}'_s Z_s \log Z_s - P_0 b'^2 \log Z_0\right] + \chi_0(z'-c) - c\phi_0(z'-c), \quad (4\cdot2b)$$

where in $\phi_0(z'-c)$, $\chi_0(z'-c)$ we put $z'_s - c$ instead of z_s and $\bar{z}'_s - c$ instead of $\bar{z}_s$ whenever they appear. From (2·34), (2·36) we find, on proceeding to the limit as c tends to infinity, that

$$\lim_{c\to\infty}\phi_0(z'-c) = -\frac{k}{\kappa}\left[\sum_{s=0}^{m} P_s Z_s \log(z' + \bar{z}'_s) - \frac{P_0 b'^2}{z' + \bar{z}'_0}\right], \quad (4\cdot3a)$$

$$\lim_{c\to\infty}\chi_0(z'-c) = \frac{k}{\kappa}\left[\sum_{s=0}^{m} P_s \bar{z}'_s Z_s \log(z' + \bar{z}'_s) + P_0 b'^2\left\{\frac{z'}{z' + \bar{z}'_0} - \log(z' + \bar{z}'_0)\right\}\right.$$
$$\left. + (1-\kappa^2)\lim_{c\to\infty} c^2 I_m\left(\frac{z'-c}{c}\right)\right]. \quad (4\cdot3b)$$

* The symbol z'_s was used earlier for the inverse of z_s in C, but no confusion should arise here.

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Let us now evaluate the last term in (4·3*b*). From (2·37) we obtain

$$\lim_{c\to\infty} c^2 I_m\left(\frac{z'-c}{c}\right) = \lim_{c\to\infty} c^2 \sum_{s=0}^{m} P_s \int_0^{z'-c} \left(1-\frac{\bar{z}_s u}{c^2}\right) \log\left(1-\frac{\bar{z}_s u}{c^2}\right)\frac{du}{u}$$

$$= \lim_{c\to\infty} c \sum_{s=0}^{m} P_s \int_0^{z'} \left(v+\bar{z}'_s-\frac{v\bar{z}'_s}{c}\right) \log\left(v+\bar{z}'_s-\frac{v\bar{z}'_s}{c}\right)\frac{dv}{v-c}$$

$$= -\sum_{s=0}^{m} P_s \int_0^{z'+\bar{z}'_s} t\log t\,dt = -\frac{1}{2}\sum_{s=0}^{m} P_s (z'+\bar{z}'_s)^2 \log(z'+\bar{z}'_s) + \text{constant}, \quad (4\cdot4)$$

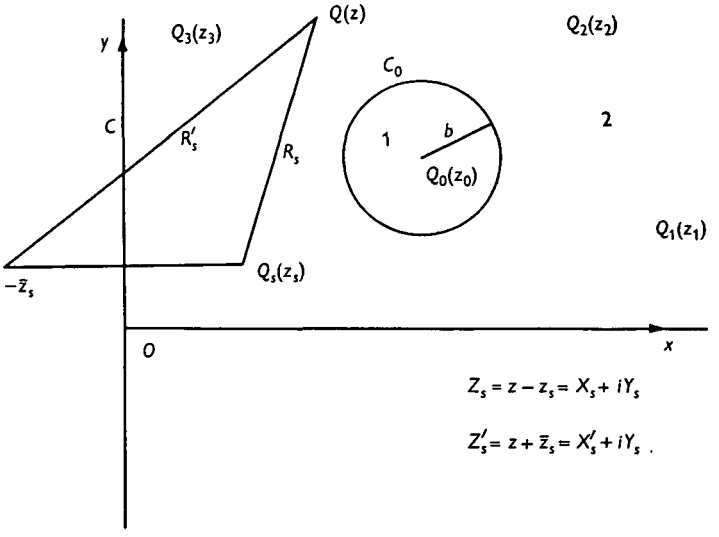


Fig. 2

on using the equilibrium conditions (2·6). Substituting from (4·3), (4·4) in (4·1), (4·2) and using the notation of Fig. 2 we get the following results for the half-plane, with a free edge, loaded normally over a circle and supported at any number of points:

$$\left.\begin{aligned}
\phi_1(z) &= k\left[\sum_1^m P_s Z_s\left(\log Z_s - \frac{1}{\kappa}\log Z'_s\right) + P_0\left\{Z_0\left(1-\frac{1}{n}+\log b-\frac{1}{\kappa}\log Z'_0\right)+\frac{b'^2}{\kappa Z'_0}\right\}\right]+C_1,\\
\chi_1(z) &= -k\left[\sum_1^m P_s\bar{z}_s Z_s\left(\log Z_s-\frac{1}{\kappa}\log Z'_s\right)+\frac{1-\kappa^2}{2\kappa}\sum_0^m P_s Z'^2_s\log Z'_s\right.\\
&\quad \left.+P_0\left\{(\bar{z}_0 Z_0-b'^2)\left(\log b-\frac{1}{\kappa}\log Z'_0\right)+\left(1-\frac{1}{n}\right)\bar{z}_0 Z_0+\left(1+\frac{1}{n'}-\frac{z}{\kappa Z'_0}\right)b'^2\right\}\right]+C_2,\\
w_1 &= 2k\left[\sum_1^m P_s R_s^2\left(\log R_s-\frac{1}{\kappa}\log R'_s\right)+\frac{\kappa^2-1}{2\kappa}\Re\sum_0^m P_s Z'^2_s\log Z'_s\right.\\
&\quad \left.+P_0\left\{\frac{2b'^2 x}{\kappa}\Re\frac{1}{Z'_0}+(R_0^2+b'^2)\left(\log b-\frac{1}{\kappa}\log R'_0\right)+\left(1-\frac{1}{n}\right)(R_0^2-b^2)+\frac{4(R_0^{n'}-b^{n'})}{nn'^2 b^n}\right\}\right]\\
&\quad +\gamma_1 x+\gamma_2 y+\gamma_3;
\end{aligned}\right\} \quad (4\cdot5)$$

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$$\left.\begin{aligned}
\phi_2(z) &= k\left[\sum_0^m P_s Z_s\left(\log Z_s - \frac{1}{\kappa}\log Z_s'\right) + \frac{P_0 b'^2}{\kappa Z_0'}\right] + C_1,\\
\chi_2(z) &= -k\left[\sum_0^m P_s \bar{z}_s Z_s\left(\log Z_s - \frac{1}{\kappa}\log Z_s'\right) + \frac{1-\kappa^2}{2\kappa}\sum_0^m P_s Z_s'^2 \log Z_s'\right.\\
&\qquad \left. - P_0 b'^2\left(\log Z_0 - \frac{1}{\kappa}\log Z_0' + \frac{z}{\kappa Z_0'}\right)\right] + C_2,\\
w_2 &= 2k\left[\sum_0^m P_s R_s^2\left(\log R_s - \frac{1}{\kappa}\log R_s'\right) + \frac{\kappa^2-1}{2\kappa}\Re\sum_0^m P_s Z_s'^2 \log Z_s'\right.\\
&\qquad \left. + P_0 b'^2\left(\log R_0 - \frac{1}{\kappa}\log R_0'\right) + \frac{2}{\kappa}P_0 b'^2 x \Re\frac{1}{Z_0'}\right] + \gamma_1 x + \gamma_2 y + \gamma_3.
\end{aligned}\right\} \tag{4·6}$$

Putting $P_0 = 0$ in (4·5) or (4·6) we get

$$\left.\begin{aligned}
\phi(z) &= k\sum_1^m P_s Z_s\left(\log Z_s - \frac{1}{\kappa}\log Z_s'\right) + C_1,\\
\chi(z) &= -k\left[\sum_1^m P_s \bar{z}_s Z_s\left(\log Z_s - \frac{1}{\kappa}\log Z_s'\right) + \frac{1-\kappa^2}{2\kappa}\sum_1^m P_s Z_s'^2 \log Z_s'\right] + C_2,\\
w &= 2k\left[\sum_1^m P_s R_s^2\left(\log R_s - \frac{1}{\kappa}\log R_s'\right) + \frac{\kappa^2-1}{2\kappa}\Re\sum_1^m P_s Z_s'^2 \log Z_s'\right] + \gamma_1 x + \gamma_2 y + \gamma_3.
\end{aligned}\right\} \tag{4·7}$$

The last equation of (4·7) gives the Green's function of the half-plane with the edge free when acted upon by any system of normal isolated forces in equilibrium. Problems of the elastic half-plane loaded in its plane have been extensively investigated by Muskhelishvili ((10), Chapter 19).

Substituting from (4·6) in (1·3) and taking $W = 0$, we get the following formulae for the moments and shearing forces in region 2:

$$M_x^2 = \frac{1-\nu}{4\pi}\left[\sum_0^m P_s\left\{\frac{1}{\beta}\log\frac{R_s'}{R_s} + \left(\frac{x}{\kappa R_s'^2} - \frac{X_s}{R_s^2}\right)X_s + \frac{x_s X_s}{R_s'^2}\left(1 + \frac{4xX_s}{\kappa R_s'^2}\right)\right\}\right.$$
$$\left. + \tfrac{1}{2}P_0 b'^2 \Re\left(\frac{1}{Z_0^2} - \frac{1}{Z_0'^2} - \frac{4x}{\kappa Z_0'^3}\right)\right], \quad \beta = \frac{1-\nu}{1+\nu}, \tag{4·8a}$$

$$M_y^2 = \frac{1-\nu}{4\pi}\left[\sum_0^m P_s\left\{\frac{1}{\beta}\left(\frac{2}{\kappa}\log R_s' - \log R_s R_s'\right) + \left(\frac{X_s}{R_s^2} + \frac{\overline{\kappa+2}\,x_s - x}{\kappa R_s'^2}\right)X_s\right.\right.$$
$$\left.\left. + \frac{2(\kappa+2)x_s^2}{\kappa R_s'^2} - \frac{4x x_s X_s'^2}{\kappa R_s'^4}\right\} + \tfrac{1}{2}P_0 b'^2 \Re\left(\frac{2\bar{Z}_0}{\kappa Z_0'^3} - \frac{1}{Z_0^2} - \frac{1}{Z_0'^2}\right)\right], \tag{4·8b}$$

$$M_{xy}^2 = \frac{1-\nu}{4\pi}\left[\sum_{s=0}^m P_s Y_s\left\{\left(\frac{1}{R_s^2} - \frac{1}{\kappa R_s'^2}\right)X_s - \frac{4x x_s X_s'}{\kappa R_s'^4}\right\} + \frac{1-\kappa^2}{4\kappa i}\sum_{s=0}^m P_s \log\frac{Z_s}{\bar{Z}_s'}\right.$$
$$\left. + \frac{P_0 b'^2}{2\kappa}\Im\left(\frac{\kappa}{Z_0^2} - \frac{1}{Z_0'^2} - \frac{2\bar{Z}_0}{Z_0'^3}\right)\right]; \tag{4·8c}$$

$$Q_x^2 = -\frac{1}{2\pi}\left[\sum_{s=0}^m P_s\left\{\left(\frac{1}{R_s^2} - \frac{1}{\kappa R_s'^2}\right)X_s - \frac{4x_s X_s'^2}{\kappa R_s'^4}\right\} + \frac{2P_0 b'^2}{\kappa}\Re\frac{1}{Z_0'^3}\right], \tag{4·9a}$$

$$Q_y^2 = -\frac{1}{2\pi}\left[\sum_{s=0}^m P_s\left\{\left(\frac{1}{R_s^2} - \frac{1}{\kappa R_s'^2}\right)Y_s - \frac{4x_s X_s'}{\kappa R_s'^4}\right\} - \frac{2P_0 b'^2}{\kappa}\Im\frac{1}{Z_0'^3}\right], \tag{4·9b}$$

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where
$$Z_s = z - z_s = X_s + iY_s, \quad |Z_s| = R_s, \tag{4·10a}$$
$$Z'_s = z + \bar{z}_s = X'_s + iY_s, \quad |Z'_s| = R'_s. \tag{4·10b}$$

It can be easily verified that the boundary conditions

$$M_x^2 = 0, \quad Q_x^2 - \partial M_{xy}^2/\partial y = 0$$

are satisfied on the edge $x = 0$ and at infinity. Expressions for the moments and shearing forces at any point of region 1 can be similarly obtained by substituting from (4·6), (2·7) in (1·3).

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