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On matrix differential equations with several unbounded delays

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The paper focuses on the matrix differential equation

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\tau_j(t)) + f(t), \quad t \in I = [t_0, \infty)$$

with continuous matrices A, B_j, a continuous vector f and continuous delays τ_j satisfying $\tau_k \circ \tau_l = \tau_l \circ \tau_k$ on I for any pair τ_k, τ_l. Assuming that the equation

$$\dot{y}(t) = A(t)y(t)$$

is uniformly exponentially stable, we present some asymptotic bounds of solutions y of the considered delay equation. A system of simultaneous Schröder equations is used to formulate these asymptotic bounds.

1 Introduction and preliminaries

This paper discusses the matrix differential equation

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\tau_j(t)) + f(t), \quad t \in I = [t_0, \infty), \tag{1.1}$$

where A, B_j $(j = 1, \ldots, m)$ are $n \times n$ real continuous matrix functions, f is a real continuous vector and τ_j $(j = 1, \ldots, m)$ satisfy either the hypothesis

$$\begin{aligned}&\tau_j \in C(I) \text{ and there exists } \tau \in C^2(I) \text{ such that } \tau(t_0) \leqslant t_0,\ \tau_j(t) \leqslant \tau(t) < t\\ &\text{for all } t > t_0,\ \lim_{t\to\infty}\tau(t) = \infty,\ 0 < \dot{\tau} \leqslant \lambda < 1 \text{ on } I,\ \dot{\tau} \text{ is nonincreasing on } I\end{aligned} \tag{H_1}$$

or the hypothesis

$$\begin{aligned}&\tau_j \in C^1(I),\ \tau_j(t_0) = t_0,\ \tau_j(t) < t \text{ for all } t > t_0,\ \lim_{t\to\infty}\tau_j(t) = \infty,\\ &0 < \dot{\tau}_j \leqslant \lambda_j < 1 \text{ on } I,\ \dot{\tau}_j \text{ are nonincreasing on } I \text{ and}\\ &\tau_k \circ \tau_l = \tau_l \circ \tau_k \text{ on } I \text{ for any pair } k, l = 1, \ldots, m.\end{aligned} \tag{H_2}$$

Let $\sigma \in \mathbb{R}$, $\sigma \geqslant t_0$ be arbitrary. By a solution of (1.1) we understand a real valued function y defined in the interval $[\sigma_0, \infty)$, where $\sigma_0 = \inf\{\tau_j(t),\ t \geqslant \sigma,\ j = 1, \ldots, m\}$ and satisfying (1.1) on $[\sigma, \infty)$. It is easy to verify that for any such σ there exists a solution of

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(1.1) defined on $[\sigma_0, \infty)$. Further, we define the norm of a vector $y = (y_1, \dots, y_n)$ as $|y| = \max\{|y_i|, i = 1, \dots, n\}$ and the norm of a matrix A as $\|A\| = \sup\{|Ay|, y \in \mathbb{R}^n, |y| = 1\}$.

Let us emphasize that the inequalities $\dot{\tau}_j \leqslant \lambda_j < 1$ on I imply that lags $t - \tau_j(t)$ have to be unbounded as $t \to \infty$. Various special cases of equation (1.1) with unbounded lags have been already studied because of numerous interesting applications as well as the specific qualitative properties. The special attention was paid to functional differential equations with rescaling, particularly to the equation

$$\dot{y}(t) = Ay(t) + By(\lambda t), \qquad 0 < \lambda < 1, \qquad t \geqslant 0, \tag{1.2}$$

where A, B are constant real or complex matrices. Equation (1.2) is usually called the pantograph equation because of its relevance to the study of the motion of the pantograph head of an electric locomotive (see Ockendon & Tayler [24]). This equation has other direct applications, e.g, in electrodynamics (see Fox *et al.* [11]) and it was also considered as the equation of the advanced type ($\lambda > 1$). In particular, Ambartsumian [1] showed that scalar advanced equation (1.2) describes the absorption of light in the Milky Way and Hall & Wake [12] presented and studied this equation as a model of cell growth. Theoretical studies of the pantograph equation can be found in papers by Kato & McLeod [17] (the scalar case), Carr & Dyson [3] and Pandolfi [25] (the vector case), Lim [20] (the nonhomogeneous scalar case) and others.

The qualitative investigation of (1.2) has been extended into several directions. Iserles [14] studied the qualitative properties of the matrix neutral equation

$$\dot{y}(t) = Ay(t) + By(\lambda t) + C\dot{y}(\lambda t), \qquad 0 < \lambda < 1, \qquad t \geqslant 0, \tag{1.3}$$

where A, B, C are complex constant matrices. Equation (1.3) with more than one proportional delay has been treated in Iserles & Liu [16] and Liu [21], whereas equation (1.3) involving a forcing term has been discussed in Lehninger & Liu [19]. The proof techniques successfully employed in these papers (mostly based on the expansion of solutions into Dirichlet series form or utilizing hypergeometric functions) have enabled to extend many results known from the "pure" delayed case to this neutral case. This approach turned out to be effective especially in the study of autonomous linear differential equations involving proportionally delayed argument (for other related results concerning differential equations with proportional delays see, for example, Derfel & Iserles [8] or Iserles [15]).

However, considering these equations with variable coefficients and/or with a general form of a delayed argument it is often very difficult to apply these techniques. The asymptotics of solutions of the higher order linear differential equation with linearly transformed arguments and polynomial coefficients

$$y^{(m)}(t) = \sum_{j=0}^{l} \sum_{k=0}^{m-1} \sum_{i=0}^{r} a_{jki} t^i y^{(k)}(\lambda_j t + \mu_j)$$

was investigated in Derfel [7] and Derfel & Vogl [9] in the connection with a problem arising in control theory. Closely related to the subject of this paper is the equation

$$\dot{y}(t) = ay(t) + by(\tau(t)), \qquad t \in I \tag{1.4}$$

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studied in Heard [13] under the assumptions $a < 0$, $b \neq 0$ are real scalars and $\tau \in C^2(I)$ is fulfilling $\tau(t_0) = t_0$, $\tau(t) < t$ for all $t > t_0$, $0 < \dot{\tau} < 1$ on I and $\dot{\tau}$ is decreasing on I. Asymptotic results derived in [13] have been generalized in several papers: equation (1.4) has been discussed in the case of special variable coefficients (Makay & Terjéki [22]), in the nonhomogeneous case [6] and the vector case [5] (for further related asymptotic results on differential equations with variable unbounded lags in the vector or neutral case see, for example, Bereketoglu & Pituk [2] or Feldstein & Liu [10]). It should be noted that some of these papers present asymptotic expansions of solutions of (1.4) (and its modifications) in terms of solutions of some auxiliary functional equations.

In this paper, we utilize proof methods based on some results of the transformation theory of delay differential equations and some results of the theory of functional equations to describe the asymptotic properties of all solutions of (1.1). In particular, we derive asymptotic bounds of solutions of

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\lambda_j t) + f(t), \quad 0 < \lambda_j < 1, \quad t \in [0, \infty)$$

and

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(t^{\gamma_j}) + f(t), \quad 0 < \gamma_j < 1, \quad t \in [1, \infty).$$

More generally, we investigate the asymptotics of solutions for the equation (1.1) involving also non-commutable delays of the above type.

Some preliminary results on functional equations will be the subject matter of Section 2. The main results are formulated in Section 3, where we derive the asymptotic bounds of all solutions of (1.1) in terms of a solution of some scalar functional equations and inequalities. In § 4 and § 5 we apply these results to some particular cases of (1.1) and show that we are able to generalize and extend several earlier asymptotic results.

2 Some functional equations and inequalities

In this chapter, we mention several basic results concerning some linear functional equations and inequalities in a single variable. These equations and inequalities are of great importance in our further investigations.

The important equation of linearization utilized in the qualitative theory of delay differential equations is that of Schröder. Its form is

$$\varphi(\tau(t)) = \lambda\varphi(t), \qquad \lambda = \dot{\tau}(t_0) < 1, \qquad t \in I = [t_0, \infty), \tag{2.1}$$

where τ is a known function and φ is unknown. The solution of this equation enables us to convert a delay differential equation involving the dependent variable y, the independent variable t and the delayed argument τ into a differential-difference equation. This transformation is realized via the changes of variables

$$s = \log \varphi(t), \qquad z(s) = y(t)$$

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converting $y(\tau(t))$ into $z(s-\log\lambda^{-1})$. This and other related problems have been treated, e.g., in Neuman [23] or Kuczma *et al.* [18]. We mention here the following existence result [18].

Theorem 2.1 *Consider the equation* (2.1), *where* $\tau \in C^r(I)$ $(r \geqslant 1)$, $\tau(t_0) \leqslant t_0$, $\tau(t) < t$ *for all* $t > t_0$, $\lim_{t\to\infty}\tau(t) = \infty$ *and* $\dot\tau$ *is positive and nonincreasing on* I. *Then there exists a solution* $\varphi \in C^r(I)$ *of* (2.1), *which is positive and unbounded on* (t_0,∞) *and has a positive derivative on* I.

Considering this transformation in a more general case, when the differential equation involves several delayed arguments $\tau_1,\dots,\tau_m$, we arrive at the system of simultaneous Schröder equations

$$\varphi(\tau_j(t)) = \lambda_j\varphi(t), \qquad \lambda_j = \dot\tau_j(t_0) < 1, \qquad j = 1,\dots,m, \quad t \in I \tag{2.2}$$

extending (2.1). If we are able to find a suitable solution φ of (2.2), then the above mentioned substitution converts a differential equation with several variable lags into a differential equation with several constant lags.

The following theorem is due to Čermák [4].

Theorem 2.2 *Consider the system of simultaneous Schröder equations* (2.2), *where* τ_j $(j = 1,\dots,m)$ *are satisfying* (H_2). *Then there exists a common solution* $\varphi \in C^1(I)$ *of* (2.2), *which is positive and unbounded on* (t_0,∞) *and has a positive derivative on* I.

In the sequel we consider the linear functional inequality

$$-a\omega(t) + \sum_{j=1}^{m} b_j(t)\omega(\tau_j(t)) \leqslant 0, \qquad t \in (t_0,\infty), \tag{2.3}$$

where $a > 0$ is a real constant, $b_j \in C((t_0,\infty))$ and τ_j $(j = 1,\dots,m)$ are satisfying (H_1). If we choose functions $b_j^* \in C^2((t_0,\infty))$ such that $b_j^*(t) > 0$ and $b_j^*(t) \geqslant b_j(t)$ for all $t \in (t_0,\infty)$ and assume that $\tau_j \in C^2((t_0,\infty))$, then we can show by the step method the existence of a function ω satisfying

$$-a\omega(t) + \sum_{j=1}^{m} b_j^*(t)\omega(\tau_j(t)) = 0, \qquad t \in (t_0,\infty)$$

(hence also satisfying (2.3)), which is positive and twice continuously differentiable on (t_0,∞).

To obtain a more applicable form of a solution ω of (2.3) we consider the case when coefficients b_j are bounded on (t_0,∞) and τ_j satisfy (H_2). The verification of the following assertion is easy.

Proposition 2.3 *Consider the inequality* (2.3), *where* $a > 0$ *is a real constant,* b_j *are continuous functions satisfying* $|b_j(t)| \leqslant b_j^*$ *for every* $t \in (t_0,\infty)$ *and suitable positive reals* b_j^* *and* τ_j $(j = 1,\dots,m)$ *are satisfying* (H_2). *If* φ *is a solution of* (2.2) *which is positive and*

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continuously differentiable on (t_0, ∞), *then the function* $\omega(t) = (\varphi(t))^{\alpha}$, *where* α *is the* (*unique*) *real root of*

$$\sum_{j=1}^{m} b_j^* \lambda_j^{\alpha} = a,$$

defines a positive and continuously differentiable function fulfilling (2.3) *on* (t_0, ∞).

We note that some previous assumptions on τ_j can be formulated in a weaker form. However, our formulation is sufficient for use in the next section.

3 Main results

In this section, we present asymptotic estimates of all solutions of equation (1.1). Throughout this section we assume that the equation

$$\dot{y}(t) = A(t)y(t) \tag{3.1}$$

is uniformly exponentially stable, i.e., there exist real constants a, $L > 0$ such that

$$\|Y(u,v)\| \leqslant L \exp\{a(v-u)\} \qquad \text{for all } u \geqslant v \geqslant t_0, \tag{3.2}$$

where Y is the evolution matrix of (3.1).

Theorem 3.1 *Consider the equation* (1.1), *where* A, B_j, $f \in C(I)$ *and* τ_j $(j = 1, \ldots, m)$ *are satisfying* (H_1). *Let equation* (3.1) *be uniformly exponentially stable with constants* a, L *given by* (3.2) *and let* $\varphi \in C^2(I)$ *be a solution of* (2.1) *with the properties guaranteed by Theorem* 2.1. *Further, let* $\omega \in C^2((t_0, \infty))$ *be a positive solution of* (2.3), *where* $b_j = \|B_j\|$ *on* (t_0, ∞) $(j = 1, \ldots, m)$ *such that* $\dot{\omega} + a\omega > 0$, $\dot{\omega}_*/(\dot{\omega} + a\omega)$ *is nonincreasing on* I *and*

$$\int_{t_0}^{\infty} \frac{\dot{\omega}_*(t)}{\dot{\omega}(t) + a\omega(t)} \frac{\dot{\varphi}(t)}{\varphi(t)}\, dt < \infty,$$

where $\dot{\omega}_* = (|\dot{\omega}| - \dot{\omega})/2$ *on* I. *If* $f(t) = O(\omega(t))$ *as* $t \to \infty$, *then the estimates*

$$\begin{aligned} y(t) &= O(\omega(t)) \quad \text{as } t \to \infty \qquad \text{if } L < 1, \\ y(t) &= O(\omega(t) \log \varphi(t)) \quad \text{as } t \to \infty \qquad \text{if } L = 1, \\ y(t) &= O\left(\omega(t)(\varphi(t))^{\log L / \log \lambda^{-1}}\right) \quad \text{as } t \to \infty \qquad \text{if } L > 1 \end{aligned} \tag{3.3}$$

hold for any solution y *of* (1.1).

Proof Let y be a solution of (1.1) defined on an interval $[\sigma_0, \infty)$, where $\sigma_0 > t_0$ is arbitrary. If we introduce the change of variables

$$s = \log \varphi(t), \qquad z(s) = \frac{y(t)}{\omega(t)},$$

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then we get the function $z(s)$ defined on the interval $[q_0, \infty)$, where $q_0 = \log \varphi(\sigma_0)$, satisfying the equation

$$\begin{aligned} z'(s) = &\left(A(h(s))h'(s) - \frac{\omega'(h(s))h'(s)}{\omega(h(s))} E \right) z(s) \\ &+ \sum_{j=1}^{m} B_j(h(s))h'(s) \frac{\omega(\tau_j(h(s)))}{\omega(h(s))} z(s - c_j(s)) + \frac{f(h(s))h'(s)}{\omega(h(s))}, \end{aligned} \tag{3.4}$$

where $h(s) = \varphi^{-1}(\exp\{s\})$, $c_j(s) \geqslant \log \lambda^{-1}$ for all $s \in [q_0, \infty)$, "$'$" stands for the derivative with the respect to s and E is the identity matrix.

Put $q_k := q_0 + k \log \lambda^{-1}$, $J_k := [q_{k-1}, q_k]$, $M_k := \sup\{|z(s)|, s \in \cup_{i=1}^{k} J_i\}$ $(k = 1, 2, \dots)$ and let k^* be such that $s - c_j(s) \geqslant q_o$ for all $s \in J_{k+1}$, $k \geqslant k^*$. Now we consider the system

$$z'(s) = \left(A(h(s))h'(s) - \frac{\omega'(h(s))h'(s)}{\omega(h(s))} E \right) z(s) \tag{3.5}$$

and let H be the evolution matrix of (3.5). Then

$$\|H(s, p)\| = \frac{\omega(h(p))}{\omega(h(s))} \|Y(h(s), h(p))\| \leqslant L \frac{\omega(h(p))}{\omega(h(s))} \exp\{-a(h(s) - h(p))\}. \tag{3.6}$$

Considering arbitrary $s \in J_{k+1}$ (where $k \geqslant k^*$) and integrating the equation (3.4) we have

$$\begin{aligned} z(s) =& H(s, q_k)z(q_k) + \int_{q_k}^{s} \sum_{j=1}^{m} H(s, u)B_j(h(u))h'(u) \frac{\omega(\tau_j(h(u)))}{\omega(h(u))} z(u - c_j(u))\, du \\ &+ \int_{q_k}^{s} H(s, u) \frac{f(h(u))h'(u)}{\omega(h(u))}\, du. \end{aligned}$$

Using (3.6) we obtain the estimate

$$\begin{aligned} |z(s)| \leqslant & LM_k \Big(\frac{\omega(h(q_k))}{\omega(h(s))} \exp\{-a(h(s) - h(q_k))\} \\ &+ \int_{q_k}^{s} \sum_{j=1}^{m} \|B_j(h(u))\| h'(u) \frac{\omega(\tau_j(h(u)))}{\omega(h(s))} \exp\{-a(h(s) - h(u))\}\, du \Big) \\ &+ L \int_{q_k}^{s} \frac{f(h(u))h'(u)}{\omega(h(s))} \exp\{-a(h(s) - h(u))\}\, du, \end{aligned} \tag{3.7}$$

i.e.,

$$\begin{aligned} \frac{|z(s)|}{L} \leqslant & M_k \left(\frac{\omega(h(q_k))}{\omega(h(s))} \exp\{-a(h(s) - h(q_k))\} \right. \\ &\left. + \int_{q_k}^{s} \frac{\omega(h(u))}{\omega(h(s))} \exp\{-a(h(s) - h(u))\} a h'(u)\, du \right) \\ &+ K_1 \int_{q_k}^{s} \frac{\omega(h(u))}{\omega(h(s))} \exp\{-a(h(s) - h(u))\} h'(u)\, du, \end{aligned}$$

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$K_1 > 0$ being a suitable real. Hence,

$$\begin{aligned}\frac{|z(s)|}{L} \leqslant & M_k \left(\frac{\omega(h(q_k))}{\omega(h(s))} \exp\{-a(h(s) - h(q_k))\} \right. \\ & \left. + \frac{\exp\{-ah(s)\}}{\omega(h(s))} \int_{q_k}^{s} \omega(h(u)) \frac{d}{du} [\exp\{ah(u)\}]\, du \right) \\ & + \frac{\exp\{-ah(s)\}}{\omega(h(s))} \frac{K_1}{a} \int_{q_k}^{s} \omega(h(u)) \frac{d}{du} [\exp\{ah(u)\}]\, du. \end{aligned} \tag{3.8}$$

Our next step is to show that

$$\begin{aligned}& \int_{q_k}^{s} \omega(h(u)) \frac{d}{du} [\exp\{ah(u)\}]\, du \\ & \leqslant \omega(h(u)) \exp\{ah(u)\}|_{q_k}^{s} \left(1 + \frac{\dot{\omega}_*(h(q_k))}{\dot{\omega}(h(q_k)) + a\omega(h(q_k))} \right). \end{aligned} \tag{3.9}$$

To show this we integrate by parts:

$$\begin{aligned}& \int_{q_k}^{s} \omega(h(u)) \frac{d}{du} [\exp\{ah(u)\}]\, du \\ & \leqslant \omega(h(u)) \exp\{ah(u)\}|_{q_k}^{s} + \int_{q_k}^{s} \dot{\omega}_*(h(u)) \dot{h}(u) \exp\{ah(u)\}\, du. \end{aligned}$$

The repeated integration by parts yields

$$\begin{aligned}& \int_{q_k}^{s} \dot{\omega}_*(h(u)) \dot{h}(u) \exp\{ah(u)\}\, du = \\ & \int_{q_k}^{s} \left(\frac{\dot{\omega}_*(h(u))}{\dot{\omega}(h(u)) + a\omega(h(u))} \frac{d}{du} \left[\omega(h(u)) \exp\{ah(u)\} \right] \right) du \\ & = \omega(h(u)) \exp\{ah(u)\} \frac{\dot{\omega}_*(h(u))}{\dot{\omega}(h(u)) + a\omega(h(u))} \Big|_{q_k}^{s} \\ & + \int_{q_k}^{s} \left(\omega(h(u)) \exp\{ah(u)\} \frac{d}{du} \left[\frac{-\dot{\omega}_*(h(u))}{\dot{\omega}(h(u))) + a\omega(h(u))} \right] \right) du \\ & \leqslant \omega(h(u)) \exp\{ah(u)\} \frac{\dot{\omega}_*(h(u))}{\dot{\omega}(h(u)) + a\omega(h(u))} \Big|_{q_k}^{s} \\ & + \omega(h(s)) \exp\{ah(s)\} \frac{-\dot{\omega}_*(h(u))}{\dot{\omega}(h(u)) + a\omega(h(u))} \Big|_{q_k}^{s} \\ & = \omega(h(u)) \exp\{ah(u)\}|_{q_k}^{s} \frac{\dot{\omega}_*(h(q_k))}{\dot{\omega}(h(q_k)) + a\omega(h(q_k))} \end{aligned}$$

and this proves (3.9). Substituting (3.9) back into (3.8) we have

$$\begin{aligned}\frac{|z(s)|}{L} \leqslant & M_k \frac{\omega(h(q_k))}{\omega(h(s))} \exp\{-a(h(s) - h(q_k))\} \\ & + \left(M_k + \frac{K_1}{a} \right) \frac{\exp\{-ah(s)\}}{\omega(h(s))} \omega(h(u)) \exp\{ah(u)\}|_{q_k}^{s} \left(1 + \frac{\dot{\omega}_*(h(q_k))}{\dot{\omega}(h(q_k)) + a\omega(h(q_k))} \right) \\ \leqslant & \left(M_k + \frac{K_1}{a} \right) \left(1 + \frac{\dot{\omega}_*(h(q_k))}{\dot{\omega}(h(q_k)) + a\omega(h(q_k))} \right). \end{aligned}$$

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Since $s \in J_{k+1}$ was arbitrary,

$$M_{k+1} \leqslant \max\left(M_k, L\left(M_k + \frac{K_1}{a}\right)\left(1 + \frac{\dot{\omega}_*(h(q_k))}{\dot{\omega}(h(q_k)) + a\omega(h(q_k))}\right)\right), \quad k = 1, 2, \ldots.$$

Moreover, utilizing the integral criterion with the respect to the substitution $s = \log \varphi(t)$ one can easily verify the convergence of the infinite product

$$\prod_{j=1}^{\infty}\left(1 + \frac{\dot{\omega}_*(h(q_j))}{\dot{\omega}(h(q_j)) + a\omega(h(q_j))}\right).$$

Consequently, if $L < 1$, then

$$M_k = O(1) \qquad \text{as } k \to \infty,$$

if $L = 1$, then

$$M_k = O(k) \qquad \text{as } k \to \infty$$

and if $L > 1$, then

$$M_k = O(L^k) \qquad \text{as } k \to \infty.$$

Using the backward substitution $y(t) = z(s)\omega(t) = z(\log \varphi(t))\omega(t)$ now the asymptotic estimates (3.3) follow from the definition of M_k.

□

In the sequel we consider the equation (1.1) with bounded coefficients B_j and delays τ_j satisfying (H_2) and impose the less restrictive asymptotic condition on the forcing term f. Then we can formulate the asymptotic bounds of solutions only in terms of a solution φ of the system (2.2).

Theorem 3.2 *Consider the equation* (1.1), *where* $A, B_j, f \in C(I)$ *and* τ_j $(j = 1, \ldots, m)$ *are satisfying* (H_2). *Further assume that the equation* (3.1) *is uniformly exponentially stable with constants* a, L *given by* (3.2), $\varphi \in C^1(I)$ *is a solution of* (2.2) *with the properties guaranteed by Theorem* 2.2, $\|B_j\| \leqslant b_j^*$ *on* I *for suitable reals* $b_j^* > 0$ $(j = 1, \ldots, m)$ *and* $f(t) = O((\varphi(t))^{\beta})$ *as* $t \to \infty$ *for a suitable real* β. *If we introduce the scalar* α *as the* (*unique*) *real root of* $\sum_{j=1}^{m} b_j^* \lambda_j^{\alpha} = a$ *and assume that* y *is a solution of* (1.1), *then*

$$\begin{aligned} y(t) &= O\left((\varphi(t))^{\max(\alpha,\beta)}\right) \quad \text{as } t \to \infty \qquad \text{if } L < 1 \quad \text{or} \quad L = 1,\ \alpha > \beta, \\ y(t) &= O\left((\varphi(t))^{\max(\alpha,\beta)} \log \varphi(t)\right) \quad \text{as } t \to \infty \qquad \text{if } L = 1,\ \alpha \leqslant \beta, \\ y(t) &= O\left((\varphi(t))^{\max(\alpha,\beta) + \log L / \log \lambda^{-1}}\right) \quad \text{as } t \to \infty \qquad \text{if } L > 1, \end{aligned} \tag{3.10}$$

where $\lambda := \max(\lambda_1, \ldots, \lambda_m)$.

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Proof First we verify that the function $\omega(t) = (\varphi(t))^\gamma$, where $\gamma = \max(\alpha, \beta)$, defines a solution of the inequality (2.3) with $b_j(t) = \|B_j(t)\|$ (see also Proposition 2.3). Indeed,

$$\sum_{j=1}^{m} \|B_j(t)\|\omega(\tau_j(t)) \leqslant \sum_{j=1}^{m} b_j^*(\varphi(\tau_j(t)))^\gamma = \sum_{j=1}^{m} b_j^* \lambda_j^\gamma (\varphi(t))^\gamma \leqslant a\omega(t).$$

The next process is a modification of the method utilized in the proof of Theorem 3.1 and therefore the related parts are only sketched. Similarly as in the proof of Theorem 3.1 we can put

$$s = \log \varphi(t), \qquad z(s) = \frac{y(t)}{(\varphi(t))^\gamma}.$$

Following this proof we derive using the same line of argument inequality (3.7) with $\omega = \varphi^\gamma$. Substituting this ω into (3.7) and using the inequalities

$$\sum_{j=1}^{m} \|B_j(h(u))\|(\varphi(\tau_j(h(u))))^\gamma \leqslant a \exp\{\gamma u\},$$

$$f(h(u)) \leqslant K_1 \exp\{\beta u\} \qquad \text{for a suitable } K_1 > 0$$

we get

$$\begin{aligned}
\frac{|z(s)|}{L} \leqslant & M_k \Bigg(\exp\{\gamma(q_k - s) - a(h(s) - h(q_k))\} \\
& + \exp\{-\gamma s - ah(s)\} \int_{q_k}^{s} \exp\{\gamma u\} \frac{d}{du} [\exp\{ah(u)\}]\, du \Bigg) \\
& + K_1 \exp\{-\gamma s - ah(s)\} \int_{q_k}^{s} \exp\{\beta u + ah(u))\} h'(u)\, du \\
\leqslant & M_k \exp\{\gamma(q_k - s) - a(h(s) - h(q_k))\} \\
& + \left(M_k + \frac{K_1}{a} \exp\{(\beta - \gamma) q_k\} \right) \exp\{-\gamma s - ah(s)\} \\
& \times \int_{q_k}^{s} \exp\{\gamma u\} \frac{d}{du} [\exp\{ah(u)\}]\, du.
\end{aligned} \tag{3.11}$$

To estimate the last integral we first note that differentiating (2.2) one can check that $\dot{\varphi}(t)$ is bounded as $t \to \infty$, hence

$$\frac{1}{h'(u)} = \frac{\dot{\varphi}(h(u))}{\varphi(h(u))} = O(\exp\{-u\}) \qquad \text{as } u \to \infty.$$

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Using this property and integrating by parts we have the following estimates:

$$\begin{aligned}
&\int_{q_k}^{s} \exp\{\gamma u\} \frac{d}{du}[\exp\{ah(u)\}]\, du \\
&\quad \leqslant \exp\{\gamma u + ah(u)\}|_{q_k}^{s} + |\gamma| \int_{q_k}^{s} \exp\{\gamma u + ah(u)\}\, du \\
&\quad = \exp\{\gamma u + ah(u)\}|_{q_k}^{s} + \int_{q_k}^{s} \frac{|\gamma|}{\gamma + ah'(u)} \frac{d}{du}[\exp\{\gamma u + ah(u)\}]\, du \\
&\quad \leqslant (1 + O(\exp\{-q_k\})) \exp\{\gamma u + ah(u)\}|_{q_k}^{s} \quad \text{as } k \to \infty.
\end{aligned}$$

Substituting this back into (3.11) we get

$$\begin{aligned}
\frac{|z(s)|}{L} &\leqslant M_k \exp\{\gamma(q_k - s) - a(h(s) - h(q_k))\} \\
&\quad + \left(M_k + \frac{K_1}{a} \exp\{(\beta - \gamma)q_k\}\right) \exp\{-\gamma s - ah(s)\}(1 + K_2 \exp\{-q_k\}) \\
&\quad \times (\exp\{\gamma s + ah(s)\} - \exp\{\gamma q_k + ah(q_k)\}) \\
&\leqslant M_k(1 + K_2 \exp\{-q_k\}) + \frac{K_1}{a} \exp\{(\beta - \gamma)q_k\}(1 + K_2 \exp\{-q_k\})
\end{aligned}$$

for a suitable real $K_2 > 0$. Since $s \in J_{k+1}$ was arbitrary,

$$M_{k+1} \leqslant \max\left(M_k, L(M_k(1 + K_2 \exp\{-q_k\}) + \frac{K_1}{a} \exp\{(\beta - \gamma)q_k\}(1 + K_2 \exp\{-q_k\}))\right).$$

If $L < 1$, then $M_k = O(1)$ as $k \to \infty$, i.e., $y(t) = O\big((\varphi(t))^{\gamma}\big)$ as $t \to \infty$. If $L \geqslant 1$, then the repetition leads to

$$\begin{aligned}
M_{k+1} \leqslant & L^k M_1 \prod_{j=1}^{k} (1 + K_2 \exp\{-q_j\}) \\
& + \frac{K_1}{a} \sum_{i=0}^{k-1} \left(L^{k-i} \exp\{(\beta - \gamma)q_{i+1}\} \prod_{j=i+1}^{k} (1 + K_2 \exp\{-q_j\})\right).
\end{aligned}$$

Setting $K_3 := \prod_{j=1}^{\infty}(1 + K_2 \exp\{-q_j\})$ one gets

$$M_{k+1} \leqslant M_1 K_3 L^k + \frac{K_1 K_3}{a} \sum_{i=0}^{k-1} L^{k-i} \exp\{(\beta - \gamma)q_{i+1}\}.$$

Now we distinguish the following cases:

If $L = 1$ and $\alpha > \beta$, then $\beta - \gamma < 0$ and the sum $\sum_{i=0}^{k-1} \exp\{(\beta - \gamma)q_{i+1}\}$ converges as $k \to \infty$. Hence, $M_k = O(1)$ as $k \to \infty$ and $y(t) = O\big((\varphi(t))^{\gamma}\big)$ as $t \to \infty$.

If $L = 1$ and $\alpha \leqslant \beta$, then $\beta - \gamma = 0$ and $M_k = O(k)$ as $k \to \infty$, i.e., $y(t) = O\big((\varphi(t))^{\gamma} \log \varphi(t)\big)$ as $t \to \infty$.

Finally, if $L > 1$, then $M_k = O(L^k)$ as $k \to \infty$ and $y(t) = O\big((\varphi(t))^{\gamma + \log L / \log \lambda^{-1}}\big)$ as $t \to \infty$.

This proves the required asymptotic properties (3.10).

□

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4 Asymptotic behaviour in particular cases

This section presents applications of Theorems 3.1 and 3.2 to some important particular cases.

To illustrate the conclusions of Theorem 3.1 we consider the scalar homogeneous equation

$$\dot{y}(t) = -ay(t) + \sum_{j=1}^{m} b_j(t)y(\tau_j(t)), \qquad t \in I = [t_0, \infty). \tag{4.1}$$

Modifying the inequality (2.3) as

$$-a\omega(t) + \sum_{j=1}^{m} |b_j(t)|\omega(\tau_j(t)) \leqslant 0, \qquad t \in (t_0, \infty) \tag{4.2}$$

we obtain the following

Corollary 4.1 *Consider the equation* (4.1)*, where* $a > 0$ *is a real scalar,* $b_j \in C(I)$*,* τ_j $(j = 1, \ldots, m)$ *are satisfying* (H_1) *and let* $\varphi \in C^2(I)$ *be a solution of* (2.1) *with the properties guaranteed by Theorem* 2.1*. Further, let* $\omega \in C^2((t_0, \infty))$ *be a positive solution of* (4.2) *with the properties mentioned in Theorem* 3.1*. Then*

$$y(t) = O(\omega(t)) \quad \text{as } t \to \infty \tag{4.3}$$

for any solution y *of* (4.1)*.*

Proof Consider Theorem 3.1 and its proof, where we put $L = 1$ and $K_1 = 0$. Then $M_k = O(1)$ as $k \to \infty$ and the second asymptotic property in (3.3) becomes (4.3). □

Example 4.2 *Consider the equation*

$$\dot{y}(t) = -ay(t) + b_1(t)y(\lambda t + \mu) + b_2(t)y(t^{\gamma}), \qquad t \in [1, \infty), \tag{4.4}$$

where $a > 0$*,* $0 < \lambda, \gamma < 1$*,* $\mu < 0$ *are real scalars,* $b_j \in C([1, \infty))$ *and* $b_j(t) = O(t^{\delta})$ *as* $t \to \infty$ *for a real* $\delta \geqslant 0$*, i.e., there are positive reals* b_j^* *such that* $|b_j(t)| \leqslant b_j^* t^{\delta}$ *for all* t *large enough* $(j = 1, 2)$*. Then the functions* $\tau_1(t) = \lambda t + \mu$*,* $\tau_2(t) = t^{\gamma}$ *are satisfying* (H_1) *with* $\tau = \tau_1$ *(the inequality* $\tau_2(t) \leqslant \tau_1(t)$ *is fulfilled for all* t *large enough) and the function* $\varphi(t) = t - \mu/(1 - \lambda)$ *is the solution of* (2.1) *having required properties. Further, if* $\delta > 0$ *or* $\delta = 0$*,* $b_1^* + b_2^* \geqslant a$*, then*

$$\omega(t) = t^{\frac{\log((b_1^* + b_2^*)/a) + \frac{\delta}{2}\log t}{\log \lambda^{-1}} + \frac{\delta}{2}}$$

defines the positive and nondecreasing solution of (4.2) *for all* t *large enough. Let us emphasize that in such a case* $\dot{\omega}_*$ *is identically zero and the assumptions imposed on* $\dot{\omega}$ *and* $\dot{\omega}_*$ *in Theorem* 3.1 *become trivial.*

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If $\delta = 0$ *and* $b_1^* + b_2^* < a$, *then the function*

$$\omega(t) = (\log t)^{\frac{\log((b_1^*+b_2^*)/a)}{\log \gamma^{-1}}}$$

defines the positive and decreasing solution of (4.2) *for all* t *large enough and it is easy to check that all the assumptions of Corollary* 4.1 *are satisfied.*

Then we can summarize that

$$y(t) = O\left(t^{\frac{\log((b_1^*+b_2^*)/a)+\frac{\delta}{2}\log t}{\log \lambda^{-1}}+\frac{\delta}{2}}\right) \quad \text{as } t \to \infty \qquad \text{if } \delta > 0 \quad \text{or} \quad \delta = 0,\ b_1^* + b_2^* \geqslant a,$$

$$y(t) = O\left((\log t)^{\frac{\log((b_1^*+b_2^*)/a)}{\log \gamma^{-1}}}\right) \quad \text{as } t \to \infty \qquad \text{if } \delta = 0,\ b_1^* + b_2^* < a$$

for any solution y *of* (4.4).

Now we formulate some immediate consequences of Theorem 3.2.

Corollary 4.3 *Consider the equation*

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\lambda_j t) + f(t), \quad t \in [0,\infty), \tag{4.5}$$

where $A, B_j, f \in C([0,\infty))$ *and* $0 < \lambda_j < 1$ $(j = 1,\dots,m)$ *are constants. Further assume that the equation* (3.1) *is uniformly exponentially stable with constants* a, L *given by* (3.2), $\|B_j\| \leqslant b_j^*$ *on* $[0,\infty)$ *for suitable reals* $b_j^* > 0$ $(j = 1,\dots,m)$ *and* $f(t) = O(t^\beta)$ *as* $t \to \infty$ *for a suitable real* β. *Let* α *be the (unique) real root of* $\sum_{j=1}^m b_j^* \lambda_j^\alpha = a$. *Then every solution* y *of* (4.5) *has the asymptotic bound*

$$\begin{aligned}
&O\left(t^{\max(\alpha,\beta)}\right) \quad \text{as } t \to \infty \qquad \text{if } L < 1 \quad \text{or} \quad L = 1,\ \alpha > \beta,\\
&O\left(t^{\max(\alpha,\beta)}\log t\right) \quad \text{as } t \to \infty \qquad \text{if } L = 1,\ \alpha \leqslant \beta,\\
&O\left(t^{\max(\alpha,\beta)+\log L/\log \lambda^{-1}}\right) \quad \text{as } t \to \infty, \quad \lambda := \max(\lambda_1,\dots,\lambda_m) \qquad \text{if } L > 1.
\end{aligned}$$

Corollary 4.4 *Consider the equation*

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(t^{\gamma_j}) + f(t), \quad t \in [1,\infty), \tag{4.6}$$

where $A, B_j, f \in C([1,\infty))$ *and* $0 < \gamma_j < 1$ $(j = 1,\dots,m)$ *are constants. Further assume that the equation* (3.1) *is uniformly exponentially stable with constants* a, L *given by* (3.2), $\|B_j\| \leqslant b_j^*$ *on* $[1,\infty)$ *for suitable reals* $b_j^* > 0$ $(j = 1,\dots,m)$ *and* $f(t) = O((\log t)^\beta)$ *as* $t \to \infty$ *for a suitable real* β. *Define the scalar* α *as the (unique) real root of* $\sum_{j=1}^m b_j^* \gamma_j^\alpha = a$. *Then every solution* y *of* (4.6) *has the asymptotic bound*

$$\begin{aligned}
&O\left((\log t)^{\max(\alpha,\beta)}\right) \quad \text{as } t \to \infty \qquad \text{if } L < 1 \quad \text{or} \quad L = 1,\ \alpha > \beta,\\
&O\left((\log t)^{\max(\alpha,\beta)}\log\log t\right) \quad \text{as } t \to \infty \qquad \text{if } L = 1,\ \alpha \leqslant \beta,\\
&O\left((\log t)^{\max(\alpha,\beta)+\log L/\log \gamma^{-1}}\right) \quad \text{as } t \to \infty, \quad \gamma := \max(\gamma_1,\dots,\gamma_m) \qquad \text{if } L > 1.
\end{aligned}$$

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Proofs of Corollaries 4.3 and 4.4 If we put $\tau_j(t) = \lambda_j t$ in (2.2), then the identity function $\varphi(t) = t$ defines the required solution of (2.2). Similarly, if $\tau_j(t) = t^{\gamma_j}$, then (2.2) with $\lambda_j = \gamma_j$ admits the common solution $\varphi(t) = \log t$. Substituting these φ into (3.10) we obtain the asymptotic estimates of solutions for equations (4.5) and (4.6). □

Remark 4.5 The essential assumption of Theorem 3.2 is the commutativity of any pair of delays following from the hypothesis (H_2). We show that this property is closely related to the theory of continuous iterations.

Let τ_j $(j = 1,\dots,m)$ be satisfying (H_2). Then, by Theorem 2.2, there exists a common solution $\varphi \in C^1(I)$ of simultaneous equations (2.2), which is positive and unbounded on (t_0,∞) and has a positive derivative on I. Hence,

$$\tau_j(t) = \varphi^{-1}(\lambda_1^u \varphi(t)), \qquad j = 1,\dots,m, \quad t \in I \tag{4.7}$$

for $\lambda_1 = \dot{\tau}_1(t_0)$ and $u = \log \lambda_j / \log \lambda_1$. In other words, τ_j can be embedded into the continuous iteration group $\{\tau_1^u,\ u \in \mathbb{R}\}$, i.e., delays τ_j are continuous iterations of a given delay.

Similarly, let $\varphi \in C^1(I)$, $\varphi(t_0) = 0$ be positive and unbounded on (t_0,∞) with a positive derivative on I, let $\tau_1(t) = \varphi^{-1}(\lambda_1 \varphi(t))$, $0 < \lambda_1 < 1$ and let $\tau_j \in \{\tau_1^u,\ u \in \mathbb{R}\}$, i.e., (4.7) holds for a suitable positive real u depending on j. Then φ is a solution of (2.2), where $\lambda_j = \lambda_1^u$. For more details concerning the iteration theory we refer to Kuczma *et al.* [18].

Our next aim is to consider the equation (1.1) in the particular case

$$\dot{y}(t) = Ay(t) + \sum_{j=1}^{m} B_j y(\tau_j(t)) + f(t), \quad t \in I = [t_0,\infty) \tag{4.8}$$

and show that under a stronger assumption on the forcing term f we can obtain a better asymptotic bound than Theorem 3.2 yields.

Theorem 4.6 *Consider the equation* (4.8), *where* $A = \operatorname{diag}(a_1,\dots,a_n)$ *is a diagonal constant matrix with* $a_i < 0$ $(i = 1,\dots,n)$, B_j *are nonzero* $n \times n$ *constant matrices,* τ_j $(j = 1,\dots,m)$ *are satisfying* (H_2) *and* $f \in C(I)$. *Further assume that* $\varphi \in C^1(I)$ *is a solution of* (2.2) *with the properties guaranteed by Theorem* 2.2 *and define* α *as a (unique) real root of* $\sum_{j=1}^{m} \|B_j\| \lambda_j^{\alpha} = a$, *where* $a := \min(-a_1,\dots,-a_n)$. *If* $f(t) = O((\varphi(t))^{\beta})$ *as* $t \to \infty$ *for a suitable real* β *and* y *is a solution of* (4.8), *then*

$$\begin{aligned} y(t) &= O\big((\varphi(t))^{\max(\alpha,\beta)}\big) \quad \text{as } t \to \infty \qquad \text{if } \alpha > \beta, \\ y(t) &= O\big((\varphi(t))^{\max(\alpha,\beta)} \log \varphi(t)\big) \quad \text{as } t \to \infty \qquad \text{if } \alpha \leqslant \beta. \end{aligned} \tag{4.9}$$

Moreover, if $f \in C^1(I)$ *and* $\dot{f}(t) = O((\varphi(t))^{\beta-1})$ *as* $t \to \infty$, *then*

$$\begin{aligned} y(t) &= O\big((\varphi(t))^{\max(\alpha,\beta)}\big) \quad \text{as } t \to \infty \qquad \text{if } \alpha \neq \beta, \\ y(t) &= O\big((\varphi(t))^{\max(\alpha,\beta)} \log \varphi(t)\big) \quad \text{as } t \to \infty \qquad \text{if } \alpha = \beta. \end{aligned} \tag{4.10}$$

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Proof The first part of Theorem 4.6 follows immediately from Theorem 3.2. To prove the second part it is enough to show that under the additional assumptions imposed on f and assuming $\beta > \alpha$ it holds

$$y(t) = O\big((\varphi(t))^{\beta}\big) \quad \text{as } t \to \infty$$

for any solution y of (4.8) defined on an interval $[\sigma_0, \infty)$, $\sigma_0 > t_0$.

Choose a real κ, $\beta < \kappa < \beta + 1$. It follows from Theorem 3.2 that

$$y(t) = O\big((\varphi(t))^{\kappa}\big) \quad \text{as } t \to \infty.$$

Differentiating (4.8) we obtain

$$\ddot{y}(t) = A\dot{y}(t) + \sum_{j=1}^{m} B_j \dot{\tau}_j(t)\dot{y}(\tau_j(t)) + \dot{f}(t).$$

The repeated application of Theorem 3.2 yields

$$\dot{y}(t) = O\big((\varphi(t))^{\kappa-1}\big) \quad \text{as } t \to \infty.$$

Now we set $s = \log \varphi(t)$, $w(s) = (\varphi(t))^{-\kappa} y(t)$. Similarly as in the proof of Theorem 3.1 we denote $h(s) = \varphi^{-1}(\exp\{s\})$ on $\varphi(I)$, $c_j = \log \lambda_j^{-1} > 0$ and let E be the identity matrix. Then w defines a solution of

$$w'(s) = (Ah'(s) - \kappa E)w(s) + \sum_{j=1}^{m} B_j \lambda_j^{\kappa} h'(s) w(s - c_j) + f(h(s))h'(s)\exp\{-\kappa s\} \tag{4.11}$$

with the properties $w(s) = O(1)$ and $w'(s) = O(\exp\{-s\}h'(s))$ as $s \to \infty$ following from the boundedness of $\dot{\varphi}$. To derive a stronger estimate of w we consider equation (4.11) as a delay nondifferential equation

$$w(s) = \sum_{j=1}^{m} M_j w(s - c_j) + p(s), \tag{4.12}$$

where $M_j = -A^{-1}B_j\lambda_j^{\kappa}$ $(j = 1, \ldots, m)$ and

$$p(s) = A^{-1}[w'(s) + \kappa E w(s) - f(h(s))h'(s)\exp\{-\kappa s\}]/h'(s).$$

Since $f(h(s)) = O(\exp\{\beta s\})$ and $\exp\{s\}/h'(s) = O(1)$ as $s \to \infty$, we get that p has the upper bound $O(\exp\{-\eta s\})$ as $s \to \infty$, where $\eta := \kappa - \beta > 0$.

Now put $q_0 := \log \varphi(\sigma_0)$ and $c := \min(c_1, \ldots, c_m)$. Let $s_0 > q_0 + \max(c_1, \ldots, c_m)$ and $K > 0$ be such that

$$|w(s)| \leqslant K \exp\{-\eta s\} \quad \text{for all } q_0 \leqslant s \leqslant s_0$$

and

$$|p(s)| \leqslant K \exp\{-\eta s\}$$

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for all $s \geqslant q_0$. If $s_0 \leqslant s \leqslant s_0 + c$, then using (4.12) we have

$$\begin{aligned}|w(s)| &\leqslant \sum_{j=1}^{m} \|M_j\| K \exp\{-\eta(s-c_j)\} + K \exp\{-\eta s\} \\ &\leqslant K \exp\{-\eta s\} \left(1 + \sum_{j=1}^{m} a^{-1} \|B_j\| \lambda_j^{\kappa} \lambda_j^{-\eta}\right) \\ &= K \exp\{-\eta s\} \left(1 + \sum_{j=1}^{m} a^{-1} \|B_j\| \lambda_j^{\beta}\right) = K \exp\{-\eta s\}(1+\rho),\end{aligned}$$

where $\rho := \sum_{j=1}^{m} a^{-1} \|B_j\| \lambda_j^{\beta} < 1$ by virtue of $\sum_{j=1}^{m} a^{-1} \|B_j\| \lambda_j^{\alpha} = 1$ and $\alpha < \beta$. Repeating this procedure we obtain

$$|w(s)| \leqslant K \exp\{-\eta s\} \sum_{l=0}^{k} \rho^l \qquad \text{for all } s_0 + (k-1)c \leqslant s \leqslant s_0 + kc \quad (k = 1, 2, \dots),$$

i.e., $w(s) = O(\exp\{-\eta s\})$ as $s \to \infty$. This implies the required asymptotic property

$$y(t) = O\left((\varphi(t))^{-\eta} (\varphi(t))^{\kappa}\right) = O\left((\varphi(t))^{\beta}\right) \qquad \text{as } t \to \infty.$$

□

Remark 4.7 Theorem 4.6 can be easily modified to the case of a diagonal constant matrix $A = \operatorname{diag}(a_1, \dots, a_n)$ with complex entries such that $\operatorname{Re} a_i < 0$ $(i = 1, \dots, n)$. More generally, assuming that A is diagonalizable with eigenvalues a_i having negative real parts, we can choose the basis such that A is in a diagonal form and thus extend the previous assertion also to this case.

5 Conclusion

The paper presents upper bounds of solutions of the equation (1.1) with continuous coefficients and unbounded lags. These bounds are formulated in terms of a solution of the auxiliary system of Schröder equations. The derived results can be applied especially to equations with linearly transformed arguments and, more generally, to equations with general unbounded lags. However, utilizing a suitable change of the independent variable we can extend the validity of the presented results also to equations with bounded (e.g. constant) lags.

To compare some of our previous results (in particular, Theorem 4.6) with those derived in papers mentioned in § 1, we consider the nonhomogeneous scalar equation

$$\dot{y}(t) = -a y(t) + \sum_{j=1}^{m} b_j y(\tau_j(t)) + f(t), \quad t \in I = [t_0, \infty), \tag{5.1}$$

where $a > 0$, $b_j \neq 0$ are scalars, τ_j $(j = 1, \dots, m)$ are satisfying (H_2), $f \in C(I)$ and $f(t) = O((\varphi(t))^{\beta})$ as $t \to \infty$. Then, using the notation of Theorem 4.6 (with $\|B_j\|$ replaced

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by $|b_j|$), every solution y of (5.1) has the asymptotic bound (4.9). Moreover, under the additional assumptions $f \in C^1(I)$ and $\dot{f}(t) = O((\varphi(t))^{\beta-1})$ as $t \to \infty$ the stronger asymptotic property (4.10) holds. This result generalizes, e.g., Theorem 1 of Lim [20] (it is enough to put $m = 1$ and $\tau(t) = \lambda t$, hence $\varphi(t) = t$) or Theorem 2.3 of Čermák [6] ($m = 1$). The list of other similar comparisons can be found in Lehninger & Liu [19].

Considering the vector case we note that our asymptotic bounds of solutions extend the validity of the corresponding results concerning the pantograph equation and its modifications (presented, for example, in Carr & Dyson [3], Makay & Terjéki [22], Pandolfi [25] or [5]).

It is an open question, whether similar asymptotic bounds are true also for the neutral equation

$$\dot{y}(t) = A(t)y(t) + \sum_{j=1}^{m} B_j(t)y(\tau_j(t)) + \sum_{j=1}^{m} C_j(t)\dot{y}(\xi_j(t)) + f(t), \quad t \in I. \tag{5.2}$$

Some particular cases of (5.2) with proportional delays were studied, for instance, in Iserles [14] or Lehninger & Liu [19], but the influence of the neutral terms on the asymptotics of solutions in the general case of (5.2) remains open.

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