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On m-ary partitions

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1. Let $g(n) = g(n; m)$ denote the number of partitions of a natural number n into powers of a given natural number m (> 2). Then the generating function of $g(n)$ is

$$F(x) = \prod_{r=0}^{\infty} (1-x^{m^r})^{-1} = \sum_{n=0}^{\infty} g(n)\, x^n \tag{1}$$

where we take $g(0) = 1$.

Evidently

$$(1-x)\, F(x) = F(x^m). \tag{2}$$

Following Churchhouse (1), we can readily show that for $r \geqslant 1$,

$$g(m^r . n) = \sum_{j=0}^{n} C_r(j)\, g(n-j) \tag{3}$$

where

$$C_{r+1}(j) = \sum_{i=0}^{mj} C_r(i) \tag{4}$$

with

$$C_1(j) = 1 \quad \text{for all} \quad j \geqslant 0.$$

Also

$$g(mn+t) = g(mn) \quad (0 \leqslant t < m). \tag{5}$$

From (3), we have

$$g(m^{r+1} . n) - g(m^r . n) = \sum_{j=1}^{n} \{C_{r+1}(j) - C_r(j)\}\, g(n-j). \tag{6}$$

In what follows, r is a positive integer, and

$$M_r = M_r(j) = C_{r+1}(j) - C_r(j). \tag{7}$$

The object of this note is to prove the

THEOREM. *For all* $j \geqslant 1$,

$$M_r \equiv 0 \pmod{m^r/b_r}$$

where $b_r = 1$ *or* 2^{r-1} *according as* m *is odd or even.*

This result is slightly stronger than that which has recently been proved by Andrews (2). Our proof is considerably simpler and shorter than his.

Our proof of the theorem depends on the

LEMMA. *For positive integers* $k \geqslant 2$, *there exist integers* f *depending only on* m, *such that*

$$\binom{mj+1}{k} = \binom{j}{k} m^k + \sum_{t=1}^{k-1} \binom{j}{k-t} f_k(t). \tag{8}$$

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Proof. Evidently $\binom{mj+1}{k}$, being a polynomial in j of degree $k \geqslant 2$, can be written in the form stated in the lemma. Putting $j = 1, 2, 3, \ldots$, in succession in (8), we obtain

$$f_k(k-1) = \binom{m+1}{k},$$

$$f_k(k-2) = \binom{2m+1}{k} - 2\binom{m+1}{k},$$

and in general, for $1 \leqslant t \leqslant k-1$,

$$f_k(k-t) = \binom{tm+1}{k} - \binom{t}{1} f_k(k-t+1) - \ldots - \binom{t}{t-1} f_k(k-1). \tag{9}$$

Induction proves our assertion that the f's are integers. Moreover, comparing the coefficients of j^{k-1} on the two sides of (8), we get

$$2f_k(1) = (k-1)\, m^k - (k-3)\, m^{k-1}. \tag{10}$$

2. *A relation in M's.* We have

$$M_1 = jm = \binom{j}{1} m. \tag{11}$$

Using (4) and the lemma, we get

$$M_2 = \binom{mj+1}{2} m = \left\{ \binom{j}{2} m^2 + \binom{j}{1} \binom{m+1}{2} \right\} m,$$

so that

$$M_2 - \binom{m+1}{2} M_1 = \binom{j}{2} m^3. \tag{12}$$

Repeating the process, we next obtain

$$M_3 - \left\{ \binom{m+1}{2} + m^3 \right\} M_2 + \left\{ \binom{m+1}{2} m^3 - \binom{m+1}{3} m^2 \right\} M_1 = \binom{j}{3} m^6. \tag{13}$$

Notice that the coefficients of the M's in (11), (12) and (13) are all integers.

Assuming that for certain integers $a_{r-1}(t)$ depending on m,

$$a_{r-1}(1)\, M_{r-1} + a_{r-1}(2)\, M_{r-2} + \ldots + a_{r-1}(r-1)\, M_1 = \binom{j}{r-1} m^{\binom{r}{2}}; \tag{14}$$

a use of (4) and the lemma will give

$$a_r(1)\, M_r + a_r(2)\, M_{r-1} + \ldots + a_r(r)\, M_1 = \binom{j}{r} m^{\binom{r+1}{2}} \tag{15}$$

where $a_r(1) = 1$, and

$$a_r(t+1) = a_{r-1}(t+1) - \sum_{h=1}^{t} m^{\binom{r}{2} - \binom{r-h+1}{2}} f_r(h)\, a_{r-h}(t-h+1), \tag{16}$$

for $1 \leqslant t \leqslant r-1$.

3. *Proof of the theorem.* From (16) and (10), we have for each positive integer $t \leqslant r-1$,

$$\begin{aligned} a_r(t+1) &\equiv a_{r-1}(t+1) - f_r(1)\, a_{r-1}(t) \pmod{m^{r-1}} \\ &\equiv a_{r-1}(t+1) \pmod{m^{r-1}/c} \end{aligned} \tag{17}$$

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where $c = 2$ if both m and r are even,

$= 1$ otherwise;

and we take $a_{r-1}(r) = 0$. Now (17) readily gives

$$a_r(t+1) \equiv 0 \pmod{m^t/d} \tag{18}$$

where $d = 2$ if m is even and t odd,

$= 1$ otherwise.

From (11), (12) and (13), it is evident that the theorem holds when $r = 1$, 2 or 3. Suppose that for $1 \leqslant t \leqslant r-1$,

$$M_{r-t} \equiv 0 \pmod{m^{r-t}/b_{r-t}}.$$

Then from (15) and (18), we at once have

$$M_r \equiv 0 \pmod{m^r/b_r}$$

and the theorem follows by induction.

REFERENCES

(1) Churchhouse, R. F. Congruence properties of the binary partition function. *Proc. Cambridge Philos. Soc.* **66** (1969), 371–376.

(2) Andrews, G. E. Congruence properties of the m-ary partition function. *J. Number Theory* **3** (1971), 104–110.

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