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Complex bounds for renormalization of critical circle maps†

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Abstract. We use the methods developed with Lyubich for proving complex bounds for real quadratics to extend de Faria's complex *a priori* bounds to all critical circle maps with an irrational rotation number. The contracting property for renormalizations of critical circle maps follows.

As another application of our methods we present a new proof of a theorem of Petersen on local connectivity of some Siegel Julia sets.

1. *Introduction*

The object of our consideration will be the family of analytic, orientation-preserving self-homeomorphisms of the circle, $f : \mathbf{T} \to \mathbf{T}$ with one critical point at 0. To fix our ideas we will assume that the critical point is cubic, although our considerations will be valid for any odd degree.

We will refer to such maps as *critical circle maps.* The rotation number of a critical circle map f will be denoted by $\rho(f)$. Examples of critical circle maps with any given rotation number are provided by maps of the standard family $x \mapsto x+\theta-(1/2\pi)\sin(2\pi x)$.

The renormalization operator for critical circle maps with irrational rotation number is defined in the language of *commuting pairs* (cf. [**La1, Ra1**]). These objects are naturally equipped with rotation numbers, on which the renormalization acts as the number-theoretical Gauss map. The questions of renormalization convergence and universality for critical circle maps under sufficient smoothness conditions were addressed by de Faria [**dF1, dF2**] in the framework of Sullivan's renormalization theory.

As in the case of interval maps, the renormalizations of smooth circle maps converge to real-analytic maps of Epstein class (see the definition below). De Faria has defined holomorphic extensions of renormalizations of analytic circle maps (see §2.5) which are appropriate for the application of Sullivan's theory. The key analytic condition for

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applicability of this theory is the existence of certain geometric bounds on the renormalized maps, the complex *a priori* bounds. These are lower bounds on the moduli of the fundamental annuli of the holomorphic extensions of the renormalizations of a critical circle map.

Applying Sullivan's methods to circle mappings de Faria has shown the existence of complex *a priori* bounds for critical circle mappings of Epstein class with *bounded combinatorics*. The last condition is equivalent to the rotation number $\rho(f)$ being Diophantine of order 2, that is having a continued fraction expansion with bounded elements.

In this paper we use the method developed with Lyubich [**LY**] to show the existence of complex *a priori* bounds for all critical circle mappings with irrational ρ. The technique was originally used in [**LY**] to treat the special case of unimodal maps with *essentially bounded combinatorics*. The geometric properties of this class of interval maps are similar to those of critical circle maps with an arbitrary irrational rotation number. Thus the methods of [**LY**] carry over to the setting of circle maps, yielding the following.

THEOREM 1.1. *Let f be a critical circle map of Epstein class. Then f has complex* a priori *bounds.*

As in the interval case, the theorem follows from a power estimate for the appropriately rescaled renormalizations of the map f:

$$|R^k f(z)| \geq c|z|^3, \tag{1.1}$$

with a universal $c > 0$. The proof of the estimate (1.1) is the main technical issue of this paper.

From Theorem 1.1 we derive the following.

THEOREM 1.2. *Let f_1 and f_2 be two critical circle mappings in the Epstein class with the same irrational rotation number. Then there exists $N > 0$ such that the nth renormalizations of f_1 and f_2 have holomorphic extensions which are K-quasiconformally conjugate with a universal bound K, for all $n > N$.*

Note that in the case of real quadratic-like maps, the existence of a bound on the moduli of fundamental annuli of two maps with the same combinatorics immediately implies the existence of a quasiconformal conjugacy with bounded 'K' (see, for example, [**McM1**]). In the circle case, however, an additional argument is required which is presented in §5. Sullivan's theory of Riemann surface laminations adapted to circle mappings by de Faria [**dF2**, Chapters VIII, IX] can now be applied to obtain the following renormalization contraction result, which complements the results obtained by de Faria for the bounded combinatorics case.

THEOREM 1.3. *Let f_1 and f_2 be two critical circle maps of Epstein class with the same irrational rotation number. Then*

$$\mathrm{dist}_{C^r}(R^n f_1, R^n f_2) \to 0,$$

for all $0 \leq r < \infty$, where dist_{C^r} denotes the distance in the C^r metric.

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As another application of our methods, in §6 we present a new proof of the result of Petersen on local connectivity of Julia sets of some Siegel quadratics.

Let us remark on the progress in the renormalization theory of critical circle maps since the preprint version of this paper [**Ya1**] appeared. In a very recent work, de Faria and de Melo [**dFdM2**] have enhanced the renormalization convergence result of [**dF1, dF2**] for critical circle mappings of bounded type, demonstrating that renormalization contracts C^r distance at a geometric rate. Their argument is based on McMullen's towers approach and incorporates our complex bounds. In combination with the results of their other recent paper [**dFdM1**] this implies that any two analytic critical circle mappings with the same irrational rotation number of bounded type are $C^{1+\alpha}$-conjugate, thus proving a version of a long-standing *$C^{1+\alpha}$-rigidity conjecture*. The possibility of generalizing these results of de Faria and de Melo to smooth maps and for the case of analytic maps with an arbitrary irrational rotation number remains open.

Also, in our new paper [**Ya2**] we take our investigation of renormalization of circle maps with arbitrary rotation numbers further, establishing the existence of a global attractor $\mathcal{A}$ for the renormalization of Epstein critical circle maps. The set $\mathcal{A}$ possesses a Cantor-like structure and the action of the (appropriately generalized) renormalization operator on $\mathcal{A}$ is conjugated to a shift on a countable set of symbols.

2. *Preliminaries*

2.1. *General notations and terminology.* The notation $D_r(z)$ will stand for the Euclidean disk with center $z \in \mathbb{C}$ and radius r. The unit disk $D_1(0)$ will be denoted $\mathbb{D}$. We use $|J|$ for the length of an interval J, dist and diam for the Euclidean distance and diameter in $\mathbb{C}$. The notation $[a, b]$ stands for the closed interval with endpoints a and b without specifying their order.

We call two real numbers a and b *K-commensurable* or simply *commensurable* if $K^{-1} \le |a|/|b| \le K$ for some $K > 1$. Two sets X and Y in $\mathbb{C}$ are K-commensurable if their diameters are. A configuration of points $x_1, \dots, x_n$ is called *K-bounded* if any two intervals $[x_i, x_j]$ and $[x_k, x_l]$ are K-commensurable. For a pair of intervals $I \subset J$ we say that I is contained *well inside of* J if there exists a universal constant K such that for each component L of $J \setminus I$ we have $|L| \ge K|I|$.

Let f be an analytic map which restricts to a self-homeomorphism of the circle T. We reserve the notation $f^{-i}(z)$ for the ith preimage of $z \in T$ under $f|_T$.

2.2. *Hyperbolic disks.* Given an interval $J \subset \mathbb{R}$, let $\mathbb{C}_J \equiv \mathbb{C}\setminus(\mathbb{R}\setminus J)$ denote the plane slit along two rays. Let $\overline{\mathbb{C}_J}$ denote the completion of this domain in the path metric in $\mathbb{C}_J$ (which means that we add to $\mathbb{C}_J$ the banks of the slits).

By symmetry, J is a hyperbolic geodesic in $\mathbb{C}_J$. Consider the *hyperbolic neighborhood* of J of radius r, that is, the set of all points in $\mathbb{C}_J$ whose hyperbolic distance to J is less than r. One verifies directly that a hyperbolic neighborhood is the union of two Euclidean discs symmetric to each other with respect to the real axis, with a common chord J. We will denote such a neighborhood $D_\theta(J)$, where $\theta = \theta(r)$ is the outer angle the boundaries of the two discs form with the real axis (an elementary computation yields

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$r = \log\tan(\pi/2 - \theta/4)$). Note that, in particular, the Euclidean disc $D(J) \equiv D_{\pi/2}(J)$ can be interpreted as a hyperbolic neighborhood of J. These hyperbolic neighborhoods were introduced into the subject by Sullivan [**S**]. They are a key tool for proving complex bounds due to the following version of the Schwarz lemma.

SCHWARZ LEMMA. *Let us consider two intervals $J' \subset J \subset \mathbb{R}$. Let $\phi : \mathbb{C}_J \to \mathbb{C}_{J'}$ be an analytic map such that $\phi(J) \subset J'$. Then for any $\theta \in (0, \pi)$, $\phi(D_\theta(J)) \subset D_\theta(J')$.*

Let $J = [a, b]$, $a < b$. For a point $z \in \overline{\mathbb{C}}_J$, the *angle between z and J*, $\widehat{(z, J)}$ is the least of the angles between the intervals $[a, z]$, $[b, z]$ and the corresponding rays $(a, -\infty]$, $[b, +\infty)$ of the real line, measured in the range $0 \le \theta \le \pi$.

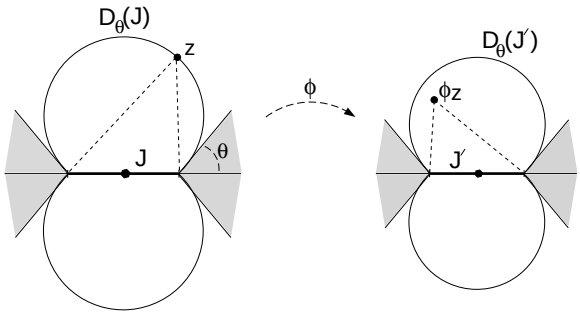


FIGURE 1.

The next observation will be used to control the expansion of the inverse branches.

LEMMA 2.1. *Under the assumptions of the Schwarz lemma, assume that ϕ admits a univalent extension $(\mathbb{C}_T, T) \to (\mathbb{C}_{T'}, T')$, where both components of $T \setminus J$ have length $2\rho|J|$. Let us consider a point $z \in \mathbb{C}_J$ such that $\widehat{(z, J)} \ge \epsilon$. Then*

$$\frac{\operatorname{dist}(\phi z, J')}{|J'|} \le C \frac{\operatorname{dist}(z, J)}{|J|}$$

for some constant $C = C(\rho, \epsilon)$.

Proof. Let us normalize the situation in this way: $J = J' = [0, 1]$. The space of univalent maps of $\mathbb{C}_T$ normalized at two points is compact in $\mathbb{C}_J$ by the Koebe theorem, thus the statement is true if $\operatorname{dist}(z, J) < \rho$. So assume $\operatorname{dist}(z, J) > \rho$.

Observe that the smallest (closed) hyperbolic neighborhood $\operatorname{cl} D_\theta(J)$ enclosing z satisfies $\operatorname{diam} D_\theta(J) \le C(\rho, \epsilon) \operatorname{dist}(z, J)$ (cf. Figure 1). Indeed, if $\theta \ge \epsilon/2$ then $\operatorname{diam} D_\theta(J) \le C(\epsilon)$, which is fine since $\operatorname{dist}(z, J) \ge \rho$.

Otherwise the straight segments $[0, z]$ and $[1, z]$ cut out arcs of angle size at least $\epsilon/2$ on the circle $\partial D_\theta(J)$. Hence the lengths of these segments are commensurable with $\operatorname{diam} D_\theta(J)$ (with a constant depending on ϵ). On the other hand, these lengths are at most $(1 + \rho^{-1}) \operatorname{dist}(z, J)$ provided that $\operatorname{dist}(z, J) \ge \rho$.

Together with the Schwarz lemma this yields

$$\operatorname{dist}(\phi z, J') \le \operatorname{diam}(D_\theta(J')) \le C(\rho, \epsilon) \operatorname{dist}(z, J)$$

and the claim follows. □

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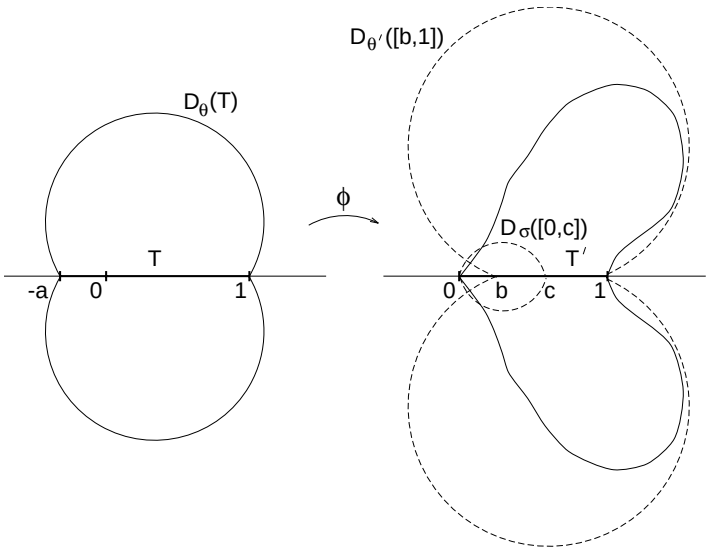


FIGURE 2.

2.3. *Cube root.* In the next lemma we collect for future reference some elementary properties of the cube root map. Let $\phi(z)$ be the branch of the cube root mapping the slit plane $\mathbb{C} \setminus \mathbb{R}_-$ into $\{z : |\arg(z)| < \pi/3\}$.

LEMMA 2.2. *Let $K > 1$, $\delta > 0$, $K^{-1} \le a \le K$, $T = [-a, 1]$, $T' = [0, 1]$. Then:*

(I) *$\phi D_\theta(T) \subset D_{\theta'}(T')$, with θ' depending on θ and K only;*

(II) *moreover, there exist $b, c \in [0, 1]$, such that $0, b, c, 1$ form a $C(K)$-bounded configuration, and $\phi D_\theta(T) \subset D_{\theta'}([b, 1]) \cup D_\sigma([0, c])$ for $\sigma < \pi/2$ (cf. Figure 2);*

(III) *if $z' \in \phi D(T) \setminus D([-\delta, 1 + \delta])$, then*

$$\widehat{(z', T')} > \epsilon(K, \delta) > 0 \quad \textit{and} \quad C(K, \delta)^{-1} < \operatorname{dist}(z', T') < C(K, \delta).$$

2.4. *Real commuting pairs and renormalization.* We present here a brief summary on renormalization of critical circle mappings. A more detailed exposition can be found, for example, in [**dF2**].

Let f be a critical circle mapping with rotation number $\rho(f)$. If $\rho(f)$ is irrational, it can be represented as an infinite continued fraction:

$$\rho(f) = [r_0, r_1, r_2, \dots] = \cfrac{1}{r_0 + \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cdots}}}$$

We say that ρ is of *bounded type* if $\sup r_i < \infty$, which is also equivalent to $\rho(f)$ being Diophantine of order 2.

For a critical circle map f denote by q_m the moments of closest return of the critical point 0. The numbers q_m appear as the denominators in the irreducible form of the

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mth truncated continued fraction expansion of $\rho(f)$, $p_m/q_m = [r_0, \dots, r_{m-1}]$. Set $I_m \equiv [0, f^{q_m}(0)]$. As a consequence of Świątek–Herman real *a priori* bounds [**Sw, H**], the intervals I_m and I_{m+1} are K-commensurable, with a universal constant K provided m is large enough.

The dynamical first return map of the union of intervals $I_m \cup I_{m+1}$ is given by f^{q_m} on I_{m+1} and by $f^{q_{m+1}}$ on I_m. The consideration of the pair of maps

$$(f^{q_{m+1}}|I_m,\, f^{q_m}|I_{m+1})$$

leads to the following general definition.

Definition 2.1. A *commuting pair* $\zeta = (\eta, \xi)$ consists of two smooth, orientation-preserving interval homeomorphisms $\eta : I_\eta \to \eta(I_\eta)$, $\xi : I_\xi \to \xi(I_\xi)$, where:

- $I_\eta = [0, \xi(0)]$, $I_\xi = [\eta(0), 0]$;
- both η and ξ have homeomorphic extensions to interval neighborhoods of their respective domains with the same degree of smoothness, which commute, i.e. $\eta \circ \xi = \xi \circ \eta$, where both sides are defined;
- $\xi \circ \eta(0) \in I_\eta$;
- $\eta'(x) \neq 0 \neq \xi'(y)$, for all $x \in I_\eta \setminus \{0\}$, and all $y \in I_\xi \setminus \{0\}$.

The notion of renormalization acting on the space of commuting pairs first appeared in a paper by Ostlund *et al* [**ORSS**], and was developed by Lanford, Rand and others (cf. [**La1, La2, Ra1, Ra2**]).

A *critical commuting pair* is a commuting pair (η, ξ), whose maps can be decomposed as $\eta = h_\eta \circ Q \circ H_\eta$, and $\xi = h_\xi \circ Q \circ H_\xi$, where $h_\eta, h_\xi, H_\eta, H_\xi$ are real analytic diffeomorphisms and $Q(x) = x^3$. Given a commuting pair $\zeta = (\eta, \xi)$ we will denote by $\widetilde{\zeta}$ the pair $(\tilde{\eta}|\widetilde{I}_\eta, \tilde{\xi}|\widetilde{I}_\xi)$ where tilde means rescaling by a linear factor $\lambda = 1/|I_\eta|$.

For a critical circle mapping f one obtains a critical commuting pair from the pair of maps $(f^{q_{m+1}}|I_m,\, f^{q_m}|I_{m+1})$ as follows. Let $\bar{f}$ be the lift of f to the real line satisfying $\bar{f}'(0) = 0$, and $0 < \bar{f}(0) < 1$. For each $m > 0$ let $\bar{I}_m \subset \mathbb{R}$ denote the closed interval adjacent to zero which projects down to the interval I_m. Let $\tau : \mathbb{R} \to \mathbb{R}$ denote the translation $x \mapsto x + 1$ and let $\eta : \bar{I}_m \to \mathbb{R}$, $\xi : \bar{I}_{m+1} \to \mathbb{R}$ be given by $\eta \equiv \tau^{-p_{m+1}} \circ \bar{f}^{q_{m+1}}$, $\xi \equiv \tau^{-p_m} \circ \bar{f}^{q_m}$. Then the pair of maps $(\eta|\bar{I}_m, \xi|\bar{I}_{m+1})$ forms a critical commuting pair corresponding to $(f^{q_{m+1}}|I_m,\, f^{q_m}|I_{m+1})$. Henceforth, we shall abuse notation and simply denote this commuting pair by

$$(f^{q_{m+1}}|I_m,\, f^{q_m}|I_{m+1}). \tag{2.1}$$

We see that return maps for critical circle homeomorphisms give rise to a sequence of critical commuting pairs (2.1). Conversely, given a critical commuting pair $\zeta = (\eta, \xi)$ we can regard the interval $I = [\eta(0), \xi \circ \eta(0)]$ as a circle, identifying $\eta(0)$ and $\xi \circ \eta(0)$ and define $f_\zeta : I \to I$ by

$$f_\zeta(x) = \begin{cases} \xi \circ \eta(x) m, & \text{if } x \in [\eta(0), 0] \\ \eta(x), & \text{if } x \in [0, \xi \circ \eta(0)]. \end{cases} \tag{2.2}$$

We perform the gluing together of $\eta(0)$ and $\xi \circ \eta(0)$ by the mapping ξ, which by the second condition in the definition of a commuting pair extends to a homeomorphism of open

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neighborhoods. The quotient of the interval I is a closed one-dimensional manifold M and the mapping f_ζ projects down to a smooth homeomorphism $F_\zeta : M \to M$. Identifying M with the circle by a diffeomorphism $\phi : M \to T$ we obtain a critical circle mapping $f^\phi = \phi \circ F_\zeta \circ \phi^{-1}$. The critical circle mappings corresponding to two different choices of ϕ are conjugated by a diffeomorphism, and thus we recover a smooth conjugacy class of circle mappings from a critical commuting pair. All mappings in the conjugacy class have the same rotation number which will be denoted $\rho(\zeta)$. We say that this number is the *rotation number of the commuting pair* $\zeta = (\eta, \xi)$.

If $\rho(\zeta) = [r, r_1, r_2, \dots]$, one verifies that the mappings $\eta|[0, \eta^r(\xi(0))]$ and $\eta^r \circ \xi|I_\xi$ again form a commuting pair.

Definition 2.2. The *renormalization* of a real commuting pair $\zeta = (\eta, \xi)$ is the commuting pair

$$\mathcal{R}\zeta = (\widetilde{\eta^r \circ \xi}|\widetilde{I}_\xi, \tilde{\eta}|[0, \widetilde{\eta^r(\xi(0))}]).$$

It is easy to see that renormalization acts as a Gauss map on rotation numbers; that is, if $\rho(\zeta) = [r, r_1, r_2, \dots]$ then $\rho(\mathcal{R}\zeta) = [r_1, r_2, \dots]$. The renormalization of the real commuting pair (2.1), associated to some critical circle map f, is the rescaled pair $(\widetilde{f^{q_{m+2}}}|\widetilde{I_{m+1}}, \widetilde{f^{q_{m+1}}}|\widetilde{I_{m+2}})$. Thus for a given critical circle map f the renormalization operator recovers the (rescaled) sequence of first return maps: $\{(\widetilde{f^{q_{i+1}}}|\widetilde{I}_i, \widetilde{f^{q_i}}|\widetilde{I_{i+1}})\}_{i=1}^\infty$.

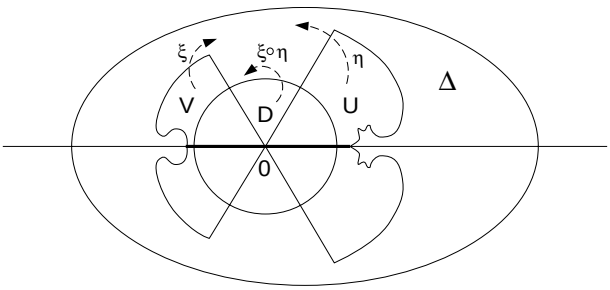


FIGURE 3.

2.5. *Holomorphic commuting pairs.* Following [**dF1, dF2**] we say that a real commuting pair (η, ξ) extends to a *holomorphic commuting pair* $\mathcal{H}$ (cf. Figure 3) if there exist four $\mathbb{R}$-symmetric domains Δ, D, U, V, such that:

- $\bar{D}, \bar{U}, \bar{V} \subset \Delta$, $\bar{U} \cap \bar{V} = \{0\}$; $U \setminus D$, $V \setminus D$, $D \setminus U$, and $D \setminus V$ are non-empty connected sets and $U \supset I_\eta$, $V \supset I_\xi$;
- mappings $\eta : U \to \Delta \cap \mathbb{C}_{\eta(J_U)}$ and $\xi : V \to \Delta \cap \mathbb{C}_{\xi(J_V)}$ are onto and univalent, where $J_U = U \cap \mathbb{R}$, $J_V = V \cap \mathbb{R}$;
- η and ξ have holomorphic extensions to D which commute, $\eta \circ \xi(z) = \xi \circ \eta(z)$ $\forall z \in D$; $\eta \circ \xi : D \to \Delta \cap \mathbb{C}_{\eta\circ\xi(J_D)}$, where $J_D = D \cap \mathbb{R}$, is a three-fold branched covering with the only critical point at 0.

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Note that $J_D = (\eta^{-1}(0), \xi^{-1}(0))$.

Set $\Omega = D \cup U \cup V$. The *shadow* of the holomorphic commuting pair is the piecewise holomorphic mapping $F : \Omega \to \Delta$ given by

$$F(z) = \begin{cases} \eta(z), & z \in U \\ \xi(z), & z \in V \\ \xi \circ \eta(z), & z \in D \setminus (U \cup V). \end{cases}$$

PROPOSITION 2.3. ([**dF2,** Proposition II.4.]) *Given a holomorphic commuting pair $\mathcal{H}$ as above, consider its shadow F. Let $I = \Omega \cap \mathbb{R}$, and $X = I \cup F^{-1}(I)$. Then:*

- *the restriction of F to $\Omega \setminus X$ is a regular three fold covering onto $\Delta \setminus \mathbb{R}$;*
- *F and $\mathcal{H}$ share the same orbits as sets.*

Definition 2.3 (Complex bounds). We say that a critical circle map f has *complex a priori bounds* if there exist M and $\mu > 0$, such that for all $m > M$ the real commuting pair (2.1) extends to a holomorphic commuting pair $(\Delta_m, D_m, U_m, V_m)$ with $\mod(\Delta_m \setminus (D_m \cup U_m \cup V_m)) > \mu > 0$.

2.6. *Epstein class.* An orientation-preserving interval homeomorphism $g : I = [0, a] \to g(I) = J$ belongs to the *Epstein class* $\mathcal{E}$ if it extends to an analytic three-fold branched covering map of a topological disk $G \supset I$ onto the double-slit plane $\mathbb{C}_{\tilde{J}}$, where $\tilde{J} \supset \operatorname{cl} J$. Any map g in the Epstein class can be decomposed as

$$g = Q_c \circ h, \tag{2.3}$$

where $Q_c(z) = z^3 + c$, and $h : I \to [0, b]$ is a univalent map $h : G \to \Delta(h)$ onto the complex plane with six slits, which triple covers $\mathbb{C}_{\tilde{J}}$ under the cubic map $Q_c(z)$.

We will equip the collection $\mathcal{E}$ with a notion of *Carathéodory convergence* (cf. [**McM1**]). Let $\{(U_n, u_n)\}$ be a sequence of open connected regions $U_n \subset \mathbb{C}$ with *marked points* $u_n \in U_n$. Recall that this sequence Carathéodory converges to a marked region (U, u) if:

- $u_n \to u \in U$; and
- for any Hausdorff limit point K of the sequence $\hat{\mathbb{C}} \setminus U_n$, U is a component of $\hat{\mathbb{C}} \setminus K$.

By a classical result of Carathéodory, the convergence of simply-connected regions $(U_n, u_n) \to (U, u)$ is equivalent to the locally uniform convergence of Riemann mappings $R_n : \mathbb{D} \to U_n$, normalized by $R_n(0) = u_n$, $R_n'(0) > 0$, to the Riemann mapping $R : \mathbb{D} \to U$ with the same normalization.

Let $g_n \in \mathcal{E}$ be a sequence of mappings in the Epstein class, with domains G_n as above. We shall say that this sequence Carathéodory converges to $g \in \mathcal{E}$ with domain G, if:

- $(G_n, 0) \to (G, 0)$ in the sense of Carathéodory convergence; and
- the maps $g_n \to g$ uniformly on compact subsets of G.

(The last condition makes sense since any compact subset of G is also contained in G_n for n sufficiently large).

For any $s \in (0, 1)$, let us introduce a smaller class $\mathcal{E}_s \subset \mathcal{E}$ of Epstein mappings $g : I = [0, a] \to J \subset \tilde{J}$ for which both $|I|$ and $\operatorname{dist}(I, J)$ are s^{-1}-commensurable

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with $|J|$, the length of each component of $\tilde{J} \setminus J$ is at least $s|J|$, and $g'(a) > s$. We will often refer to the space $\mathcal{E}$ as *the* Epstein class, and to each $\mathcal{E}_s$ as *an* Epstein class.

We say that a commuting pair (η, ξ) belongs to the (an) Epstein class if both of its maps do. The next lemma immediately follows from the definitions.

LEMMA 2.4. *The space of commuting pairs in the Epstein class $\mathcal{E}$ is invariant under renormalization.*

Let us make a note of an important compactness property of $\mathcal{E}_s$.

LEMMA 2.5. *Let $s \in (0, 1)$. The collection of normalized maps $g \in \mathcal{E}_s$ with $I = [0, 1]$ is sequentially compact with respect to Carathéodory convergence.*

Proof. Let $g_n : I = [0, 1] \to J_n \subset \tilde{J}_n$ be a normalized sequence in $\mathcal{E}_s$. These maps extend to triple branched coverings $g_n : G_n \to \mathbb{C}_{\tilde{J}_n}$, and can be decomposed (2.3) as $g_n = Q_n \circ h_n$, where $Q_n(z) = Q_{c_n}(z) = z^3 + c_n$, and $h_n : I \to [0, b]$ is a univalent map $G_n \to \Delta(h_n)$. By passing to a subsequence we can ensure that $\tilde{J}_n$ converge to an interval $\tilde{J}$, and $c_n \to c$. Then $(\Delta(h_n), 0)$ Carathédory converges to a slit domain $(\Delta, 0)$.

Since J_n is s^{-1}-commensurable with I_n, $h_n(1)$ is bounded. As $g_n'(1) > s > 0$, the derivatives $(h_n^{-1})'(h_n(1))$ are bounded from above. The points $h_n(1)$ stay away from the boundary of Δ, and it follows from the Koebe distortion theorem that $\{h_n^{-1}\}$ form a normal family in Δ.

Let us select a locally uniformly converging subsequence h_n^{-1}. Since $\{h_n(I)\}$ have bounded length, the limit of this sequence is non-constant, and therefore univalent, function $h^{-1} : \Delta \to G$. Let $R_n : \mathbb{D} \to G_n$ be the normalized Riemann map, $R_n(0) = 0$, $R_n'(0) > 0$. It can be decomposed as $R_n = h_n^{-1} \circ R_n'$, where R' is the normalized Riemann map $\mathbb{D} \to \Delta(h_n)$. By the above, the maps R_n converge locally uniformly to the Riemann map $R : \mathbb{D} \to G$, which is equivalent to Carathéodory convergence $(G_n, 0) \to (G, 0)$.

Finally, note that the convergence of h_n^{-1} implies that there is a point $c \in I$ where the derivatives of h_n are bounded from above. It follows from the Koebe theorem, that $\{h_n\}$ is a normal family in G, and hence $h_n \to h \equiv (h^{-1})^{-1}$. □

The above proof also yields the following.

LEMMA 2.6. *For any $s \in (0, 1)$, there exists a domain $O_s \supset [0, 1]$, such that for any $g \in \mathcal{E}_s$ with normalization $I = [0, 1]$, the univalent map h in (2.3) is well-defined and has $K(s)$-bounded distortion in O_s.*

We will further refer to the above property by saying that a map $g \in \mathcal{E}_s$ is *cubic up to bounded distortion*.

The following statement is a weakened version of the convergence to an Epstein class of the renormalizations of a smooth map, discussed in [**dFdM1**].

LEMMA 2.7. *Let $f \in C^r$ $(r \geq 3)$ be a critical circle map with an irrational rotation number. Then the collection of real commuting pairs $(\widetilde{f^{q_{m+1}}|I_m}, \widetilde{f^{q_m}|I_{m+1}})$ is precompact in the C^r topology, and all its partial limits are contained in $\mathcal{E}_s$, for some universal constant $s > 0$, independent of the original map f.*

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In particular, for a critical circle map $f \in \mathcal{E}$ there exists $\sigma > 0$ such that all its renormalizations are contained in $\mathcal{E}_\sigma$. Moreover, the constant σ can be chosen independent of f, after skipping the first few renormalizations.

In the remainder of this paper we will only work with maps in the Epstein class. It will be shown that a renormalization of such a map has an extension to a holomorphic commuting pair with modulus bound depending only on the Epstein constants of the few previous renormalizations and, thus, in the view of the above lemma, eventually universal. Therefore we obtain universal complex *a priori* bounds for critical circle maps. Note that even for maps of bounded type this improves the complex *a priori* bounds obtained by de Faria, which also depend on the rotation number of the map.

3. *The main lemma*

Let f be an Epstein critical circle map with an irrational rotation number. Its nth renormalization is the commuting pair (2.1) $(f^{q_n}|_{I_{n+1}}, f^{q_{n+1}}|_{I_n}) \in \mathcal{E}$. The map $f^{q_{n+1}}$ extends to a three-fold branched covering,

$$f^{q_{n+1}} : G_n \to \mathbb{C}_{f^{q_{n+1}}(I_n)}.$$

Let us decompose:

$$f^{q_{n+1}} = \psi_n \circ f, \tag{3.1}$$

where ψ_n is a univalent map from a neighborhood $U_n = f(G_n)$ of $f(I_n)$ onto $\mathbb{C}_{f^{q_{n+1}}(I_n)}$.

Denote by D_m the Euclidean disc with diameter $[f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)]$. Let $\sigma(m)$ be the smallest positive number such that $\mathcal{R}^i(f) \in \mathcal{E}_{\sigma(m)}$ for all $i \geq m$.

LEMMA 3.1. (MAIN LEMMA) *Fix integers $n > M > 0$, and let $z \in \mathbb{C}_{f^{q_{n+1}}(I_n)} \cap D_M$. Then the following estimate holds:*

$$\frac{\operatorname{dist}(\psi_n^{-1} z, f(I_n))}{|f(I_n)|} \leq C \frac{\operatorname{dist}(z, I_n)}{|I_n|}, \tag{3.2}$$

where ψ_n is the map from (3.1), and the constant $C = C(\sigma(M))$.

Thus the maps ψ_n^{-1} after appropriate rescaling (that is normalizing $|I_n| = |f(I_n)| = 1$) have at most linear growth. Note that if $\widehat{(z, I_n)} > \epsilon > 0$, then the inequality (3.2) follows directly from Lemma 2.1. Our strategy for proving Lemma 3.1 will be to monitor the inverse orbit of a point z together with the interval I_n until the conditions of Lemma 2.1 are satisfied.

In view of Lemma 2.7, for sufficiently large M the constant C in Lemma 3.1 becomes universal. In particular, this implies that for large M, $\psi_n^{-1}(D_M \cap \mathbb{C}_{f^{q_{n+1}}(I_n)})$ is contained in the domain where f^{-1} is a cube root up to bounded distortion (see Lemma 2.6). This yields the cubic estimate (1.1) announced in the introduction:

$$\begin{gathered}\frac{\operatorname{dist}(f^{q_{n+1}}(z), f^{q_{n+1}}(I_n))}{|f^{q_{n+1}}(I_n)|} + d \geq c \left(\frac{\operatorname{dist}(z, I_n)}{|I_n|} \right)^3, \\ f^{q_{n+1}}(z) \in D_M \cap \mathbb{C}_{f^{q_{n+1}}(I_n)},\end{gathered} \tag{3.3}$$

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with $c = c(\sigma(M))$, $d = d(\sigma(M))$ and thus universal for M sufficiently large. Now fixing M and choosing sufficiently large l (depending only on $\sigma(M)$), for $n = M + l$ we have $\Omega_n = f^{-q_{n+1}}(D_l) \cap G_n$ compactly contained in G_n and $\mod(D_l \setminus \Omega_n) > \mu = \mu(\sigma(M)) > 0$, and Theorem 1.1 follows.

For future reference let us note the following.

Remark 3.1. It follows from our argument that the domains Δ_m, D_m, U_m, and V_m in the definition of complex bounds (2.3) can be chosen K-commensurable with I_m, with a universal constant K, for all m sufficiently large.

4. *Proof of the main lemma*

We begin by introducing some notation. For a map $g : U \to \mathbb{C}_T$ in the Epstein class and $x \in \mathbb{R} \cap U$ which is not a critical point of g^n, let $V_n(x, g)$ denote the maximal domain containing x which is univalently mapped by g^n to $\mathbb{C}_T$. Let $g_x^{-n} : \mathbb{C}_T \to V_n(x, f)$ denote the corresponding inverse branch of g^{-n} (continuous up to the boundary of the slits, with different values on different banks). Take an $x \in \mathbb{R}$, and $z \in \mathbb{C}_T$. If we have a backward orbit of $x \equiv x_0, x_{-1}, \ldots, x_{-l}$ which does not contain 0, the *corresponding* backward orbit $z \equiv z_0, z_{-1}, \ldots, z_{-l}$ is obtained by applying the appropriate inverse branches: $z_{-i} = g_{x_{-i}}^{-1}(z_{-(i-1)})$. The same terminology is applied if we have a monotone pull-back of an interval $J \equiv J_0, J_{-1}, \ldots, J_{-l}$.

Let us formulate a straightforward generalization of Lemma 2.2(III) to maps in an Epstein class.

LEMMA 4.1. *Fix $\sigma \in (\pi/3, \pi/2)$, $K > 1$ and let $\theta \in (0, \pi)$. Let $g : I \to J \subset \tilde{J}$ be a map in $\mathcal{E}_s$ for some $s \in (0, 1)$. Let $H \subset \tilde{J}$ be an interval such that both components of $\tilde{J} \setminus H$ have length at least $K^{-1}|H|$, and the critical value $g(0)$ divides H into K-commensurable parts.*

Denote by ψ the $\mathbb{R}$-symmetric univalent branch of g^{-1} defined in $\mathbb{C}_{\tilde{J}} \setminus [g(0), +\infty)$. Set $\psi(H) \cap \mathbb{R} = [r, 0)$. Then there exist points a_1, a_2 in $[r, 0]$ such that $0, r, a_1$ and a_2 form a C-bounded configuration and

$$\psi(D_\theta(H)) \subset D_{\theta'}([r, a_1]) \cup D_\sigma([a_2, 0]),$$

where the constants C and θ' depend only on K and s.

Let $f \in \mathcal{E}$ be a critical circle map. Fix an integer $M > 0$ and $n \geq M$. Recall from the previous section, that D_m denotes the Euclidean disc $D([f^{q_{m+1}}(0), f^{q_m - q_{m+1}}(0)])$. Consider the inverse orbit:

$$J_0 \equiv f^{q_{n+1}}(I_n), J_{-1} \equiv f^{q_{n+1}-1}(I_n), \ldots, J_{-(q_{n+1}-1)} \equiv f(I_n). \tag{4.1}$$

For a point $z \in D_M \cap \mathbb{C}_{f^{q_{n+1}}(I_n)}$ look at the corresponding inverse orbit

$$z_0 \equiv z, z_{-1}, \ldots, z_{-(q_{n+1}-1)}. \tag{4.2}$$

We say that an element of the orbit z_{-i} *ϵ-jumps* if $\widehat{(z_{-i}, J_{-i})} > \epsilon$. Let us say that $-i$ is a *good moment* if the interval J_{-i} is commensurable with J_0 with a quantifier depending only on $\sigma(M)$. It follows from the compactness of an Epstein class that the first few

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returns of the orbit (4.1) to each I_m with $m \geq M$ happen at good moments. We would like to argue that the points of the orbit (4.2) either ϵ-jump at a good moment, or follow closely the corresponding intervals of the orbit (4.1). The first step towards this assertion is the following lemma.

LEMMA 4.2. *Let $J \equiv J_{-k}$, $J_{-k-q_{m+1}} \equiv J'$ be two consecutive returns of the backward orbit (4.1) to I_m, for $m \geq M$, and let ζ and ζ' be the corresponding points of the orbit (4.2). Suppose $\zeta \in D_m$. Then either $\zeta' \in D_m$, or $\widehat{(\zeta', J')} > \epsilon$, and* $\operatorname{dist}(\zeta', J') < C|I_m|$, *where the quantifiers ϵ and C depend only on the Epstein constant $\sigma(M)$.*

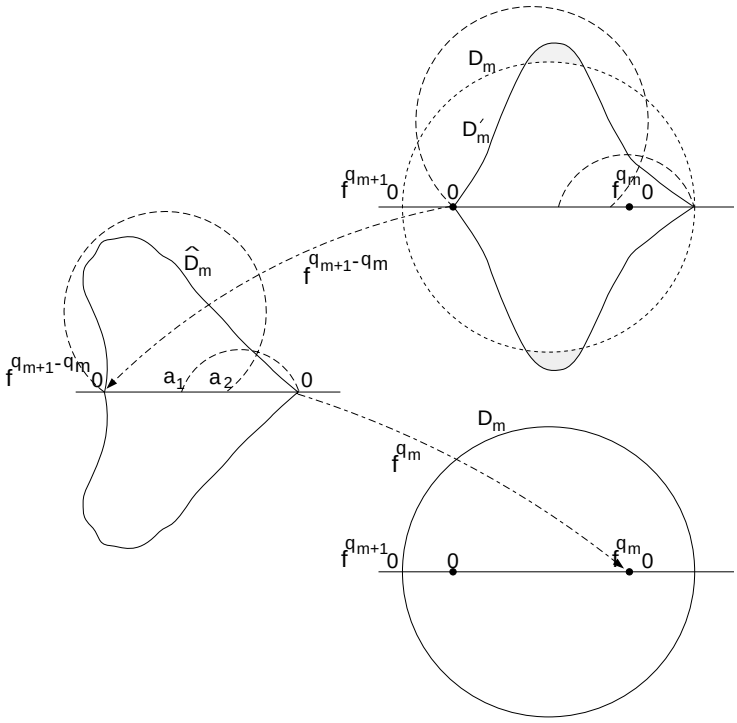


FIGURE 4.

Proof. Let D'_m denote the pull-back of D_m corresponding to the piece of backward orbit $J_{-k}, \ldots, J_{-k-q_{m+1}}$, and let $\hat{D}_m$ denote the pull-back of D_m along the piece of the orbit $J_{-k} \to \cdots \to J_{-k-q_m}$ (cf. Figure 4). By the compactness of an Epstein class, the points 0 and $f^{q_m}(0)$ divide the interval $D_m \cap \mathbb{R}$ into $B(\sigma(M))$-commensurable pieces. By the Schwarz lemma and by Lemma 4.1, there exist points $a_1, a_2 \in [f^{q_{m+1}-q_m}(0), 0]$, such that $f^{q_{m+1}-q_m}(0), a_1, a_2$, and 0 form a $K = K(\sigma(M))$-bounded configuration, and an angle θ depending only on $\sigma(M)$, for which $\hat{D}_m \subset D_\theta([f^{q_{m+1}-q_m}(0), a_1]) \cup D_\sigma([a_2, 0])$, where $\sigma < \pi/2$. Applying the Schwarz lemma, we obtain that $D'_m \subset D_m \cup D_\theta([0, f^{-q_{m+1}+q_m}(a_1)])$ and the claim immediately follows. □

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A saddle-node phenomenon. The above lemma puts us in a position to apply Lemma 2.1 if the inverse orbit (4.2) ϵ-jumps during one of the first few (or the last few) returns of (4.1) to some I_m. There is a danger, however, that when q_{m+1}/q_m is large, some of the returns of the inverse orbit of J_0 to I_m will be small compared to the original size. If the orbit (4.2) ϵ-jumps at such a bad moment $-i$, although in view of the previous lemma $\operatorname{dist}(z_{-i}, J_{-i}) < C|I_m|$, the ratio $\operatorname{dist}(z_{-i}, J_{-i})/|J_{-i}|$ may become unbounded.

The situation when q_{m+1}/q_m is large is, however, well understood (see, for example, [**H**]). By Lemma 2.7, the family of rescaled restrictions $\{\widetilde{g_j^{q_{m_j+1}}}|\widetilde{I_{m_j}}\}$ of Epstein circle maps, for which $g^{q_{m+1}}$ belongs to a fixed $\mathcal{E}_s$, is pre-compact in the sense of Carathéodory convergence. Any partial limit of a sequence of maps in this family with $q_{m+1}/q_m \to \infty$ has a fixed point in the interval. Since this fixed point is destroyed by a small perturbation, it is necessarily parabolic, and by the Schwarz lemma it is unique.

Let us now handle the possibility of a jump at a bad moment.

LEMMA 4.3. *Let us consider the map $f^{q_{m+1}}|I_m$. Let $P_0, P_{-1}, \dots, P_{-k}$ be the consecutive returns of the backward orbit (4.1) to I_m, and denote by $\zeta_0, \dots, \zeta_{-k}$ the corresponding moments of the backward orbit of a point $\zeta_0 = z \in D_m$. Then either $z' \equiv \zeta_{-k} \in D_m$, or $\widehat{(z', P_{-k})} > \epsilon$ and $\operatorname{dist}(z', P_{-k}) \le C|I_m|$, where $C = C(\sigma(M))$.*

Proof. To be definite, let us assume that the intervals P_{-i} lie on the left of 0. Without loss of generality we can assume that $z \in \mathbb{H}$. Let $\phi = f^{-q_{m+1}}$ be the branch of the inverse for which $\phi P_{-i} = P_{-(i+1)}$. As ϕ is orientation preserving on $(-\infty, f^{q_{m+1}+q_m}(0)]$, it maps the upper half-plane $\mathbb{H}$ into itself: $\phi(\mathbb{H}) \subset \{z = re^{i\theta} \mid r > 0, \pi > \theta > 2\pi/3\}$. By the Denjoy–Wolf theorem, ϕ has an attracting fixed point $\eta_\phi \in \mathbb{H}$, whose basin is the whole of $\mathbb{H}$. As we have seen, for q_{m+1}/q_m large enough the map ϕ is a small perturbation of a parabolic map. A compactness argument implies that there exists $B = B(\sigma(M))$ such that the fixed point $\eta_\phi \in D(I_m) \subset D_m$, provided $q_{m+1}/q_m > B$.

If $q_{m+1}/q_m < N$ the lemma readily follows from Lemmas 4.2 and 2.1. Otherwise, by the Denjoy–Wolf theorem the iterates of ϕ uniformly converge to η_ϕ on any compact subset $K \Subset \mathbb{H}$. By compactness, there exists $N = N(K, \sigma(M))$ such that $\phi^N(K) \subset D_m$.

Suppose $\zeta_{-i} \notin D_m$. By Lemma 4.2 the set $(D_m \setminus \phi(D_m)) \cap \mathbb{H}$ is contained in a compact region $K = K(\sigma(M)) \Subset \mathbb{H}$. For N as above we have $z' \in \cup_{i=0}^{N-1} \phi^i(K) \cup D_m$, and the lemma is proved. □

The inductive step. The next lemma provides a step of induction for our argument.

LEMMA 4.4. *Let J be the last return of the backward orbit (4.1) to the interval I_m before the first return to I_{m+1}, and let J' and J'' be the first two returns of (4.1) to I_{m+1}. Let ζ, ζ', ζ'' be the corresponding moments in the backward orbit (4.2), $\zeta = f^{q_m}(\zeta')$, $\zeta' = f^{q_{m+2}}(\zeta'')$.*

Suppose $\zeta \in D_m$. Then either $\widehat{(\zeta'', I_{m+1})} > \epsilon(\sigma(M))$ and $\operatorname{dist}(\zeta'', J'') < C(\sigma(M))|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

Proof. Note that $J \subset [f^{q_m+q_{m+1}}(0), f^{q_m}(0)]$. By Lemma 2.2, $\zeta' \in D_\theta([f^{q_{m+1}-q_m}(0), 0])$ for some θ depending only on the Epstein constants. Denote by $\tilde{J}$, and $\hat{J}$ the intervals

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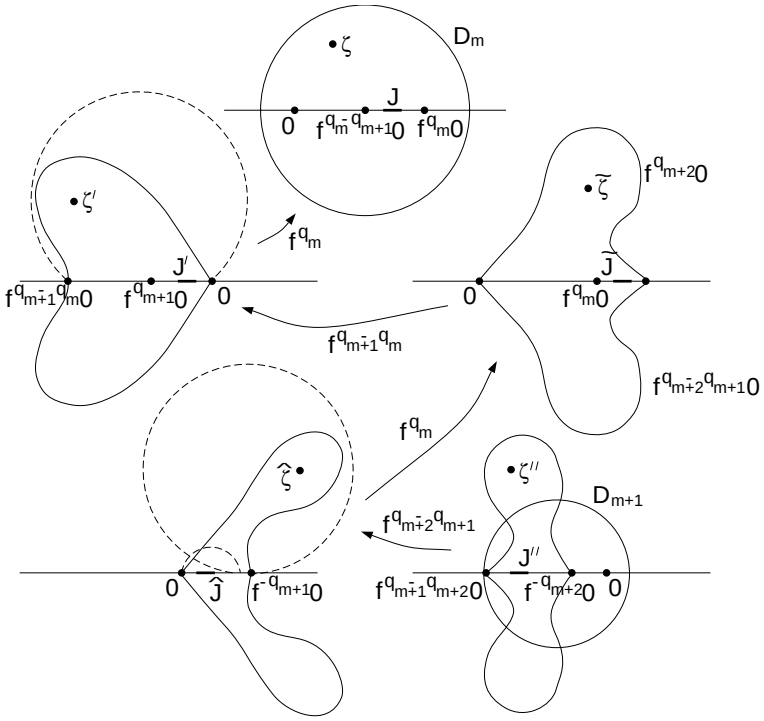


FIGURE 5.

of (4.1), such that $f^{q_{m+1}-q_m}(\tilde{J}) = J'$, $f^{q_m}(\hat{J}) = \tilde{J}$, and let $\tilde{\zeta}$, $\hat{\zeta}$ be the corresponding points in the orbit (4.2) (cf. Figure 5). The interval $\tilde{J} \subset [f^{q_m}(0), f^{q_m-q_{m+1}}(0)]$, $\tilde{\zeta} \in D_\theta([f^{q_m}(0), f^{q_m-q_{m+1}}(0)])$. By the Schwarz lemma and Lemma 4.1 there are points $b_1, b_2 \in [0, f^{-q_{m+1}}(0)]$ such that $0, b_1, b_2$, and $f^{-q_{m+1}}(0)$ form a $K(\sigma(M))$-bounded configuration, and $\hat{\zeta} \in D_\delta([0, b_1]) \cup D_\gamma([b_2, f^{-q_{m+1}}(0)])$ for $\gamma = \gamma(\sigma(M))$, and $\delta < \pi/2$. The claim now follows by the Schwarz lemma. □

Inductive argument. We start with a point $z \in D_M$. Consider the largest m such that D_m contains z. We will carry out induction on m. Let $P_0, \dots, P_{-k}$ be the consecutive returns of the backward orbit (4.1) to the interval I_m until the first return to I_{m+1}, and denote by $z = \zeta_0, \dots, \zeta_{-k} = \zeta'$ the corresponding points of the orbit (4.2). By Lemmas 4.2 and 4.3, ζ_{-i} either ϵ-jumps at a good moment when P_{-i} is commensurable with J_0, and $\operatorname{dist}(\zeta_{-i}, P_{-i}) \le C|I_m|$, or $\zeta' \in D_m$.

In the former case we are done by Lemma 2.1. In the latter case consider the point ζ'' which corresponds to the second return of the orbit (4.1) to I_{m+1}. By Lemma 4.4, either $\widehat{(\zeta'', I_{m+1})} > \epsilon$ and $\operatorname{dist}(\zeta'', I_{m+1}) \le C|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

In the first case we are done again by Lemma 2.1. In the second case, the argument is completed by induction on m.

Let us make several remarks.

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Remark 4.1. It follows directly from the argument, that every point $z \in f^{-q_{n+1}}(D_M) \cap G_n$ is either contained in the disc $D([0, f^{q_n - q_{n-1}}(0)])$ or satisfies $\widehat{(z, I_n)} > \epsilon = \epsilon(\sigma(M))$.

Remark 4.2. In Lemma 3.1 and the estimate (3.3) the domain D_M can be replaced with $D_\alpha([f^{q_{M+1}}(0), f^{q_M - q_{M+1}}(0)])$ for some $\alpha > \pi/2$, following an obvious change in the argument.

5. *Construction of a quasiconformal conjugacy*

We proceed to prove Theorem 1.2. Let f_1, f_2 be two critical circle mappings in the Epstein class, having irrational rotation numbers. We will use the subscript i to denote the objects corresponding to the mapping f_i for $i = 1, 2$. By choosing M sufficiently large, we can ensure that all renormalizations $\mathcal{R}^n f_i$ for $n \geq M$ are contained in an Epstein class $\mathcal{E}_s$ with a universal $s > 0$. Furthermore, in view of Remark 3.1, there exists a universal constant $N > 0$ such that the real commuting pair $(f_i^{q_{n+1}}, f_i^{q_n})$ extends to a holomorphic commuting pair $\mathcal{H}_i^n$ with complex *a priori* bounds (2.3), ranging over $\Delta_i^n \equiv D([f_i^{q_{n-N+1}}(0), f_i^{q_{n-N}}(0)])$, provided $n - N \geq M$. We will denote the shadow of this holomorphic pair, as defined in §2.5, by $F_i^n : \Omega_i^n \to \Delta_i^n$. The precise formulation of the statement of Theorem 1.2 is the following.

Let n be as above, and assume that the pairs $(f_i^{q_{n-1}}, f_i^{q_{n-2}})$ have the same irrational rotation number. Then there exists a K-quasiconformal $\mathbb{R}$-symmetric homeomorphism $H^n : \Delta_1^n \to \Delta_2^n$, conjugating the piecewise-holomorphic maps F_1^n and F_2^n, with a universal dilatation bound K.

By a theorem of Herman [**H**] there exists a K_1-quasisymmetric map

$$h : [f_1^{q_{n-N+1}}(0), f_1^{q_{n-N}}(0)] \to [f_2^{q_{n-N+1}}(0), f_2^{q_{n-N}}(0)]$$

conjugating the real commuting pairs $(f_1^{q_{n+1}}, f_1^{q_n})$ and $(f_2^{q_{n+1}}, f_2^{q_n})$, with a universal bound on K_1. We will show that this map can be extended to a quasiconformal conjugacy with the required properties.

As we work with a fixed value of n, let us omit the superscript n where possible, to simplify the notation. The mapping h can be extended to an $\mathbb{R}$-symmetric quasiconformal homeomorphism $\psi : \Delta_1 \to \Delta_2$ whose dilatation bound depends only on K_1 and therefore is universal. As follows from Proposition 2.3, the map ψ can be lifted to an $\mathbb{R}$-symmetric mapping $\tilde{\psi} : \Omega_1 \to \Omega_2$, such that $F_2 \circ \tilde{\psi} = \psi \circ F_1$, which is still an extension of h. Theorem 1.2 will follow via the standard pull-back argument (as presented in [**dF2,** pp. 15–17]) from the following statement.

PROPOSITION 5.1. (Quasiconformal interpolation) *There exists a K-quasiconformal mapping $H_1(z) : \Delta_1 \to \Delta_2$, with a universal K, satisfying the following:*

$$H_1(z) = \begin{cases} \psi(z), & \text{if } z \in \partial\Delta_1 \\ \tilde{\psi}(z), & \text{if } z \in \Omega_1. \end{cases}$$

Let us comment on the nature of the difficulties we are faced with in proving this proposition. First recall that in the case of quadratic-like maps, complex *a priori* bounds

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imply the existence of a fundamental annulus with modulus bounded below, both of whose boundary curves are quasicircles (see, for example, [**McM1**]). The existence of a quasiconformal interpolation between two such annuli is standard.

Consider, however, a maximal extension $\mathcal{H} : \Omega \to \Delta$ of a real commuting pair $(f^{q_{m+1}}|I_m, f^{q_m}|I_{m+1})$. It is easy to see that the intersection $\Omega \cap \mathbb{R}$ is the interval $T = [f^{q_{m-1}-q_m}(0), f^{q_m-q_{m+1}}(0)]$, and its image $T' = [f^{q_{m-1}}(0), f^{q_m}(0)] \subset T$. Thus on the real line we see exactly the opposite to what happens in the complex plane: a region which is compactly contained in its preimage. This precludes us from finding a fundamental annulus for a holomorphic commuting pair.

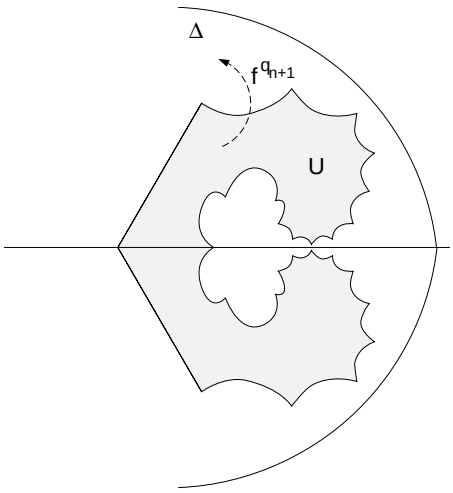


FIGURE 6. The domains Δ and U when q_{n+1}/q_n is large.

Instead, we should construct the quasiconformal interpolation in the fundamental regions $(\Delta_i \setminus \Omega_i) \cap \pm\mathbb{H}$. This does not create a problem in the case of ρ of bounded type, as in this case these regions are $K(B)$-quasidiscs, where B is the bound on q_{n+1}/q_n (see [**dF1, dF2**]). As seen before, if we select the rotation number ρ, for which the ratio q_{n+1}/q_n is large, the restriction $f_i^{q_n}|I_{n-1}$ is a small perturbation of a parabolic map. Following the logic of Lemma 4.3, we see that for the corresponding holomorphic pair, the domain U_i is pinched in a neighborhood of the ghost of the parabolic point (see Figure 6). Thus the bound on the quasicircle $(\Delta_i \setminus \Omega_i) \cap \pm\mathbb{H}$ spoils, and we require a new argument to show the existence of the quasiconformal interpolation.

We will establish Proposition 5.1 by analysing in some detail the geometry of domains Ω_i and showing that the pinching occurs in the same way for all such domains.

Some auxiliary lemmas. Let $f \in \mathcal{E}$ be a critical circle mapping. By the main result of §4, the pair of maps $(f^{q_n+1}|I_n, f^{q_n}|I_{n+1})$ extends to a holomorphic commuting pair $\mathcal{H}$ with domains Δ, D, U, and V, commensurable with I_n, and having complex bounds for all n sufficiently large.

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Consider the inverse orbit:

$$J_0 \equiv f^{q_{n+1}}(I_n),\ J_{-1} \equiv f^{q_{n+1}-1}(I_n),\ \dots,\ J_{-(q_{n+1}-1)} \equiv f(I_n), \tag{5.1}$$

and the corresponding inverse orbit for the domain $\Delta_0 \equiv \Delta \cap \mathbb{H}$:

$$\Delta_0,\ \Delta_{-1},\ \dots,\ \Delta_{q_{n+1}-(-1)} \equiv f(U) \cap \mathbb{H}. \tag{5.2}$$

We make several observations.

LEMMA 5.2. *Let $J' \equiv J_{-k}$, $J'' \equiv J_{-k-q_m}$ be two consecutive returns of the backward orbit (5.1) to the interval I_{m-1} before the first return to I_m, and denote by Δ', Δ'' the corresponding moments in the orbit (5.2). Then*

$$f^{q_m}(\partial\Delta'' \cap \mathbb{H}) = (\partial\Delta' \cap \mathbb{H}) \cup [f^{q_{m-1}-q_m}(0),\ f^{q_{m-1}}(0)].$$

The following two lemmas are illustrated by Figure 7.

LEMMA 5.3. *Let $J' = J_{-q_{m+1}}$ and $J'' = J_{-2q_{m+1}}$ be the first two returns of the backward orbit (5.1) to the interval I_m, and let Δ', Δ'' be the corresponding domains in the orbit (5.2). Then for any $z \in \Delta''$,*

$$\widehat{(z,\ P)} > \epsilon,$$

for some fixed angle $\epsilon > 0$, where $P = [f^{-q_{m+1}}(0),\ f^{q_m-q_{m+1}}(0)] = \Delta'' \cap \mathbb{R}$.

Proof. It follows from Remark 4.1 that the domain Δ' is contained in $W = D_\theta([0,\ f^{q_m-q_{m-1}}(0)])$ for some fixed θ. Note that the smallest iterate of the critical point 0 contained in W is $f^{q_n(0)}$. The claim now follows from the Schwarz lemma and Lemma 2.2. □

LEMMA 5.4. *Let $J = J_{-q_{m+1}+q_{m-1}}$ be the last return of the backward orbit (5.1) to the interval I_{m-1} before the first return to I_m, and denote again by J' and J'' the first two returns of the orbit (5.1) to I_m. Let Δ, Δ' and Δ'' be the corresponding domains in the orbit (5.2). Let $\zeta'' \in \partial\Delta''$, and let $f^{q_{m+1}}(\zeta'') = \zeta' \in \partial\Delta'$, $f^{q_{m-1}}(\zeta') = \zeta \in \partial\Delta$.*

Suppose, that $\zeta \in \mathbb{H}$. Then,

$$\operatorname{dist}(\zeta'', I_m) > K|I_m|.$$

Proof. Note, that the disk $D([f^{q_{m+1}-q_m}(0),\ f^{q_m}(0)])$ is pulled back univalently along the piece of the inverse orbit (5.1)

$$J_0,\ J_{-1},\ \dots,\ J_{-q_{m+1}+q_{m-1}} = J.$$

By the Świątek–Herman real *a priori* bounds, we can choose an interval

$$[f^{q_{m+1}}(0),\ f^{q_m+q_{m+1}}(0)] \subset T_0 \subset [f^{q_{m+1}-q_m}(0),\ f^{q_m}(0)]$$

such that each of the intervals is contained well inside the next one. Let T and $D \subset \Delta$ denote the pullbacks of the interval T_0 and the disk $D(T_0)$ correspondingly along the orbit $J_0, J_{-1}, \dots, J_{-q_{m+1}+q_{m-1}} = J$. By the Koebe distortion theorem, $D \supset D_\theta(T)$ for some fixed value of θ. Thus, if $\widehat{(\zeta, [f^{q_{m-1}}(0),\ f^{q_m+q_{m-1}}(0)])} > \epsilon$,

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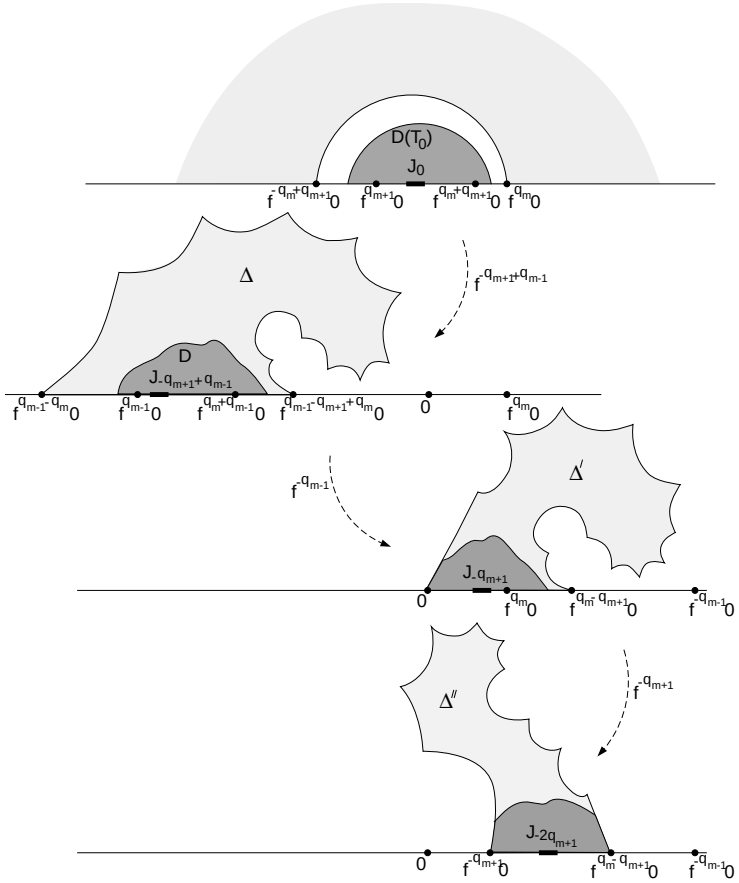


FIGURE 7.

then $\operatorname{dist}(\zeta, [f^{q_{m-1}}(0),\ f^{q_m+q_{m-1}}(0)]) > K|I_{m-1}|$, and the claim follows by the Koebe distortion theorem.

Otherwise, the point ζ is contained in an ϵ-wedge, with the vertex at an end point of the interval $[f^{q_{m-1}}(0),\ f^{q_m+q_{m-1}}(0)]$. Assume first that ζ is contained in the wedge attached to $f^{q_{m-1}}(0)$. Then ζ' is contained in a triangle with angle $\pi/3$ at 0, at a distance $K|I_m|$ from the interval I_m. The claim follows from the Koebe distortion theorem and the elementary properties of the cube root map.

In the other case, ζ' is contained in an ϵ'-wedge at the point $f^{q_m}(0)$, at a distance $C|I_m|$ from the point $f^{q_m}(0)$. The lemma follows for the same reason. □

Perturbations of parabolic maps. As we know, when q_{m+1}/q_m is large, the restriction $f^{q_m}|I_{m-1}$ is a small perturbation of a parabolic map. We summarize here several facts about such mappings. A detailed account may be found, for example, in [**Sh**].

Consider the consecutive returns of the backward orbit of 0 to the interval I_{m-1} before

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the first return to I_m,

$$f^{-q_m}(0),\ f^{-2q_m}(0),\ \dots,\ f^{q_{m-1}-q_{m+1}+q_m}(0),\ f^{q_{m-1}-q_{m+1}}(0).$$

For a bounded value of s, the interval $[f^{q_{m-1}-q_{m+1}+sq_m}(0),\ f^{-sq_m}(0)] \subset I_{m-1}$ is $K = K(s)$-commensurable with I_{m-1}.

LEMMA 5.5. (Perturbed Fatou coordinate) *There exists a holomorphic $\mathbb{R}$-symmetric mapping Ψ from a neighborhood $W \supset [f^{q_{m-1}-q_{m+1}+sq_m}(0),\ f^{-sq_m}(0)]$ to a vertical strip $\{0 < \Re(z) < (q_{m+1}-q_{m-1})/q_m - 2s\}$, conjugating f^{-q_m} to the unit translation $z \mapsto z+1$.*

By a compactness argument, the constant s in the above statement can be chosen independent of f.

A direct transcription of the proof of Lemma 4.3 yields the following statement.

LEMMA 5.6. *Consider the sequence of returns of the backward orbit (5.1) to the interval I_{m-1} until the first return to I_m, $J_{-q_m}, J_{-2q_m}, \dots, J_{-l_m q_m}$, and let $\zeta_{-1}, \dots, \zeta_{-l_m}$ be the corresponding inverse orbit of a point $\zeta_0 \in \mathbb{H}$ under the mapping f^{q_m}.*

Let $\operatorname{dist}(\zeta_{-2}, J_{-2q_m})$ *be K-commensurable with I_{m-1} and $\angle(\zeta_{-2}, J_{-2q_m}) > \epsilon$. Then $\zeta_{-s} \in W$ for some s depending only on K and ϵ.*

Now consider a curve $t \subset \mathbb{C}$ such that for every $z \in t$, $\operatorname{dist}(z, J_{-2q_m})$ is K-commensurable with I_{m-1} and $\angle(z, J_{-2q_m}) > \epsilon$. Let $T = J_{-q_m} \cup J_{-2q_m} \cup J_{-3q_m}$. There exists an annulus $A \subset \mathbb{C}_T$ with $\operatorname{mod} A > \mu > 0$, containing the curve t together with the interval J_{-2q_m}. The domain $\mathbb{C}_T$ can be pulled back univalently along the inverse orbit $J_{-2q_m}, J_{-2q_m-1}, \dots, J_{-(l_m-3)q_m}$. Applying the Koebe distortion theorem, and noting that the intervals J_{-2q_m} and $J_{-(l_m-3)q_m}$ are commensurable, we obtain the following corollary.

COROLLARY 5.7. *Let a curve $t \subset \mathbb{C}$ be as above. Denote by t' the pull-back of the curve t along the backward orbit $J_{-2q_m}, J_{-2q_m-1}, \dots, J_{-l_m q_m}$. Then* $\operatorname{dist}(t', I_{m-1}) > K|I_{m-1}|$, $\angle(\zeta, I_{m-1}) > \delta(\epsilon) > 0$, *for all $\zeta \in t'$, and the curves t and t' have commensurable lengths.*

The shape of the domain of a holomorphic pair. Let $L \subset \mathbb{C}$ be a rectifiable simple closed curve. For any two points z_1, z_2 contained in L denote by $l(z_1, z_2)$ the length of the shorter subarc of L connecting z_1 and z_2. We say that the curve L is of *bounded* (by K) *turning*, if there exists $K > 1$, such that $l(z_1, z_2) < K \operatorname{dist}(z_1, z_2)$, for all pairs of points z_1, z_2. We recall the following fact [**LV,** Theorem 8.6]:

There exists an increasing function $B(K) > 1$, such that a Jordan curve of K-bounded turning is a $B(K)$-quasicircle.

We will further simply say 'bounded turning' implying the existence of some universal constant K. Consider the consecutive returns of the orbit (5.1) to I_{m-1} before the first return to I_m,

$$J_{-q_m}, J_{-2q_m}, \dots, J_{-l_m q_m}. \tag{5.3}$$

Consider the curve segment $\gamma^m \subset f^{-q_m}([f^{q_{m-1}-q_m}(0),\ f^{q_{m-1}}(0)]) \cap \mathbb{H}$, $\operatorname{cl}(\gamma^m) \ni f^{q_{m-1}-q_m}(0)$, and let

$$\gamma^m \equiv \gamma^m_{-1}, \gamma^m_{-2}, \dots, \gamma^m_{-l_m} \tag{5.4}$$

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be the corresponding inverse orbit of the curve segment γ^m under f^{q_m}. Let the curve Γ^m be the union of the segments γ_i^m, for $i = -1, \dots, -l_m$.

LEMMA 5.8. *The curve Γ^m is of bounded turning.*

Proof. Let Ψ_m, as in Lemma 5.5, be the perturbed Fatou coordinate for the mapping f^{-q_m}. The curve segments $\Psi_m(\gamma_i^m)$ are obtained by a horizontal translation of the segment γ^m, and therefore the whole curve $\Psi_m(\Gamma^m)$ is of bounded turning. An explicit estimate can be given for the mapping Ψ_m (see [**Sh,** pp. 19, 30]) which shows that it is a small perturbation of a Möbius map. Thus the curve Γ^m itself is of bounded turning. □

LEMMA 5.9. *There exists a topological disk $\hat{D} \subset D([f^{sq_n - q_{n+1}}(0), f^{-q_{n-1}-sq_n}(0)])$ commensurable with I_{n-1}, such that:*

- *the domain $U \setminus \hat{D}$ is a K-quasidisk for some fixed K;*
- *the intersection $\partial U \cap \hat{D} \cap \mathbb{H}$ is contained in the curve $f^{-(q_{n-1}-1)}(\Gamma^n)$, where Γ^n is as above.*

Proof. The proof of the lemma consists of several steps. First we partition the boundary of the domain U into a finite number of curves with bounded turning, attached to each other at angles $\pi/3$. Each of these curves is 'created' between the moments of the first return of the inverse orbit (5.1) respectively to I_{m-1} and to I_m, for some m. We proceed to show that after we cut out a piece of the boundary of U as in the statement of the lemma, the union of the remaining curves is of bounded turning.

So let $\zeta \equiv \zeta_{-p} \in \partial U \cap \mathbb{H}$, and consider the forward orbit:

$$\zeta_{-(p-1)} \equiv f(\zeta), \zeta_{-(p-2)}, \dots, \zeta_0 \equiv f^p(\zeta).$$

Take the smallest $k = k(\zeta)$ such that $\zeta_{-k} \in \mathbb{H}$. Let the curve $t_m^+ \subset \partial U \cap \mathbb{H}$ consist of all ζ, such that the interval J_{-k} is contained in the backward orbit (5.1) between the first return to the interval I_{m-1} and the first return to I_m. Let t_m^- denote the corresponding curve in the lower half-plane. Thus we obtain a finite partition of the boundary of the domain U into a collection of curves $t_m^{\pm}$. If we let the curve Γ^m be as above, then by Lemma 5.2, $f^{p-1-(q_{m+1}-q_{m-1})}(t_m^+) = \Gamma^m$.

As we know from Lemma 5.8, each curve Γ^m is of bounded turning. Suppose that $m < n$. Recall that $J_{-q_{m+1}+q_{m-1}}$ is the last return of the inverse orbit (5.1) to the interval I_{m-1} before the first return to I_m, and that $J_{-q_{m+1}}$, $J_{-2q_{m+1}}$ are the first two returns of the same orbit to I_m. Let $\tilde{\Gamma}^m$ be the pullback of Γ^m along the backward orbit $J_{-q_{m+1}+q_{m-1}}, \dots, J_{-2q_{m+1}}$. By Lemmas 5.3 and 5.4, $\operatorname{dist}(\tilde{\Gamma}^m, I_m) > K|I_m|$, and $\widehat{(\zeta, I_m)} > \epsilon > 0$ for any $\zeta \in \tilde{\Gamma}^m$. Now applying the Koebe distortion theorem to the further pullbacks of $\tilde{\Gamma}^m$, we conclude that the curve t_m^+ is of bounded turning as well.

The curve $\tilde{\Gamma}^m$ satisfies the assumptions of Corollary 5.7. Thus if we denote by $\hat{\Gamma}^m$ the further pullback of $\tilde{\Gamma}^m$ until the first return of the orbit (5.1) to the interval I_{m+1}, then the curves $\hat{\Gamma}^m$ and $\tilde{\Gamma}^m$ have commensurable sizes. Proceeding to apply Corollary 5.7 to $\hat{\Gamma}^m$, etc, we establish that the sizes of the curves $t_m^{\pm}$ are commensurable with I_n, and $\operatorname{dist}(t_m^{\pm}, I_n) > K|I_n|$, provided that $m > n$.

Choose a domain $\tilde{G}_m \supset \tilde{\Gamma}^m$, such that $\operatorname{mod}(\tilde{G} \setminus \tilde{\Gamma}^m) > \mu > 0$, and $\tilde{G}_m \subset \mathbb{H}$. Setting $G_m \supset t_m^+$ to be the pullback of $\tilde{G}_m$, we have $\operatorname{mod}(G_m \setminus t_m^+) = \operatorname{mod}(\tilde{G} \setminus \tilde{\Gamma}^m) > \mu$.

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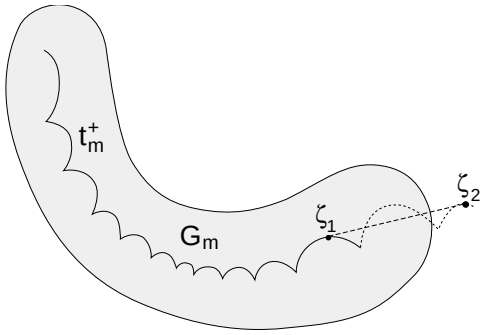


FIGURE 8.

The domain G_m can be chosen in such a way as to intersect only the first segments of the adjacent curves t_i^+ (see Figure 8). Let ζ_1 and ζ_2 be any two points on the boundary of U contained in different curves $t_m^\pm$, with $m > n$. It follows from elementary considerations, that $\operatorname{dist}(\zeta_1, \zeta_2)$ is commensurable with the shortest segment of the boundary of U between the points ζ_1, ζ_2. Therefore, the whole curve $\cup_{m>n} t_m^+$ is of bounded turning.

Since $\operatorname{dist}(t_m^\pm, I_n) > K|I_n|$ for $m > n$, we can choose a bounded s such that $D \equiv D([f^{sq_n - q_{n+1}}(0), f^{-q_{n-1}-sq_n}(0)]) \cap \partial U \subset t_n^+ \cup t_n^-$. By Lemma 5.6, all but finitely many curves γ_i^n are contained in the disk D. Adjust the disk D slightly to obtain a smaller disk $\hat{D}$ with the same property, and such that $\partial(U \setminus \hat{D})$ is connected and is of bounded turning. Then the domain $U \setminus \hat{D}$ is a K-quasidisk, and the proof is complete. □

The existence of a quasiconformal interpolation. We now present the proof of Proposition 5.1. Assume that q_{n+1}/q_n is large. For the map $f_i^{-q_n}$ consider its perturbed Fatou coordinate $\Psi_i|W_i$ as in Lemma 5.5. Consider the orbit (5.4) corresponding to $m = n$ and $f \equiv f_i$. By Lemma 5.6, we can choose r bounded and independent on the map, so that $\gamma_{-r}^n \subset W_i$. Let $c_i \subset \mathbb{H}$ be a curve connecting an endpoint of γ_{-r}^n with $f_i^{-q_{n+1}+q_{n-1}+sq_n}(0)$, such that c_i does not cross γ_{-r}^n, and the translate of c_i under $f_i^{-q_n}$ is disjoint from c_i. The segments $[f_i^{-q_{n+1}+q_{n-1}+sq_n}(0), f_i^{-q_{n+1}+q_{n-1}+(s+1)q_n}(0)]$, γ_{-r}^n, c_i and $f_i^{q_n}(c_i)$ bound a fundamental rectangle $R_i \subset W_i$ for the map Ψ_i (see Figure 9).

Since all four sides of the rectangle R_i are commensurable, there exists a K-quasiconformal mapping $\kappa : R_1 \to R_2$ with a bounded dilatation K, satisfying $\kappa \circ f_1^{q_n}|c_1 \equiv f_2^{q_n} \circ \kappa|c_2$. We extend κ to all domains $R_1^k \subset W_1$, $f_1^{kq_n}(R_1^k) = R_1$ by $\kappa|R_1^k \equiv f_2^{-kq_n} \circ \kappa \circ f_1^{kq_n}|R_1^k$, provided that $R_2^k \equiv \kappa(R_1^k) \subset W_2$. Thus we obtain the desired interpolation near the pinched region of the domain U_1. The pinched region of V_1 is treated in a similar fashion.

By Lemma 5.9, the interpolating map can be extended to the rest of $\Delta_1 \setminus \Omega_1$ with bounded dilatation, which completes the proof.

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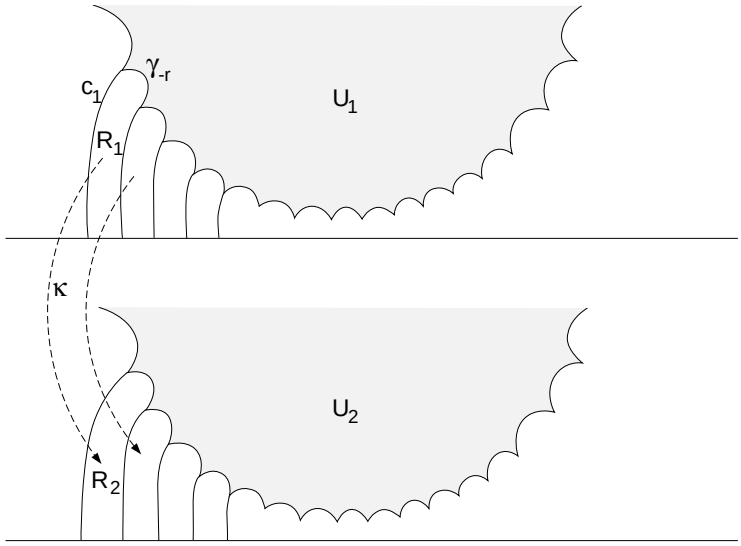


FIGURE 9.

6. *Application to proving local connectivity*

In this section we will use the methods developed in §4 to give a new proof for the following theorem of Petersen.

THEOREM 6.1. (Petersen [**P**]) *Let P_θ denote the quadratic polynomial $z \mapsto e^{2\pi i\theta}z + z^2$. If θ is an irrational number of bounded type, then the Julia set $J(P_\theta)$ is locally connected and has Lebesgue measure zero.*

As a consequence of a classical result of Siegel, for an irrational θ of bounded type, the polynomial P_θ has a Siegel disc Δ around the origin. The action of P_θ on the domain Δ is conformally conjugated to that of the rigid rotation $z \mapsto e^{2\pi i\theta}z$ on the open unit disc $\mathbb{D}$. The methods of quasiconformal surgery allow us in this case to relate the dynamics of P_θ to that of a rational critical circle map. Namely, consider the following one-parameter family of Blaschke products:

$$f^\tau(z) = e^{i2\pi\tau}z^2\frac{z-3}{1-3z}. \tag{6.1}$$

A map in this family is a degree three branched covering of the Riemann sphere with superattracting fixed points at 0 and ∞. The only other singularity of f_τ is a cubic critical point at 1. Since the unit circle T is invariant, the restriction $f^\tau|T$ is a homeomorphism with one cubic critical point; that is, a critical circle map. By the standard monotonicity considerations, for each irrational θ in $(0, 1)$ there is a unique parameter $\tau = \tau(\theta) \in (0, 1)$ for which the circle restriction f^τ has rotation number θ; we set $f_\theta \equiv f^{\tau(\theta)}$. The connection between a Blaschke product f_θ and the quadratic polynomial P_θ is given by the following theorem.

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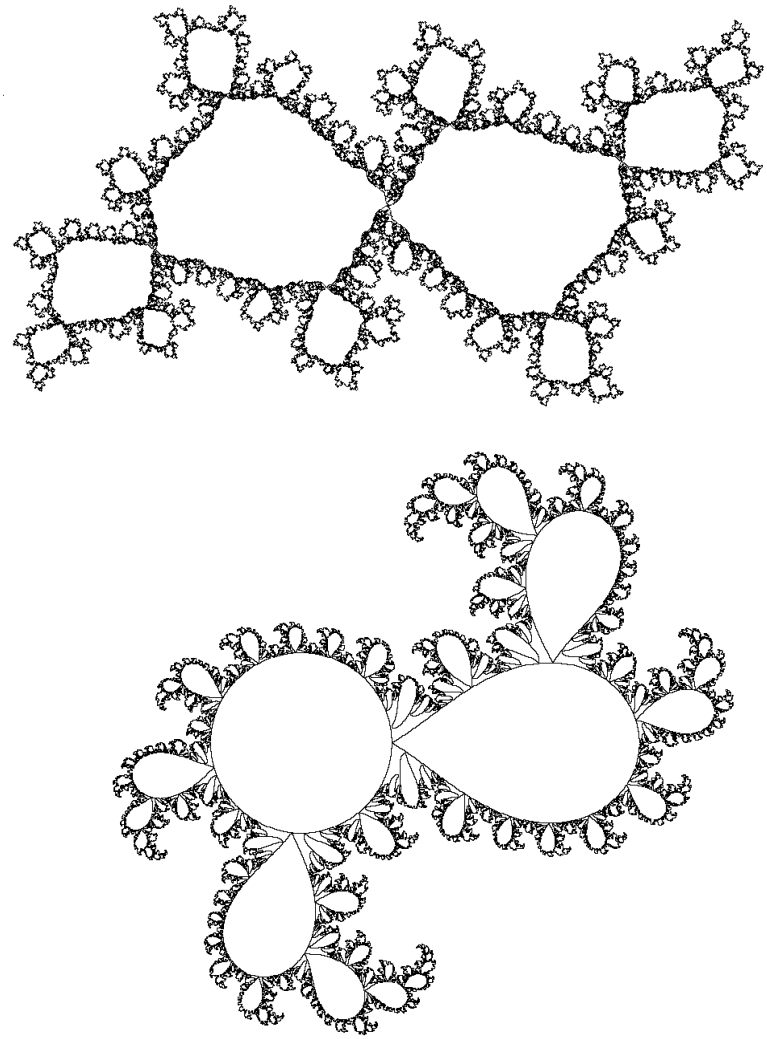


FIGURE 10. A quadratic Siegel Julia set (top) and the corresponding set J_θ.

THEOREM 6.2. (Douady, Ghys, Herman, Shishikura [**D**]) *Let θ be an irrational number of bounded type, and let Δ denote the Siegel disc of the quadratic polynomial P_θ. Then there exists a quasiconformal homeomorphism $\phi : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, conformal on the immediate basin of infinity of f_θ, such that $\phi(\mathbb{D}) = \Delta$, and $\phi \circ f_\theta = P_\theta \circ \phi$ on $\mathbb{C} \setminus \mathbb{D}$.*

Denote by W the f_θ^{-1}-preimage of the unit disc $\mathbb{D}$ not contained in $\mathbb{D}$. Set $J_\theta = \phi^{-1}(J(P_\theta))$. This set is obtained by removing from the Blaschke Julia set $J(f_\theta)$ all points whose orbits eventually land in the interior of $\mathbb{D}$:

$$J_\theta = J(f_\theta) \setminus (\cup_{n\geq 0} f_\theta^{-n}(W) \cup \mathbb{D}).$$

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Petersen [**P**] proved the following result, which together with Theorem 6.2 implies Theorem 6.1.

THEOREM 6.3. (Petersen [**P**]) *For any irrational rotation number θ the set J_θ is locally connected and has Lebesgue measure zero*†.

Our proof of Theorem 6.3 bears a strong resemblance to the proofs of local connectivity of quadratic Julia sets with complex bounds developed in [**HJ, McM2**]. Following Petersen, we construct a plethora of *puzzle-pieces* for the map f_θ, which partition J_θ into connected subsets. By a version of the cubic estimate (3.3) for the map f_θ, the diameters of the puzzle-pieces around the critical point 1 shrink to zero. Thus we obtain a basis of connected neighborhoods around the critical point. Finally, we use bounded distortion considerations to transfer this basis to other points of the set J_θ.

Let us choose an irrational $0 < \theta < 1$ and in what follows work with the fixed map $f \equiv f_\theta$.

For two points a and b on the unit circle T, $[a, b] \subset T$ will denote the shorter arc with these endpoints; $|[a, b]|$ will stand for its length. For $[a, b] \subset T$ let $D_\alpha([a, b])$ be the preimage of $D_\alpha([0, 1])$ under a Möbius map sending T to $\mathbb{R}$ and $[a, b]$ to $[0, 1]$. As ∞ is a superattracting fixed point of f, its immediate basin is parametrized by $\mathbb{C} \setminus \bar{\mathbb{D}}$ via Böttcher coordinates. The images of polar coordinate curves in this parametrization will be referred to as the external rays and equipotentials of f; the former have the same landing properties as in the polynomial case. For a point $\zeta \in T$ with $f^i(\zeta) = 1$ denote by $W(\zeta)$ the preimage of the domain W attached to ζ.

Construction of puzzle-pieces. Petersen's construction of puzzle-pieces begins as follows. Let $\gamma_0' \subset \partial W$, $\gamma_0 \subset \partial W$, and $f(\gamma_0') = [f(1), 1]$, $f(\gamma_0) = T \setminus [f(1), 1]$. Let $\gamma_1' \cap \gamma_0' \neq \emptyset$, $\gamma_1 \cap \gamma_0 \neq \emptyset$, and $f(\gamma_1') = \gamma_0'$, $f(\gamma_1) = f(\gamma_0)$. Inductively let $\gamma_i' \cap \gamma_{i-1}' \neq \emptyset$, $\gamma_i \cap \gamma_{i-1} \neq \emptyset$, and $f(\gamma_i') = \gamma_{i-1}'$, $f(\gamma_i) = f(\gamma_{i-1})$. Considerations of hyperbolic geometry (compare with the Douady–Hubbard–Sullivan landing theorem) imply that the curves γ_i converge to a fixed point of f which we will denote by β; this point is necessarily repelling.

Let $\Gamma' = \cup_i \gamma_i' \cup \beta$, $\Gamma = \cup_i \gamma_i \cup \beta$, and let $\hat{\Gamma}$ be the component of $f^{-1}(\Gamma')$ attached to $f^{-1}(1)$. Being the only fixed point of f in the complement of the unit disc, β is the landing point of the external ray R of the external argument 0. Let R' be the preimage of R landing at the end point of $\hat{\Gamma}$. Fix any equipotential curve E, and let the puzzle-piece $P_0 \supset W$ be the closed domain cut out by the curves $R \cup \Gamma \cup [1, f^{-1}(1)] \cup \hat{\Gamma} \cup R'$ and E (cf. Figure 11).

As before, denote the moments of the closest returns of the critical point 1 by q_i, these numbers appear as the denominators in the truncated continued fraction expansions of the rotation number θ. Let the *nth central puzzle-piece* P_n be the univalent pull-back of P_{n-1} along the inverse orbit $[1, f^{q_n}(1)] \subset P_{n-1}, [f^{-1}(1), f^{q_n-1}(1)], \ldots, [f^{-q_n}(1), 1]$. Note that $P_n \cap T = [f^{-q_n}(1), 1]$, $f^{q_n}(P_n \cap \partial W) = [f^{q_n}(1), f^{-q_{n-1}}(1)]$, and $f^{q_n+q_{n-1}+q_{n-2}}(P_n \cap \partial W(f^{-q_n}(1))) = [1, f^{q_{n-1}+q_{n-2}}(1)]$. As follows from disjointness properties of external

† The original argument of Petersen showed measure zero for θ of bounded type only. The argument for the general case was suggested by Lyubich.

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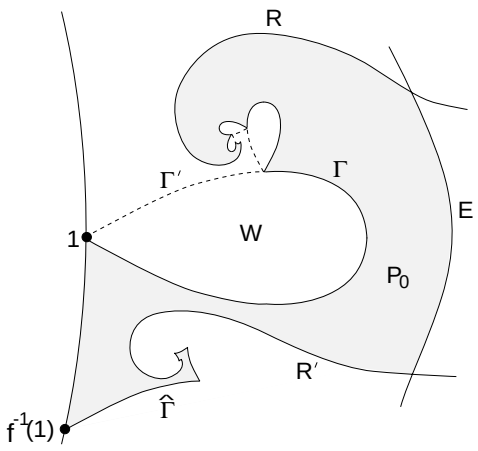


FIGURE 11.

rays and equipotentials, the central puzzle-pieces form two nested sequences inside the first two puzzle-pieces P_0 and P_1. By construction we have the following.

PROPOSITION 6.4. *The intersection $P_n \cap J_\theta$ is connected.*

Let us also note the following.

LEMMA 6.5. *The puzzle-piece P_n contains a Euclidean disc $D_r(z)$ with $D_r(z) \cap J_\theta = \emptyset$, $D_{r/2}(z) \cap J(f) \neq \emptyset$, and $r > K$ diam P_n, for some $K > 0$, independent of n.*

Proof. Note that by construction, $W(f^{-q_{n+2}}(1)) \subset P_n$. The claim now easily follows from the Świątek–Herman real *a priori* bounds and the Koebe distortion theorem. □

A cubic estimate for the map f. Our goal now is to adapt the methods of §4 to obtain a cubic estimate for the inverse branches of f, similar to (3.3). Let us first observe that

$$f(z) - f(1) = e^{2\pi i \tau(\theta)} \frac{(z-1)^3}{1-3z}.$$

Therefore, for all points $z \in J_\theta$ we have

$$c|z-1|^3 < |f(z) - f(1)| < C|z-1|^3 \tag{6.2}$$

for a suitable choice of positive constants c and C.

The map f is not a self-map of the Riemann sphere with points 0 and ∞ removed, and therefore cannot be lifted to the universal covering. Nevertheless, consider the multivalued meromorphic function

$$F(z) = \frac{1}{2\pi i} \operatorname{Log} f(e^{2\pi i z}).$$

It preserves the real axis and has singularities at the integer points, whose images are the integer translations of $\tau = \tau(\theta)$. By the monodromy theorem, in the domain $\mathcal{C}_{(\tau+i,\tau+i+1)}$

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with the critical values removed, we have well-defined univalent branches of the inverse map F^{-1}, mapping the open interval $(\tau+i, \tau+i+1)$ homeomorphically onto an interval between two consecutive integers $(m, m+1)$. These maps range over the simply-connected regions $\mathbb{C}_{(m,m+1)} \setminus \cup \mathcal{W}_j$, where $\mathcal{W}_j$ denotes the component of $(1/2\pi i)\operatorname{Log}(W)$ attached to j.

Denote by $\Upsilon(z) : T \setminus \{f^{-1}(1)\}$ the branch of $(1/2\pi i)\operatorname{Log}(z)$ mapping 1 to 0. Consider the lifts of the inverse branches $f^{-q_1}|[f^{-1}(1), 1]$ and $f^{-1}|[1, f^{-q_1}(1)]$ given by $\Upsilon \circ f^{-q_1} \circ \Upsilon^{-1}$ and $\Upsilon \circ f^{-1} \circ \Upsilon^{-1}$ respectively. It is convenient for us to abuse notation and label these branches ϕ^{-q_1} and ϕ^{-1}. Thus we obtain a commuting pair of inverse maps

$$\Phi = (\phi^{-q_1}|\Upsilon([f^{-1}(1), 1]), \phi^{-1}|\Upsilon([1, f^{-q_1}(1)])).$$

As we have seen above, these maps have univalent extensions to the double slit plane, however, they are not inverses of Epstein maps, and we will re-state Lemmas 4.2–4.4 for them.

Set $I_m = \Upsilon([1, f^{q_m}(1)])$, $T_m = \Upsilon(f^{q_{m+1}}([1, f^{q_m}(1)]))$. Let D_m denote the hyperbolic neighborhood $D_\alpha(\Upsilon([f^{q_{m+1}}(1), f^{q_m-q_{m+1}}(1)]))$, where $0 < \alpha < \pi/2$ will be specified later. For an arc $[a, b] \subset T \setminus \{f^{-1}(1)\}$ denote $D_\theta([a, b])$ the image of $D_\theta(\Upsilon([a, b]))$ under $z \mapsto e^{2\pi i z}$. Set $J_0 = T_n$ and consider the orbit of this interval under Φ. For a point $z \in \mathbb{C}_{T_n} \cap (1/2\pi i)\operatorname{Log}(J_\theta)$ the *corresponding* inverse orbit is obtained by applying the appropriate univalent branches in the slit plane. We begin with the following version of Lemma 4.2.

LEMMA 6.6. *Let J and J' be two consecutive returns of the orbit of J_0 to I_m, for $m > 1$, and let ζ and ζ' be the corresponding points of the inverse orbit of z as above. Suppose $\zeta \in D_m$. Then either $\zeta' \in D_m$, or $\widehat{(\zeta', J')} > \epsilon$, and $\operatorname{dist}(\zeta', J') < C|I_m|$, where the quantifiers ϵ and C are independent of m.*

We remark that the constants ϵ and C will in general depend on the choice of the Blaschke product f.

Proof. Let D'_m denote the pull-back of D_m corresponding to the inverse orbit $J, \dots, J'$, and let $\hat{D}_m$ denote the pull-back of D_m along the piece of the orbit $J, \dots, \phi^{-q_m}(J)$. By Świątek–Herman real *a priori* bounds, the points 0 and $\Upsilon(f^{q_m}(1))$ divide the interval $D_m \cap \mathbb{R}$ into K-commensurable pieces, where K becomes universal for large m, and therefore can be chosen simultaneously for all m.

As the absolute value of the derivative of the logarithmic map is bounded away from 0 and ∞ on the set J_θ, the estimate (6.2) is still valid for the lifted map near the critical point. Together with the Schwarz lemma and Lemma 2.2 this implies that $\hat{D}_m \subset D_\beta([\Upsilon(f^{q_{m+1}-q_m}(1)), 0])$ for some $\beta > 0$ independent of m. Moreover, since the boundary of $\hat{D}_m$ contains a segment of $\partial\mathcal{W}_0$, which forms the angle $\pi/3$ with $\mathbb{R}$ at 0, we have $\hat{D}_m \subset D_\gamma([\Upsilon(f^{q_{m+1}-q_m}(1)), a_1]) \cup D_\sigma([a_2, 0])$, where the points $\Upsilon(f^{q_{m+1}-q_m}(1)), a_1, a_2, 0$ form a B-bounded configuration and with B, $\gamma > 0$ and $\sigma < \pi/2 < \alpha$ independent of m. Applying the Schwarz lemma we have $D'_m \subset D_m \cup D_\gamma([0, \Upsilon(f^{-q_{m+1}+q_m}(a_1))])$ and the claim follows. □

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Lemma 4.4 is also re-formulated in the obvious way.

LEMMA 6.7. *Let J be the last return of the orbit of J_0 to the interval I_m before the first return to I_{m+1}, and let J' and J'' be the first two returns to I_{m+1}. Let ζ, ζ', and ζ'' be the corresponding points in the inverse orbit of z, $\zeta' = \phi^{-q_m}(\zeta)$, $\zeta'' = \phi^{-q_{m+2}}(\zeta')$.*

Suppose $\zeta \in D_m$. Then either $\widehat{(\zeta'', I_{m+1})} > \epsilon = \epsilon(f)$ and $\operatorname{dist}(\zeta'', J'') < C(f)|I_{m+1}|$, or $\zeta'' \in D_{m+1}$.

The proof is again obtained by a direct transcription of the proof of Lemma 4.4. Finally, let us address the saddle-node phenomenon.

LEMMA 6.8. *Let $P_0, P_{-1}, \dots, P_{-k}$ be the consecutive returns of the orbit of J_0 to I_m under the iterates of $\phi^{-q_{m+1}}$, and denote by $\zeta_0, \zeta_{-1}, \dots, \zeta_{-k}$ the corresponding moments in the backward orbit of z. Suppose that $\zeta_{-i} \in D_m$ for some $-k \le -i \le 0$. Then either $\zeta_{-k} \in D_m$, or $\widehat{(\zeta_{-k}, P_{-k})} > \epsilon = \epsilon(f)$ and $\operatorname{dist}(\zeta_{-k}, P_{-k}) \le C(f)|I_m|$.*

Proof. Let $\widetilde{\phi^{-q_{m+1}}}$ denote the rescaling of the appropriate inverse branch by a linear factor $1/|I_m|$. Since the maps $\phi^{-q_{m+1}}$ do not assume values in $\mathcal{W}_0$, the sequence $\{\widetilde{\phi^{-q_{m+1}}}\}$ forms a normal family by Montel's theorem. The partial limits of subsequences with $q_{m+1}/q_m \to \infty$ are necessarily parabolic maps. By the Denjoy–Wolf theorem each of the maps $\phi^{-q_{m+1}}$ has an attracting fixed point both in the upper and the lower half-plane, whose basin is the whole half-plane; by compactness, this point is contained in $D(I_m) \subset D_m$ provided q_{m+1}/q_m is large enough.

The proof is now completed as in Lemma 4.3. □

We are now in a position to repeat the inductive argument of §4, to obtain the following estimate for the inverse branches:

$$\frac{\operatorname{dist}(\phi^{-q_{n+1}+1}(z), \phi^{-q_{n+1}+1}(T_n))}{|\phi^{-q_{m+1}+1}(T_n)|} \le C\frac{\operatorname{dist}(z, T_n)}{|T_n|} + D, \tag{6.3}$$

where $z \in \mathbb{C}_{T_n} \cap D_1$. The constants C, D depend on the map f as well as on the choice of the angle α in the definition of D_m.

Let us now choose α in such a way that the components of $(1/2\pi i)\operatorname{Log}(P_2)$ and $(1/2\pi i)\operatorname{Log}(P_3)$ attached to 0 are contained in D_1. For $n > 3$ let $G_n \supset P_{n+1}$ be the univalent pull-back of $P_2 \cup P_3$ along the orbit $[1, f^{q_{n+1}}(1)], \dots, [f^{-q_{n+1}}(1), 1]$. Using the estimate (6.3), and the fact that the absolute value of the derivative of $(1/2\pi i)\operatorname{Log}(z)$ is bounded away from 0 and ∞ on J_θ, we arrive at the following estimate:

$$\frac{\operatorname{dist}(f(z), f([1, f^{q_n}(1)]))}{|f([1, f^{q_n}(1)])|} \le c\frac{\operatorname{dist}(f^{q_{n+1}}(z), f^{q_{n+1}}([1, f^{q_n}(1)]))}{|f^{q_{n+1}}([1, f^{q_n}(1)])|} \tag{6.4}$$

for any $z \in G_n \cap J_\theta$. Together with the cubic estimate (6.2) for the map f this yields the desired cubic estimate for $f^{q_{n+1}}$:

$$\frac{\operatorname{dist}(f^{q_{n+1}}(z), f^{q_{n+1}}([1, f^{q_n}(1)]))}{|f^{q_{n+1}}([1, f^{q_n}(1)])|} \ge B\left(\frac{\operatorname{dist}(z, [1, f^{q_n}(1)])}{|[1, f^{q_n}(1)]|}\right)^3 \tag{6.5}$$

for some $B > 0$.

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By the Świątek–Herman real *a priori* bounds the arcs $[f^{-q_{n+1}}(1), 1]$, $[f^{q_n}(1), 1]$, $[1, f^{q_{n+1}}(1)]$ and $f^{q_{n+1}}([1, f^{q_n}(1)])$ are all K-commensurable, with a universal constant K for sufficiently large n. By the above cubic estimate (6.5), for $n > 3$, we have

$$\operatorname{diam} P_{n+1} \le b\sqrt[3]{\frac{\operatorname{diam} P_n}{|[1, f^{-q_n}(1)]|}} \cdot |[f^{-q_{n+1}}(1), 1]|.$$

Hence, if $\operatorname{diam} P_n/|[1, f^{-q_n}(1)]| > K_1$ for a large K_1, then $\operatorname{diam} P_{n+1}/|[f^{-q_{n+1}}(1), 1]| < \frac{1}{2} \cdot (\operatorname{diam} P_n)/(|[1, f^{-q_n}(1)]|)$. It follows that for all sufficiently large n, the piece P_n is K_2-commensurable with $[f^{-q_n}(1), 1]$ with a universal constant K_2. Together with Proposition 6.4 this implies the following.

PROPOSITION 6.9. *The set J_θ is locally connected at the critical point 1.*

'Spreading around' argument. Choose any $z \in J_\theta$. First assume that there exists n such that $f^i(z) \notin P_k$ for any $i \ge 0$ and $k \ge n$. As f has an irrational rotation number on the circle this implies that the forward orbit $z_0 \equiv z, z_1 \equiv f(z), z_2 \equiv f(z_1), \dots$ does not accumulate on the circle, i.e. there exists $\epsilon > 0$ such that $z_i \in V_\epsilon \equiv \{\zeta, |\zeta| > 1 + \epsilon\}$.

As the set J_θ is locally connected at the critical point 1, there exist two external rays r_1 and r_2 landing at this point on different sides of W. For a point $\zeta \in J_\theta$ whose forward orbit lands at the critical point 1 let $r_1(\zeta)$, $r_2(\zeta)$ be the preimages of the rays r_1 and r_2, landing on ζ. Denote by $Z(\zeta)$ the component of $\mathbb{C} \setminus (r_1(\zeta) \cup r_2(\zeta) \cup \{\zeta\})$ not containing 1. The *limb* L_ζ of the set J_θ is the set $Z_\zeta \cap J_\theta$. Since external rays do not intersect the Julia set, L_ζ is a connected component of $J_\theta \setminus \{\zeta\}$. Let a be an accumulation point of the sequence $\{z_k\}$. Choose a limb $L \equiv L_\zeta$ containing a, with $L \cap T = \emptyset$. Denote by k_n the moments when $z_{k_n} \in L$. Let $L_n \ni z$ be the pull-back of L along the backward orbit $z_{k_n}, z_{k_n-1}, \dots, z_1, z_0 \equiv z$. We refer to the following general principle to assert that $\operatorname{diam}(L_n) \to 0$.

LEMMA 6.10. ([**L2,** Proposition 1.10]) *Let f be a rational map. Let $\{f_i^{-m}\}$ be a family of univalent branches of the inverse functions in a domain U. If $U \cap J(f) \ne \emptyset$, then for any V such that* $\operatorname{cl}(V) \subset U$,

$$\operatorname{diam}(f_i^{-m} V) \to 0.$$

This yields the desired nest of connected neighborhoods around z, and we are done.

In the complementary case, let z_k be the first point in the orbit $z_0, z_1, z_2, \dots$ contained in the puzzle-piece P_n. Denote by

$$\Pi_0 \equiv P_n, \Pi_{-1}, \dots, \Pi_{-k} \tag{6.6}$$

the preimages of P_n corresponding to the inverse orbit $z_k, z_{k-1}, \dots, z_0$.

LEMMA 6.11. *The inverse orbit (6.6) hits the critical point* 1 *at most once.*

Proof. To be definite, assume that P_n is above the critical point 1. Note that if $\Pi_{-i} \cap T = \emptyset$ for some $i \le q^{n+1}$, then the inverse orbit (6.6) never hits the critical point. Otherwise, denote by A and $B \equiv P_{n+1}$ the 'above' and 'below' $f^{q_{n+1}}$-preimages of P_n, i.e.

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$f^{q_{n+1}}(A) = P_n$, $A \cap T \neq \emptyset$, and A is above 1, and similarly for B. Notice that $A \cap T = [f^{-q_{n+1}-q_n}(1), 1] \subset [f^{-q_n}(1), 1]$. Let $L_1 = P_n \cap \partial W$, and $L_2 = A \cap W$. Then

$$\begin{aligned} f^{q_{n-1}+q_n+q_{n+1}}(L_1) &= f^{q_{n+1}}([f^{q_{n-1}+q_n}(1), 1]) = [f^{q_{n-1}+q_n+q_{n+1}}(1), f^{q_{n+1}}(1)] \\ &\supset [f^{q_{n-1}+q_n}(1), f^{q_{n-1}+q_n+q_{n+1}}(1)] = f^{q_{n-1}+q_n}([1, f^{q_{n+1}}(1)]) \\ &= f^{q_{n-1}+q_n+q_{n+1}}(L_2). \end{aligned}$$

Thus $L_1 \supset L_2$, and as two different preimages of W cannot cross, it follows that $A \subset P_n$. Hence $\Pi_{-q_{n+1}} \neq A$.

Finally, denote by $B' \cap T \neq \emptyset$ the f^{q_n} preimage of B, $f^{q_n}(B') = B$. $B' \cap T = [f^{-q_n}(1), f^{-q_n-q_{n+1}}(1)] \subset [f^{-q_n}(1), 1]$. Let $L_2 = P_n \cap W(f^{-q_n}(1))$, and $L_3 = B' \cap W(f^{-q_n}(1))$. Then

$$\begin{aligned} f^{q_n+q_{n-1}+q_{n-2}}(L_2) &= [1, f^{q_{n-1}+q_{n-2}}(1)] \\ &\supset [f^{q_{n-1}+q_{n-2}-q_n-q_{n+1}}(1), f^{q_{n-1}+q_{n-2}}(1)] \\ &= f^{q_n+q_{n-1}+q_{n-2}}(L_3). \end{aligned}$$

Therefore, $L_2 \supset L_3$, and $B' \subset P_n$. Thus, $\Pi_{-q_{n+1}-q_n} \cap T = \emptyset$ and the claim follows. □

As follows from the inductive argument (compare with Remark 4.1), for n large enough the nth puzzle-piece P_n is contained in $D_\beta([f^{-q_{n-2}-q_{n-1}}(1), 1])$ for some $\beta > 0$ independent of n. Suppose first, that the pull-back $\Pi_0, \ldots, \Pi_{-q_{n-1}}$ does not correspond to the inverse orbit

$$[f^{-q_n}(1), 1], \ldots, [f^{-q_{n-1}-q_n}(1), f^{-q_{n-1}}(1)].$$

The Koebe distortion theorem combined with real *a priori* bounds then imply that $\Pi_{-q_{n-1}}$ is a bounded distortion image of Π_0. Moreover, there is an annulus of definite modulus separating $\Pi_{-q_{n-1}}$ from the circle (and thus from the whole postcritical set of f), so that Π_{-k} is a bounded distortion image of Π_0.

Otherwise, by the Schwarz lemma (compare with Figure 7),

$$\Pi_{-q_{n-1}} \subset D_{\beta'}([f^{-q_{n-1}-q_n}(1), f^{-q_{n-1}}(1)])$$

for β' depending only on β. By real *a priori* bounds and Lemma 6.11, Π_{-k} is the image of $\Pi_{-q_{n-1}}$ under a composition of one iterate of f^{-1} near the critical value, and two maps with bounded distortion.

In either case we can apply Lemma 6.5 to see that the set $P_n(z) \equiv \Pi_{-k}$ contains a Euclidean disc B_n of commensurable size, whose center belongs to $J(f)$. The diameters of B_n necessarily converge to zero, as any disc around a point in $J(f)$ would eventually map over the whole of $J(f)$. Thus $\cap P_n(z) = \{z\}$, and the set J_θ is locally connected at z.

Proof of measure zero statement. The following argument was suggested by Lyubich. We will make use of the following ergodic principle of Lyubich [**L1**].

Let f be a rational map, whose Julia set $J(f)$ is not the whole Riemann sphere. Then the forward orbit of almost every $z \in J(f)$ converges to the postcritical set of f.

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First consider the set of points $J_1 \subset J_\theta$, $J_1 \equiv \{\zeta \in J_\theta \mid \exists n,\, f^i(\zeta) \notin P_k, \forall i \geq 0, k \geq n\}$. Since $f|T$ has irrational rotation number, the points in J_1 do not accumulate on the critical orbit. Thus J_1 has zero Lebesgue measure.

Now consider a point $z \in J_\theta \setminus J_1$, and let $P_n(z)$ be as above. We have shown that $P_n(z)$ contains a Euclidean disc B_n of commensurable diameter, not intersecting the set J_θ. As diam $P_n(z) \to 0$, z cannot be a Lebesgue density point of J_θ. This completes the proof of Theorem 6.3.

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