SHRINKING CONTINUA IN 3-SPACE

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It is the purpose of this note to prove the following result.

Theorem. *Let C be a continuum in E^3, and let X be the quotient space obtained by collapsing C to a point. If X is a homotopy manifold, then X is homeomorphic with E^3.*

The homotopy manifold referred to here is that defined by Griffiths (5). Since this refers to local properties, it is an easy corollary to our theorem that if one shrinks any set of continua to points such that the image points form a discrete set in X, and X is a homotopy manifold, then X is E^3. It is interesting to contrast this with the Bing space B which is obtained from E^3 by shrinking arcs such that the set of image points in B is a zero-dimensional set. For B is a homotopy manifold, but it is not locally Euclidean at any of these points. See (1) and (3).

Our theorem will be proved by making use of a result of Brown (2) and a version of the sphere theorem due to Whitehead (6).

Definition 1. A separable metric n-dimensional space X is an *n-homotopy manifold* if each x in X has arbitrarily small pairs of neighbourhoods $V \subset U$ such that $\overline{V} \subset U$, $\overline{V} - V$ is connected, and $i_*(\pi_k(V - x)) \approx \pi_k(S^{n-1})$ for each k, where $i\colon V - x \to U - x$ is the inclusion map.

Note. Our definition is somewhat stronger than that of Griffiths in that he does not require $\overline{V} - V$ to be connected. We note, however, that without this requirement the condition on the homotopy groups is satisfied by boundary points of a manifold as well as interior points. For let x be a point of the boundary $\dot{M}$ of an n-manifold M, and let W be any neighbourhood of x. Take half-ball neighbourhoods $C_1 \subset C_2 \subset W$ of x. Let y be an interior point of C_1, and let $R_1 \supset R_2$ be concentric balls about y in C_1. Then $V = C_1 - R_1$ and $U = C_2 - R_2$ satisfy the required condition.

Definition 2. A subset C in E^n is *cellular* if there exists a sequence $K_1, K_2, \ldots$ of closed n-cells in E^n such that K_{i+1} lies in the interior of K_i for each i, and the intersection of the K_i is C.

Theorem (Brown). *If C is cellular subset of E^n, then E^n/C is homeomorphic with E^n.*

Sphere Theorem. *Let M be a triangulated 3-manifold (possibly with boundary) and let Y be a topological space containing M. Let $f\colon S^2 \to M - \dot{M}$ be such that jf is essential where $j\colon M \to Y$. Then there exists a 2-sphere S arbitrarily close to $f(S^2)$ such that S is essential in Y.*

Given any open set Q containing C in E^3, we will find a closed 3-cell K containing C in its interior, with K lying in Q. In view of Brown's result quoted above, this will

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suffice to prove that E^3/C is homeomorphic with E^3. Let β: $E^3 \to X$ be the natural map and $x = \beta(C)$.

By hypothesis X is a homotopy manifold, so that there exists a pair of neighbourhoods $V \subset U$ in $\beta(Q)$ satisfying the conditions of Definition 1. In particular then, $V - x$ contains a singular 2-sphere g: $S^2 \to V - x$ which is essential in $V - x$ (since it is essential in $U - x$). It follows that $\beta^{-1}g$: $S^2 \to E^3$ is essential in $\beta^{-1}(V) - C$. In order to apply the sphere theorem we construct a triangulated 3-manifold M with boundary such that M lies in $P = \beta^{-1}(V)$ and $\beta^{-1}g(S^2)$ lies in $M - \dot{M}$. Let d be the least distance measurable between any pair of the sets C, $E^3 - P$, $\beta^{-1}g(S^2)$. Choose a cubical decomposition of E^3 with cubes of diameter less than $\frac{1}{3}d$. Let L_1 be the union of all cubes which intersect C, and L_2 be the union of all cubes which intersect the boundary of P. Let T_1 and T_2 be regular neighbourhoods of L_1 and L_2 which lie in $\frac{1}{6}d$ neighbourhoods of L_1 and L_2, respectively. Then $M = P - T_1 - T_2$ is a triangulated 3-manifold with the desired properties. We now apply the sphere theorem with $P - C = Y$.

It follows that there is a polyhedral 2-sphere S in $M - \dot{M}$ which is essential in $P - C$. By the Schoenflies theorem, the bounded complementary domain D of S is a 3-cell and $S \cup D$ is a closed 3-cell. Now S does not intersect the boundary B of P, and B is connected, so that D lies in P. Since S links $B \cup C$, it follows that D intersects C and hence D contains C. This proves the theorem.

Remark. Instead of assuming that X is a homotopy manifold we could use a notion introduced in (4). For any space X, let $\hat{X} = \{(x_1, x_2) | x_1 \neq x_2\}$ and define p: $\hat{X} \to X$ by $p(x_1, x_2) = x_1$. We say that X is *homotopically homogeneous* if $(\hat{X}, X, p)$ is a Hurewicz fibre space. With this hypothesis on X it is even easier to find an appropriate singular 2-sphere in a neighbourhood of $x = \beta(C)$, and the rest of the proof is the same.

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