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On the Complexity of Computing the Tutte Polynomial of Bicircular Matroids

OMER GIMÉNEZ† and MARC NOY†

Departament de Matemàtica Aplicada II, Universitat Politècnica de Catalunya,
Pau Gargallo 5, 08028 Barcelona, Spain
(e-mail: omer.gimenez@upc.es, marc.noy@upc.es)

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We show that evaluating the Tutte polynomial for the class of bicircular matroids is #P-hard at every point (x, y) except those in the hyperbola $(x-1)(y-1)=1$ and possibly those on the lines $x=0$ and $x=-1$. Since bicircular matroids form a rather restricted subclass of transversal matroids, our results can be seen as a partial strengthening of a result by Colbourn, Provan and Vertigan, namely that the evaluation of the Tutte polynomial for the class of transversal matroids is #P-hard for all points except those in the hyperbola $(x-1)(y-1)=1$.

1. Introduction

The Tutte polynomial $T(M; x, y)$ of a matroid M is a well-known invariant that has been studied from many different points of view. A thorough exposition of its properties and relationship to other areas can be found in [5] and [19]; see also [20] for a more recent survey.

A question that has received much attention is whether the evaluation of $T(M; x, y)$ at a particular point of the (x, y) plane can be done in polynomial time. The seminal paper in the area is by Jaeger, Vertigan and Welsh [8], who showed that evaluating the Tutte polynomial of a graph is #P-hard at every point except those lying in the hyperbola $(x-1)(y-1)=1$ and eight special points. In each of the exceptional cases the evaluation can be done in polynomial time. In a later paper Vertigan and Welsh [18] showed that the same statement is true when the problem is restricted to bipartite graphs.

Another result relevant to this paper is that the evaluation of $T(M; x, y)$ for the class of transversal matroids is #P-hard for all points except those in the hyperbola $(x-1)(y-1)=1$. This was proved by Colbourn, Provan and Vertigan in [6] using the construction of free extensions and coextensions of matroids. The situation for the class of

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matroids representable over a given field is more complex and is fully studied by Vertigan in [17].

In a different direction, it has been shown that the full Tutte polynomial can be computed in polynomial time in several cases. First it was proved for matroids of bounded width by Oxley and Welsh in [13]; this includes the class of series-parallel graphs, which are precisely the graphs of tree-width at most two. The extension to graphs of bounded tree-width was obtained independently by Andrzejak [1] and Noble [11]. Recently Hliněný [7] has obtained the same result for matroids of bounded branch-width representable over a given finite field, which is a substantial generalization of the previous results. The few other cases we know where $T(M;x,y)$ can be computed in polynomial time include matroids with a high degree of symmetry (see for instance [10]) and the class of lattice-path matroids introduced in [3] (this has been extended recently to a larger class in [4]).

In this paper we study the complexity of evaluating $T(M;x,y)$ for the class of bicircular matroids. If $G=(V,E)$ is a graph, the bicircular matroid $B(G)$ has E as ground set and has as bases the edge sets of spanning subgraphs of G in which every connected component has a unique cycle. The circuits of $B(G)$ are the edge sets inducing a connected subgraph with exactly two independent cycles of G and no vertex of degree one. We refer to [9] for additional properties of bicircular matroids.

We show that evaluating $T(M;x,y)$ for the class of bicircular matroids is #P-hard at every point (x,y) except those in the hyperbola $(x-1)(y-1)=1$ and possibly those on the lines $x=0$ and $x=-1$. For these two lines we are unable to prove either that it is #P-hard or that it can be done in polynomial time. Since bicircular matroids form a rather restricted subclass of transversal matroids, our results can be seen as a strengthening of the results in [6] (except for the two lines $x=0$ and $x=-1$ where we must leave the complexity issue undecided). In this paper we always assume that bicircular matroids are given in terms of a graph G; working with alternative representations as transversal matroids arising from a given set system does not change our results since we can switch from one representation to the other one in polynomial time.

The proof has three ingredients. In Section 2 we show that if G is a graph with exactly one loop at every vertex, then

$$T(B(G);x,0) = x\,T(G;x+1,1),$$

where $T(G;x,y)$ is the Tutte polynomial of the cycle matroid associated to G. This, together with the results in [8], imply that computing the Tutte polynomial of bicircular matroids at $(a,0)$ is #P-hard for every $a\neq 0$. In Section 3 we prove that computing the number of bases in a bicircular matroid is #P-hard. This is done by reduction from the problem of computing the number of perfect matchings in a graph, a problem well known to be #P-hard [16]. Since the number of bases in a matroid M is equal to $T(M;1,1)$, this covers the point $(1,1)$. Finally in Section 4 we use the matroid operation of stretching and polynomial interpolation to obtain the complexity results for the remaining points of the (x,y) plane.

In the last section we find an explicit expression for the exponential generating function associated to $T(B(K_n);x,y)$, where K_n is a complete graph. This is similar to the result

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obtained by Tutte [15] for the exponential generating function associated to $T(K_n; x, y)$. In particular, our expression shows that $T(B(K_n); x, y)$ can be computed in polynomial time.

We refer to [19] for an introduction to the complexity of counting and the definition of #P-hardness, and to [12] for general background on matroids.

2. The line $y = 0$

In this section we need the expression of the Tutte polynomial in terms of activities of bases. Let M be a matroid whose ground set E is linearly ordered. For a basis A of M, the internal activity $i(A)$ is the number of elements e in A such that e is the minimum element in the unique cocircuit contained in $(E\backslash A) \cup \{e\}$; the external activity $e(A)$ is the number of elements e in $E\backslash A$ such that e is the minimum element in the unique circuit contained in $A \cup \{e\}$. A classical result of Tutte says that

$$T(M; x, y) = \sum_A x^{i(A)} y^{e(A)}, \tag{2.1}$$

where the sum is over the set of all bases A of M.

Now let $G = (V, E)$ be a graph with exactly one loop at every vertex. If G has n vertices, we order the edges of E so that those corresponding to the n loops have the smallest labels. We also label the vertices of G with the same labels as the loops attached to them.

Lemma 2.1. *Let G be as above, and let A be a basis of $B(G)$. Then $e(A) = 0$ if and only if A is the edge-set of a spanning forest of G having one loop attached to the minimum vertex of every component of the forest. Moreover, the internal activity $i(A)$ is equal to the number of components.*

Proof. Let A be a basis of $B(G)$. We know that A is the edge-set of a spanning subgraph of G consisting of disjoint unicyclic subgraphs of G. If any of them has a cycle of length at least three, the loops attached to the vertices of the unique cycle are externally active and then $e(A) > 0$. Hence if $e(A) = 0$, the only unicyclic subgraphs allowed in A are trees with one loop attached and A is indeed a spanning forest with one loop at every component. But this loop must have the minimum label, otherwise the loop with minimum label would be externally active. This proves the first part of the claim.

If A is as above, the only active elements in A are the loops attached to the components of the forest, hence the internal activity is the number of such components. □

Theorem 2.2. *If G is as above, then*

$$T(B(G); x, 0) = x\, T(G; x + 1, 1).$$

Proof. We know from equation (2.1) that $T(B(G); x, 0) = \sum e_k x^k$, where e_k is the number of bases B with $e(B) = 0$ and $i(B) = k$. On the other hand, it is well known [5] that

$$T(G; x + 1, 1) = \sum_{k \geqslant 0} f_k x^k,$$

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where f_k is the number of forests of G with $k+1$ components. The result follows directly from the previous lemma. □

3. The number of bases

In this section we prove that counting the number of bases in a bicircular matroid is a #P-hard problem. Our proof is by means of a two step reduction from the problem of counting the number of perfect matchings in graphs, which is known to be #P-hard [16]. Next we state the relevant counting problems in detail.

Problem: #PM (perfect matchings)
Instance: A graph G.
Output: The number $\mathrm{pm}(G)$ of perfect matchings of G.

Problem: #BB (bicircular bases)
Instance: A graph G.
Output: The number $\mathrm{bb}(G)$ of bases of $B(G)$.

Problem: #SBB (signed bicircular bases)
Instance: A graph G plus an assignment of a sign $+1$ or -1 to every edge of G.
Output: The weighted sum $\mathrm{sbb}(G)$ of the bases of $B(G)$, where the weight of a basis is the product of the sign of its elements.

In what follows we identify the bases of a bicircular matroid $B(G)$ with the edge sets of spanning subgraphs of G in which every connected component has a unique cycle.

Proposition 3.1. *#PM is polynomial-time reducible to #SBB.*

Proof. Let $G=(V,E)$ be an instance of #PM and assume $n=|V(G)|$ is even. Define a signed graph $G'=(V,E^+\cup E^-)$, where both E^+ and E^- are copies of E with signs 1 and -1, respectively. In this way an edge $e\in E$ gives raise to two parallel edges e^+ and e^- in G', and we call them *twins*. We claim that

$$\mathrm{sbb}(G')=(-1)^{n/2}\mathrm{pm}(G).$$

Clearly this is enough to prove the proposition.

To prove the claim, take any linear ordering of the edges of G and extend it to an ordering of the edges of G' in the following way: if $e<f$ are edges of G then $e^+<e^-<f^+<f^-$ in G'. Define also the following permutation ϕ of the bases of $B(G')$. If A is a basis of $B(G')$, then $\phi(A)$ is the basis obtained from A by removing the smallest edge e^* whose twin is not in A, and replacing it with the twin of e^* (see Figure 1 for an example). If there is no such edge, then set $\phi(A)=A$. It is clear that $\phi(A)$ is indeed a basis of $B(G')$, and that the bases fixed by ϕ consist precisely of $n/2$ disjoint pairs of twin edges, so that they are in one-to-one correspondence with the perfect matchings of G.

From the definition it follows that ϕ is an involution. If $\phi(A)\neq A$ then the weights of A and $\phi(A)$ have opposite signs, hence they cancel in the weighted sum $\mathrm{sbb}(G')$. On

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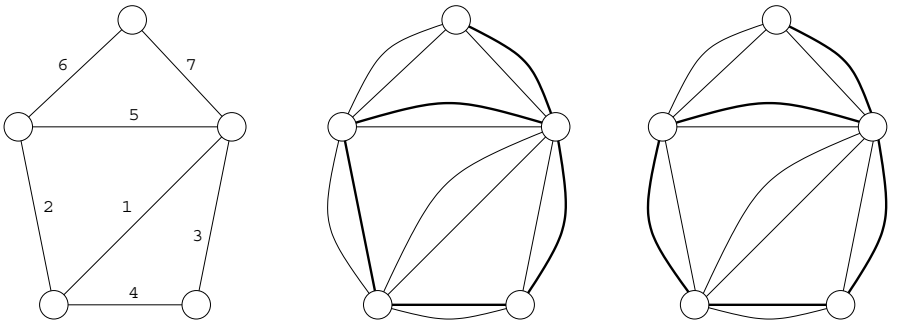


Figure 1. (a) A graph G with a linear ordering of the edges. (b) The graph G', where positive edges are drawn straight and negative edges curved; a basis A of $B(G')$ is shown with thick lines. (c) The basis $\phi(A)$.

the other hand, if $A = \phi(A)$ then the weight of A is equal to $(-1)^{n/2}$ since every pair of twin edges contributes -1 to the weight. From these observations the claim follows directly. □

Proposition 3.2. *#SBB is polynomial-time reducible to #BB.*

Proof. Let $G = (V, E)$ be an instance of #SBB and let $m = |E|$. Recall that the edges of E are signed: edges with sign $+1$ are called positive and those with sign -1 negative. Let $\mathrm{bb}_i(G)$ be the number of bases of $B(G)$ that contain exactly i negative edges. Then we have

$$\mathrm{sbb}(G) = \sum_{i=0}^{m} (-1)^i \mathrm{bb}_i(G). \tag{3.1}$$

Define a polynomial

$$p(x) = \sum_{i=0}^{m} \mathrm{bb}_i(G) x^i.$$

The degree of $p(x)$ is at most m and equation (3.1) implies that $\mathrm{sbb}(G) = p(-1)$.

For every positive integer k define the unsigned graph G_k obtained from G by replacing every negative edge by a path of length k (see Figure 2). Notice that G_k has size polynomial in the size m of G. If v is the number of negative edges in G, we claim that

$$p(1/k) = \mathrm{bb}(G_k)\, k^{-v}$$

for every positive k. Assuming the claim, with at most $m + 1$ calls to #BB we can evaluate $p(x)$ at $m + 1$ different values $1, 1/2, \ldots, 1/(m + 1)$, hence we can interpolate and compute the desired value $\mathrm{sbb}(G) = p(-1)$ in polynomial time.

To prove the claim, consider a basis A of $B(G_k)$, a negative edge e of G, and let P_e be the path of length k that replaces e in G_k. At least $k - 1$ edges of P_e are in A, since

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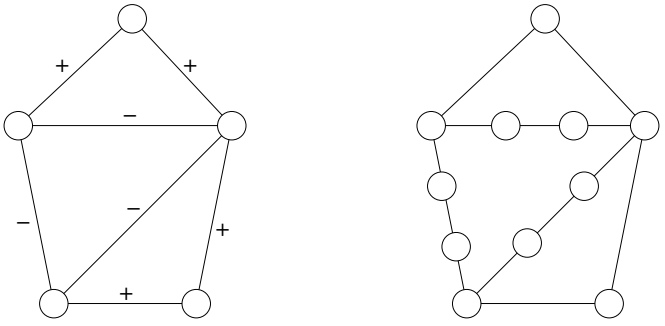


Figure 2. (a) A signed graph G. (b) The associated graph G_3.

otherwise A would have a connected component consisting of a path and would not be a basis. We say that A is *e-disconnected* if exactly $k-1$ edges of P_e are in A, and *e-connected* if it contains all the k edges of P_e.

Given a basis A of $B(G_k)$, define a basis $\phi(A)$ of $B(G)$ as follows: $e \in E$ is an edge of $\phi(A)$ if and only if either e is positive and e belongs to A (in G_k), or e is negative and A is e-connected. This defines a mapping ϕ from the set of bases of $B(G_k)$ onto the set of bases of $B(G)$. The key fact is that $|\phi^{-1}(C)| = k^{v-i}$ for every basis C of $B(G)$, where v is as before and i is the number of negative edges of G that are in C. Indeed, if $\phi(A) = C$ then A has $v-i$ different 'disconnections', and any one of them can be specified in k different ways (the missing edge from the corresponding path of length k), hence there are k^{v-i} possibilities for A.

Now we have

$$\mathrm{bb}(G_k) = \sum_C |\phi^{-1}(C)| = \sum_i k^{v-i}\,\mathrm{bb}_i(G) = k^v p(1/k)$$

as was to be shown. This proves the claim and the proposition. □

From the previous propositions we obtain the main result in this section.

Corollary 3.3. *Computing the number of bases of a bicircular matroid is a #P problem.*

4. Complexity results

We have proved in the previous section that evaluating the Tutte polynomial of a bicircular matroid at $(1,1)$ is #P-hard. A similar result can be obtained from Theorem 2.2.

Lemma 4.1. *Evaluating the Tutte polynomial of a bicircular matroid at* $(a,0)$ *is #P-hard for any* $a \neq 0$.

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Proof. Let G be a loopless graph with n vertices and let G' be the graph obtained from G by attaching one loop at every vertex. It is well known that in this situation $T(G';x,y) = y^n T(G;x,y)$. By Theorem 2.2, we have $T(B(G);a,0) = aT(G';a+1,1)$, and by the previous remark this is exactly $aT(G;a+1,1)$, since we are evaluating at $y=1$. It follows from the results in [8] that computing $T(G;a+1,1)$ is hard if $a \neq 0$, hence the same is true for $T(B(G);a,0)$. □

The β invariant $\beta(M)$ of a matroid M is the coefficient of x in $T(M;x,y)$. Annan [2] showed that computing $\beta(M)$ for a graphic matroid is a #P-hard problem. The following corollary of the previous proof shows a different situation for bicircular matroids.

Proposition 4.2. *If G is a loopless graph, then $\beta(B(G))$ can be computed in polynomial time.*

Proof. From the previous proof it follows that $\beta(B(G)) = T(G;1,1)$. Since $T(G;1,1)$ is the number of spanning trees of G, it can be computed in polynomial time using the well-known matrix-tree theorem; hence the same is true for $\beta(B(G))$. □

We now proceed towards the main result in this section. A pointed matroid N_e consists of a matroid N and a distinguished element e of N which is neither a loop nor an isthmus. If M is any matroid, the tensor product $M \otimes N_e$ is the matroid obtained from M by performing a 2-sum with N_e at every element of M (see [12, Chapter 7] for the definition and properties of 2-sums). When N is the uniform matroid $U_{k,k+1}$ (a circuit of size $k+1$), the tensor product $M \otimes U_{k,k+1}$ is called the k-stretching of M and is denoted by $s_k(M)$. Since the 2-sum operation is also defined for graphs, we can define the k-stretching $s_k(G)$ of a graph G as $G \otimes C_{k+1}$, that is, the graph obtained from G by replacing each edge by a path of length k.

Lemma 4.3. *For any graph G, we have $s_k(B(G)) \cong B(s_k(G))$.*

Proof. The k-stretching of a matroid or a graph can be seen as the consecutive application of 2-sums. Hence it is enough to prove that $B(G) \oplus_2 U_{k,k+1} \cong B(G \oplus_2 C_{k+1})$, where both 2-sums are based on an edge e of G, considered also as an element of the matroid $B(G)$.

Since $U_{k,k+1}$ has a unique circuit, it follows that the circuits of $B(G) \oplus_2 U_{k,k+1}$ are $\mathscr{C}_1 \cup \mathscr{C}_2$, where

$$\mathscr{C}_1 = \{(C - e) \cup (U_{k,k+1} - e) \mid C \text{ circuit of } B(G), e \in C\},$$
$$\mathscr{C}_2 = \{C \mid C \text{ circuit of } B(G), e \notin C\}.$$

We next show that the matroid $B(G \oplus_2 C_{k+1})$ has the same circuits as $B(G) \oplus_2 U_{k,k+1}$. Recall that a circuit in a bicircular matroid is a subset of edges of the underlying graph whose induced subgraph is connected, has exactly two independent cycles and no vertices of degree one. Any such subset of edges in $G \oplus_2 C_{k+1}$ either contains all the edges of C_{k+1}, or does not contain any of them; in the first case they are in correspondence with the circuits in $\mathscr{C}_1$, while in the second case they correspond to those in $\mathscr{C}_2$. □

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The key fact about the stretching operation (proved by Brylawski; see [8]) is that evaluating $T(s_k(M); x, y)$ is, up to a trivial factor, equivalent to evaluating $T(M; X, Y)$, where

$$X = x^k, \qquad Y = \frac{y + x + \cdots + x^{k-1}}{1 + x + \cdots + x^{k-1}}. \tag{4.1}$$

The relation $(X-1)(Y-1) = (x-1)(y-1)$ allows us to interpolate along the hyperbolas

$$H_q = \{(x, y) : (x-1)(y-1) = q\}.$$

This fact, together with Lemma 4.3 and the fact that we already know many hard points for the Tutte polynomial, yields our main result.

Theorem 4.4. *Evaluating the Tutte polynomial of bicircular matroids is #P-hard at every point (a, b), except those points lying on the trivial hyperbola H_1, and possibly those on the lines $a = 0$ and $a = -1$.*

Proof. Let M be a bicircular matroid on n elements. We recall that the degree of $T(M; x, y)$ is at most n.

Previous results Lemma 4.1 and Corollary 3.3 prove the theorem for $(a, 0)$ with $a \neq 0$, and $(1, 1)$, respectively. These points have the property that any hyperbola H_q with $q \neq 1$ contains one of them. Hence computing the restriction of the Tutte polynomial along any of these hyperbolas, even along the lines $x = 1$ or $y = 1$ of the degenerate hyperbola H_0, is #P-hard.

We proceed to prove the theorem. Let (a, b) be a point different from $(1, 1)$, not in H_1 and such that $a \neq 0$ and $a \neq -1$, and let $q = (a-1)(b-1)$. We claim that, with the help of an oracle for evaluating the Tutte polynomial at the point (a, b), we can evaluate the Tutte polynomial along the hyperbola H_q or, if $q = 0$, along one of the lines $x = 1$ or $y = 1$.

Lemma 4.3 shows that the k-stretching of a bicircular matroid is also bicircular, so we can use the oracle for evaluating $T(s_k(M); a, b)$ at $2n + 1$ different values of k. By the previous remarks, from these evaluations we obtain $T(M; A_k, B_k)$, where A_k and B_k are defined as in (4.1):

$$A_k = a^k, \qquad B_k = \frac{b + a + \cdots + a^{k-1}}{1 + a + \cdots + a^{k-1}}.$$

The points (A_k, B_k) belong to the same hyperbola H_q as (a, b); if $q = 0$ they also belong to the same line $x = 1$ or $y = 1$ as (a, b). On the other hand, if $a \notin \{-1, 0, 1\}$ then all A_k are different; if $a = 1$ with $b \neq 1$, then all B_k are different. By the hypothesis on (a, b), all cases are covered.

Thus we have enough evaluations to interpolate the $(2n+1)$-degree polynomial $t^n T(M; t+1, q/t+1)$, that is, the restriction of $T(M; x, y)$ to the hyperbola H_q with $q \neq 0$. If $q = 0$ we would have interpolated either $T(M; t, 0)$ or $T(M; 0, t)$, and in fact $n + 1$ different points would have been enough. □

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Note. Unfortunately, the two lines in the statement above are still undecided. Those familiar with [8] may wonder why we do not use the operation of thickening (dual to stretching); the reason is that the thickening of a bicircular matroid is in general not a bicircular matroid, hence we cannot use the interpolation technique in the theorem above. More generally, we have not found other (connected) pointed matroids besides circuits that guarantee that taking tensor products with arbitrary bicircular matroids again produce bicircular matroids.

5. Complete graphs

In this section we show how to compute $T(B(K_n); x, y)$ using generating functions. The result is similar to the one for computing $T(K_n; x, y)$ derived by Tutte [15]. We need the following expression for the Tutte polynomial of a matroid M with base set E:

$$T(M; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A)$ is the rank function in M. In our case the rank of $A \subseteq E(K_n)$ in $B(K_n)$ is

$$r(A) = n - t(A),$$

where $t(A)$ is the number of connected components of the spanning subgraph induced by A which are trees, including isolated points; this was proved in [9]. We also need the fact that the rank of $B(K_n)$ is n.

Then we have

$$\begin{aligned}\sum_n T(B(K_n); x, y) \frac{z^n}{n!} &= \sum_n \sum_{A \subseteq E(K_n)} (x-1)^{n-(n-t(A))} (y-1)^{|A|-(n-t(A))} \frac{z^n}{n!} \\ &= \sum_{n,A} (y-1)^{|A|} ((x-1)(y-1))^{t(A)} \frac{(z(y-1)^{-1})^n}{n!}.\end{aligned}$$

Our task is to find the exponential generating function (EGF for short) of labelled graphs with respect to the number of vertices, number of edges and number of tree components

$$H(z, u, v) = \sum_{n \geqslant 0} \sum_{A \subseteq E(K_n)} u^{|A|} v^{t(A)} \frac{z^n}{n!}, \tag{5.1}$$

and evaluate $H(z(y-1)^{-1}, y-1, (x-1)(y-1))$.

Henceforth we use the language and machinery of generating functions as in [14]. In particular, as we are dealing with labelled structures, the operation of taking sets of a combinatorial class corresponds to taking the exponential of the associated generating function. From now on all graphs considered are assumed to be labelled.

Let $G(z, u)$ be the EGF for graphs, where the variable z marks vertices and u marks edges, and let $C(z, u)$ and $T(z, u)$ be defined analogously for connected graphs and trees, respectively. Then we have

$$G(z, u) = \sum_{n \geqslant 0} \sum_k \binom{\binom{n}{2}}{k} u^k \frac{z^n}{n!} = \sum_n (1+u)^{\binom{n}{2}} \frac{z^n}{n!},$$

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$$C(z,u) = \log G(z,u),$$
$$T(z,u) = \sum_{n\geqslant 1} n^{n-2}u^{n-1}\frac{z^n}{n!}.$$

The first equation is because the number of graphs with n vertices and k edges is equal to $\binom{\binom{n}{2}}{k}$. The second equation is because a graph is a set of connected graphs, hence

$$G(z,u) = \exp(C(z,u)).$$

Finally, the last equation follows from the fact that there are n^{n-2} trees with n vertices and any of them has $n-1$ edges.

Distinguishing whether a connected graph is or is not a tree and marking tree components with the variable v as in equation (5.1), we obtain

$$\begin{aligned} H(z,u,v) &= \exp((C-T)+vT) = G\exp((v-1)T) \\ &= \left(\sum_{n\geqslant 0}(1+u)^{\binom{n}{2}}\frac{z^n}{n!}\right)\exp\left((v-1)\sum_{n\geqslant 1} n^{n-2}u^{n-1}\frac{z^n}{n!}\right). \end{aligned}$$

The substitution $H(z(y-1)^{-1}, y-1, (x-1)(y-1))$ gives us the following result.

Theorem 5.1. *The Tutte polynomials of the matroids $B(K_n)$ have the exponential generating function*

$$\sum_n T(B(K_n);x,y)\frac{z^n}{n!} = \left(\sum_{n\geqslant 0}(y-1)^{-n}y^{\binom{n}{2}}\frac{z^n}{n!}\right)\exp\left(\frac{xy-x-y}{y-1}\sum_{n\geqslant 1} n^{n-2}\frac{z^n}{n!}\right).$$

From the previous formula one can compute $T(B(K_n);x,y)$ in polynomial time in n. Indeed, truncate the two series above to polynomials in z of degree n, and compute the Taylor development up to order n, the result being a polynomial in x and y of degree n^2. For instance:

$$T(B(K_4);x,y) = x^4+2x^3+3x^2+4x+y^2+4y;$$
$$T(B(K_5);x,y) = x^5+5x^4+15x^3+35x^2+(20y+45+5y^2)x+y^5+15y^3+5y^4+30y^2+45y;$$
$$\begin{aligned} T(B(K_6);x,y) = {} & x^6+9x^5+45x^4+165x^3+\left(15y^2+60y+420\right)x^2 \\ & +\left(30y^4+6y^5+195y^2+90y^3+330y+546\right)x \\ & +y^9+6y^8+21y^7+56y^6+120y^5+216y^4+336y^3+456y^2+546y. \end{aligned}$$

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