Math. Proc. Camb. Phil. Soc. (1979), **85**, 43
Printed in Great Britain

An application of p-factorization methods to symmetric graphs

BY RICHARD WEISS
Freie Universität Berlin, W. Germany

(*Received* 14 *March* 1978)

Let Γ be an undirected graph and G a subgroup of aut (Γ) acting transitively on the vertex set $V(\Gamma)$ of Γ. Let x be an arbitrary vertex of Γ. We denote by $\Gamma(x)$ the set of vertices adjacent to x and by $G(x)^{\Gamma(x)}$ the permutation group induced by the stabilizer $G(x)$ of x in G on $\Gamma(x)$; $G(x)^{\Gamma(x)}$ is called the subconstituent of G (with respect to Γ). Let $G_1(x) = \{a \in G(x) | a \in G(y) \text{ for each } y \in \Gamma(x)\}$. For each $y \in \Gamma(x)$, let $G(x,y) = G(x) \cap G(y)$ and $G_1(x,y) = G_1(x) \cap G_1(y)$. An s-path is an $(s+1)$-tuple $(x_0, x_1, \ldots, x_s)$ of vertices such that $x_{i-1} \in \Gamma(x_i)$ if $1 \leqslant i \leqslant s$ and $x_{i-2} \neq x_i$ if $2 \leqslant i \leqslant s$. Γ is called (G,s)-transitive if G acts transitively on the set of all s-paths but intransitively on the set of all $(s+1)$-paths in Γ.

In (13) Tutte showed that if Γ is finite, connected, trivalent and (G,s)-transitive with $s \geqslant 1$, then $s \leqslant 5$ and $|G(x)| = 3.2^{s-1}$. In this paper we continue the attempt to extend this result to graphs of arbitrary valency. Several authors (see, for instance, (1), (2) and (15)) beginning, to my knowledge, with Quirin (11) have successfully applied Thompson's p-Factorization Theorem (12) and generalizations of this theorem (for instance (4), (5) and (6)) to various special cases. We handle here the case that $G(x)^{\Gamma(x)}$ contains a regular abelian normal subgroup. Our proof, although derived from the proofs of various versions of the p-Factorization Theorem (principally from the proof in (6)), is largely self-contained. The result is as follows:

THEOREM: *Let Γ be a finite, connected, undirected graph regular of valency $d \geqslant 3$ and $\{x,y\}$ an edge of Γ. Suppose there exists a subgroup G of aut (Γ) such that Γ is (G,s)-transitive with $s \geqslant 1$ and such that $G(x)^{\Gamma(x)}$ is primitive and contains a regular abelian normal subgroup. Then*:

(i) *If $G_1(x,y) \neq 1$, then for some n, $d = q^n$ with $q = 2$ or 3, $G_1(x,y)$ is a p-group with $p = 6/q$, and n is even if $p = 3$; $G(x)^{\Gamma(x)}$ contains m distinct subgroups $K_1, \ldots, K_m$ with $m = n/(p-1)$, each isomorphic to the symmetric group S_3 if $p = 2$ or the alternating group A_4 if $p = 3$, forming a complete conjugate class of subgroups with*

$$\langle K_1, \ldots, K_m \rangle = K_1 \times \ldots \times K_m.$$

(ii) *If $d \geqslant 5$ and $s \geqslant 2$, then $G_1(x,y) = 1$.*

(iii) *If $d \geqslant 5$, then $s \leqslant 3$.*

Before beginning the proof of the Theorem, we note that $G_1(x,y) = 1$ implies that $|G(x)|$ divides $|G(x)^{\Gamma(x)}| \cdot |G(x,y)^{\Gamma(y)}| = |G(x)^{\Gamma(x)}|^2/d$ and hence $d!(d-1)!$. For a given subconstituent H^X of degree d, the bound $|H|^2/d$ can be obtained (with $s = 3$ if H acts 2-transitively on X) by taking for Γ the complete bipartite graph with $V(\Gamma) = X \times \mathbb{Z}_2$ and edge set $E(\Gamma)$ the set of pairs $\{(x,i),(x',i')\}$ with $i \neq i' \pmod 2$ and for G the wreath product $H \wr \mathbb{Z}_2$ (acting as in (8), I·15·3).

https://doi.org/10.1017/S030500410005547X Published online by Cambridge University Press

When $d = 3$ or 4, neither $G_1(x, y) = 1$ nor $s \leqslant 3$ need hold. Consider, for example, the incidence graph of the projective plane $PG(2, p)$ with $p \leqslant 3$, G the group of all collineations and correlations. When $d - 1$ is a prime, however, bounds for s and $|G_1(x, y)|$ are known; see, for instance, (3). I have not been able to determine whether Γ and G exist fulfilling the hypotheses of the Theorem with $d \geqslant 5$ and $G_1(x, y) \neq 1$.

Note finally that if we do not require $G(x)^{\Gamma(x)}$ to be primitive, $G_1(x, y)$ can be arbitrarily large. Let Γ, for instance, be the graph with $V(\Gamma) = \mathbb{Z}_m \times \mathbb{Z}_k$ (where $m \geqslant 2$ and $k \geqslant 4$) and $E(\Gamma)$ the set of pairs $\{(i, j), (i', j')\}$ with $j - j' \equiv 1$ or $-1 \pmod{k}$. Let

$$G = \mathbb{Z}_m \wr D_{2k}$$

where D_{2k} (the dihedral group of order $2k$) is to be considered as acting in the usual fashion on $\mathbb{Z}_k$ and $\mathbb{Z}_m$ regularly on itself. Then Γ and G satisfy the hypotheses of the Theorem (with $d = 2m$) except that $G(x)^{\Gamma(x)}$ is not primitive; $G_1(x, y)$ is a group of order m^{k-4}.

We now begin the proof. Suppose that Γ and G fulfil the hypotheses and that $G_1(x, y) \neq 1$. By ((3), (2·3)), there is a prime p such that $G_1(x, y)$ is a p-group. Since $G_1(x, y) \trianglelefteqslant G(x, y)$, $G_1(x, y) \leqslant O_p(G(x, y))$ (where $O_p(G(x, y))$ denotes the largest normal p-subgroup of $G(x, y)$). We will save parentheses from now on when there is no risk of ambiguity by writing, for instance, $O_p G(x, y)$ instead of $O_p(G(x, y))$. Suppose that $O_p G(x, y) \leqslant G_1(x)$. Then $O_p G(x, y) = O_p G_1(x)$ and hence $O_p G(x, y) \trianglelefteqslant G(x)$. Thus $O_p G(x, y) \trianglelefteqslant \langle G(x), G\{x, y\}\rangle$ where $G\{x, y\}$ denotes the stabilizer in G of the edge $\{x, y\}$. Since $G\{x, y\}$ contains elements exchanging x and y, and since Γ is connected,

$$\langle G(x), G\{x, y\}\rangle$$

acts transitively on $V(\Gamma)$. Thus $O_p G(x, y) = 1$ and hence $G_1(x, y) = 1$, a contradiction. Thus $O_p G(x, y) \not\leqslant G_1(x)$. In particular, $G(x)^{\Gamma(x)}$ is not regular and, by ((8), II·3·2), $(p, d) = 1$ so that $O_p G(x) \leqslant G_1(x)$ and hence $O_p G(x) = O_p G_1(x)$.

Let $w \in \Gamma(x)$ be arbitrary. We claim that $ZO_p G(x, w)$ (i.e. the centre of $O_p G(x, w)$) is contained in $ZO_p G(x)$. Otherwise, $ZO_p G(x, w) \not\leqslant G_1(x)$ since

$$O_p G(x) = O_p G(x, w) \cap G_1(x).$$

Let a be an element in $ZO_p G(x, w) - G_1(x)$ and u and $v \in \Gamma(x)$ such that $v = a(u) \neq u$. Since $G_1(x, u) \trianglelefteqslant G_1(x)$, $G_1(x, u) \leqslant O_p G_1(x) \leqslant O_p G(x, w)$. Hence

$$G_1(x, u) = aG_1(x, u)\, a^{-1} = G_1(x, v) \quad \text{and so} \quad G_1(x, u) \trianglelefteqslant \langle G(x, u), G(x, v)\rangle.$$

Since $G(x)^{\Gamma(x)}$ is primitive but not regular, $\langle G(x, u), G(x, v)\rangle = G(x)$. Thus

$$G_1(x, u) \trianglelefteqslant \langle G(x), G\{x, u\}\rangle$$

and hence $G_1(x, u) = 1$. With this contradiction, we conclude that

$$ZO_p G(x, w) \leqslant ZO_p G(x)$$

as claimed.

Let $U = \langle ZO_p G(x, w) | w \in \Gamma(x)\rangle$ and let V be the subgroup generated by the elements of U of order p. By the previous paragraph, $U \leqslant ZO_p G(x)$. In particular, U is an abelian p-group and $O_p G(x) \leqslant C(U) \leqslant C(V)$ where $C(U) = C_{G(x)}(U)$ (i.e. the central-

https://doi.org/10.1017/S030500410005547X Published online by Cambridge University Press

izer of U in $G(x)$) and $C(V) = C_{G(x)}(V)$. Since U and V are normal in $G(x)$, so are $C(U)$ and $C(V)$. Suppose that $C(V) \not\leqslant G_1(x)$. Since $(p, d) = 1$ and $C(V) \trianglelefteqslant G(x)$, $C(V) - G_1(x)$ contains an element a whose order is relatively prime to p. By ((7), (5·2·4)), $a \in C(U)$. Choose u and v in $\Gamma(x)$ such that $v = a(u) \neq u$. Since $ZO_p G(x, u) \leqslant U$,

$$ZO_p G(x, u) = aZO_p G(x, u) a^{-1} = ZO_p G(x, v)$$

and thus $ZO_p G(x, u) \trianglelefteqslant \langle G(x, u), G(x, v)\rangle = G(x)$. Hence $ZO_p G(x, u) \trianglelefteqslant \langle G(x), G\{x, u\}\rangle$ so that $ZO_p G(x, u) = 1$ although $O_p G(x, u) \neq 1$, a contradiction. It follows that $C(V) \leqslant G_1(x)$.

The set of abelian subgroups of maximal order of a group H will be denoted by $A(H)$ (as in ((7), p. 271)); let $J(H)$ denote the subgroup of H generated by the subgroups in $A(H)$. Suppose that $JO_p G(x, y) \leqslant G_1(x)$. Then

$$JO_p G(x, y) \leqslant O_p G(x, y) \cap G_1(x) = O_p G(x).$$

But then clearly $AO_p G(x, y) = AO_p G(x)$ and hence $JO_p G(x, y) = JO_p G(x)$. It follows that $JO_p G(x, y) \trianglelefteqslant \langle G(x), G\{x, y\}\rangle$ so that $JO_p G(x, y) = 1$ although $O_p G(x, y) \neq 1$, a contradiction. Therefore $JO_p G(x, y) \not\leqslant G_1(x)$. Choose $P \in AO_p G(x, y)$ among those elements of $AO_p G(x, y)$ not contained in $G_1(x)$ such that $|P \cap U|$ is maximal.

Let Q be the regular abelian normal subgroup of $G(x)^{\Gamma(x)}$. By ((8), II·3·2), $|Q| = d$ is a power of some prime q, say $d = q^n$, and Q is elementary abelian and self-centralizing in $G(x)^{\Gamma(x)}$. Let Q_0 be a minimal P-invariant but not P-centralized subgroup of Q and let $P_0 = C_P(Q_0)$, i.e. the subgroup consisting of those elements of P acting trivially on Q_0. Of course, $G_1(x) \cap P \leqslant P_0$. Since P/P_0 acts faithfully and irreducibly on Q_0, P/P_0 is cyclic (by ((7), (3·2·2))). Let L be the intersection of $G(x)$ with the stabilizers $G(u)$ for all u in the Q_0-orbit containing y. Let $b \in G(x)$ be an arbitrary q-element (i.e. of order a power of q) inducing on $\Gamma(x)$ a non-trivial permutation contained in Q_0. Then

$$bP_0 b^{-1} \leqslant L \cap O_p G(x, b(y)) \trianglelefteqslant L \quad \text{and therefore} \quad bP_0 b^{-1} \leqslant O_p L.$$

Since $P_0 \leqslant L \cap O_p G(x, y) \leqslant O_p L, [P_0, b] \leqslant G_1(x) \cap O_p L = O_p G(x) \leqslant C(V)$. Thus P_0 and $\langle b\rangle$ induce groups on V which commute (and, in particular, $C_V(P_0)$ is b-invariant). Let $Y = C_V(P_0)$. Since $b \notin G_1(x)$, b acts non-trivially on V. By ((7), (5·3·4)), b acts non-trivially on Y.

Since P normalizes P_0 and V it also normalizes Y. Suppose that P acts trivially on Y. Then $[P, b]$, too, acts trivially on Y. Since P acts irreducibly but not trivially on Q_0, $[P, b]$ contains an element c inducing on $\Gamma(x)$ a non-trivial permutation contained in Q_0. Replacing c by a power of itself if necessary, we may assume that c is a q-element. But then c acts non-trivially on Y as we have seen in the previous paragraph. With this contradiction, we conclude that P acts non-trivially on Y.

Choose $a \in P$ such that the coset aP_0 generates P/P_0; the element a acts non-trivially on Y. By considering the Jordan form, it is clear that a^p has more fixed points in Y than a. If we set $P_1 = \langle a^p, P_0\rangle$, then $C_V(P) = C_Y(P) < C_Y(P_1) = C_V(P_1)$ and hence, since $V \leqslant U$, $C_U(P) < C_U(P_1)$. Since P is abelian, we have $P \cap U \leqslant C_U(P)$. It follows that $C_U(P_1) \not\leqslant P$. Therefore $P \cap U = (P \cap G_1(x)) \cap U \leqslant P_1 \cap U < P_1 C_U(P_1) \cap U$. Since

https://doi.org/10.1017/S030500410005547X Published online by Cambridge University Press

$|P_1C_U(P_1)| > |P_1| = |P|/p$ and $P_1C_U(P_1)$ is abelian, $P_1C_U(P_1) \in AO_p G(x,y)$. By the choice of P, $P_1C_U(P_1) \leqslant G_1(x)$. Therefore $P_1 \leqslant G_1(x)$ and hence

$$P_0 = P_1 \leqslant O_p G(x,y) \cap G_1(x) = O_p G(x) \leqslant C(U) \quad \text{and} \quad |P/P_0| = p.$$

Since P_0 and U are both abelian and $P_0 \leqslant C(U)$, P_0U is also abelian. Since P is abelian and a acts non-trivially on U, $P_0 \cap U \leqslant C_U(a) < U$ and, in particular, $U \nleqslant P_0$. Hence $|P_0U| > |P_0| = |P|/p$. Since $P \in AO_p G(x,y)$, $|P_0U| = |P|$. Thus $|U/P_0 \cap U| = p$ and so $|U/C_U(a)| = p$ too.

Let $T = \{g \in G(x) | g \in O_p G(x,u)$ for some $u \in \Gamma(x)$ and $|U/C_U(g)| = p\}$. If

$$g \in O_p G(x,u) \cap G_1(x)$$

for some $u \in \Gamma(x)$, then $g \in O_p G(x) \leqslant C(U)$ and hence $g \notin T$. Therefore $T \cap G_1(x) = \varnothing$. Since $a \in T$, $T \neq \varnothing$. Let H be the subgroup of $G(x)$ generated by T and $C(U)$. Since $H \trianglelefteqslant G(x)$, $Q \leqslant H^{\Gamma(x)}$ by ((8), II·3·2) (where, again, Q is the regular abelian normal subgroup of $G(x)^{\Gamma(x)}$). Since clearly $H^{\Gamma(x)} \leqslant O_p(G(x,y)^{\Gamma(x)})Q$, $|H^{\Gamma(x)}| = dp^k$ for some k.

Set $1 = U_0 < U_1 < \ldots < U_r = U$ be a composition series of U as an H-module and let

$$H_1 = \bigcap_{i=1}^{r} C_H(U_i/U_{i-1}).$$

By ((7), (5·3·3)), $H_1/C(U)$ is a p-group. Since $|H^{\Gamma(x)}|$ is divisible by q and

$$C(U) \leqslant H \cap G_1(x),$$

there exists a t with $1 \leqslant t \leqslant r$ such that H acts non-trivially on U_t/U_{t-1}. Let

$$W = U_t/U_{t-1}.$$

Since $H/C(U)$ is generated by p-elements, $|W| \geqslant p^2$. For each $g \in T$,

$$|W/C_W(g)| \leqslant |U/C_U(g)| = p.$$

Thus H induces an irreducible group on W which is generated by transvections. Let m be the index of the H-invariant subgroup $(G_1(x) \cap H)C_H(W)$ in H. We have

$$m \Big| |H/G_1(x) \cap H| = dp^k.$$

Since $T \nsubseteq C_H(W)$, $p|m$. Suppose $(q,m) = 1$. Then $d \Big| |C_H(W)^{\Gamma(x)}|$. Since Q is the only q-Sylow subgroup of $H^{\Gamma(x)}$, $Q \leqslant C_H(W)^{\Gamma(x)}$. Thus $C_H(W)$ acts transitively on $\Gamma(x)$ and so for every element $g \in T$ there exists an element $h \in C_H(W)$ such that $hgh^{-1} \in T \cap O_p G(x,y)$. Since $T \cap O_p G(x,y)$ certainly induces a p-group on W, so does H. Since H acts irreducibly on W, H acts trivially on W by ((7), (2·6·3)). This contradicts the choice of t. Hence $q|m$. Thus m is divisible by precisely two primes. By (9) and (10), $SL(2,2)$ and $SL(2,3)$ are the only irreducible groups of finite degree $\geqslant 2$ over a prime field generated by transvections and containing a normal subgroup of index divisible by precisely two primes. Hence $pq = 6$, $|W| = p^2$ and $H/C_H(W) \cong SL(2,p)$. (Note that when p is odd, ((9), lemmas 3 and 5) and ((10), lemma 2·1) suffice since H acts irreducibly on W.)

https://doi.org/10.1017/S030500410005547X Published online by Cambridge University Press

Choose $a_1 \in T$ acting non-trivially on W. Since $H/C_H(W)$ is not a p-group, there exists an $a_2 \in T$ such that $a_2 \notin \langle a_1, C_H(W)\rangle$. Since $SL(2,p)$ is generated by any two of its p-Sylow subgroups, $\langle a_1, a_2\rangle$ induces a group on W isomorphic to $SL(2,p)$. Setting $L = \langle a_1, a_2\rangle$, we have $H = LC_H(W)$.

Since $|U/C_U(a_i)| = p$ for $i = 1$ and 2, $|U/C_U(L)| \leqslant p^2$. Since $U_t \cap C_U(L) \leqslant U_{t-1}$, $|U_t \cap C_U(L)| \leqslant |U_t|/p^2$. Thus

$$|U| \geqslant |U_t C_U(L)| = |C_U(L)| \cdot |U_t/U_t \cap C_U(L)| \geqslant |C_U(L)| \cdot p^2 \geqslant |U|.$$

It follows that $U_t C_U(L) = U$ and $U_t \cap C_U(L) = U_{t-1}$. Hence

$$L \leqslant C_H(U/U_t) \cap C_H(U_{t-1}).$$

Let $B = C_H(U/U_t) \cap C_H(U_{t-1})$. Since $L \leqslant B$, $B/C_B(W) \cong SL(2,p)$. Since $C_B(W)$ stabilizes the series $U \geqslant U_t > U_{t-1} \geqslant 1$, $C_B(W)/C(U)$ is a p-group by ((7), (5·3·3)); hence $C_B(W) \leqslant G_1(x)$ since $C(U) \leqslant G_1(x)$, $C_B(W) \trianglelefteqslant H$ and (because $O_p(H^{\Gamma(x)})$ centralizes Q and Q is self-centralizing) $O_p(H^{\Gamma(x)}) = 1$. Therefore $B^{\Gamma(x)}$ is isomorphic to a factor group of $SL(2,p)$, the order of which is divisible by p. It follows that $B^{\Gamma(x)} = S_3$ if $p = 2$ and A_4 if $p = 3$ since the q-Sylow subgroup Q of $H^{\Gamma(x)}$ is elementary abelian.

Let $K_1, \ldots, K_m$ be the distinct conjugates of $B^{\Gamma(x)}$ in $G(x)^{\Gamma(x)}$ and $K = \langle K_1, \ldots, K_m\rangle$. Since $K \trianglelefteqslant G(x)^{\Gamma(x)}$, $Q \leqslant K$ by (8), II·3·2). Since $H \trianglelefteqslant G(x)$, $K_i \leqslant H^{\Gamma(x)}$ (and, in fact, $K_i \trianglelefteqslant H^{\Gamma(x)}$) for $1 \leqslant i \leqslant m$. Since $K \leqslant H^{\Gamma(x)}$, Q is a q-Sylow subgroup of K. Suppose $1 \leqslant i \neq j \leqslant m$. Then $K_i \cap K_j \trianglelefteqslant K_i$ and hence either $K_i \cap K_j = 1$ or $K_i \cap K_j = O_q(K_i)$, i.e. $O_q(K_i) = O_q(K_j)$. If $K_i \cap K_j = 1$ then $[K_i, K_j] = 1$. If $O_q(K_i) = O_q(K_j)$, we can choose non-trivial p-elements $a_i \in K_i$ and $a_j \in K_j$ such that $a_i a_j^{-1}$ acts trivially on $O_q(K_i)$. Then for each h with $1 \leqslant h \leqslant m$, either $K_i \cap K_h = 1$ and hence $K_j \cap K_h = 1$ or $O_q(K_i) = O_q(K_h)$; in either case, $a_i a_j^{-1}$ acts trivially on $O_q(K_h)$. Since

$$Q = \langle O_q(K_h) | 1 \leqslant h \leqslant m\rangle,$$

$a_i a_j^{-1}$ acts trivially on Q. Hence $a_i a_j^{-1} \in Q \cap K_i K_j$ since Q is self-centralizing. Since $|K_i K_j| = |K_i| \cdot |K_j|/|O_q(K_i)|$, $O_q(K_i)$ is a q-Sylow subgroup of $K_i K_j$. Hence

$$a_i a_j^{-1} \in O_q(K_i) = O_q(K_j).$$

It follows that $a_i \in O_q(K_j) a_j \subseteq K_j$. Since $K_i = \langle O_q(K_i), a_i\rangle$, we have $K_i = K_j$, a contradiction. Thus $K_i \cap K_j = 1$. Therefore a_i centralizes K_j for $j \neq i$ and acts without fixed points on $O_q(K_i)$. Hence $K_i \cap K_1 \ldots K_{i-1} K_{i+1} \ldots K_m = 1$. In other words,

$$K = K_1 \times \ldots \times K_m.$$

In particular, $d = q^m$ if $p = 2$ and $d = q^{2m}$ if $p = 3$ since Q is a q-Sylow subgroup of K. Thus we have proved (i).

If $d \geqslant 5$, then $m \geqslant 2$. Since $O_q(K_1)$ is conjugate in $G(x)^{\Gamma(x)}$ only to the other $O_q(K_i)$, $1 \leqslant i \leqslant m$, $G(x)^{\Gamma(x)}$ acts intransitively on the non-trivial elements of Q. In other words, $G(x)^{\Gamma(x)}$ is not 2-transitive. Hence $s < 2$. (ii) follows.

To prove (iii), we assume that $s \geqslant 4$. By (ii), $G_1(x, y) = 1$ and hence

$$|G(x,y)| = |G(x,y)^{\Gamma(x) \cup \Gamma(y)}| \quad \text{divides} \quad |GL(n,q)|^2$$

https://doi.org/10.1017/S030500410005547X Published online by Cambridge University Press

(where $d = q^n$). Since $s \geqslant 4$, $|G(x,y)|$ is a multiple of $(d-1)^3$, the number of 4-paths $(x_0, x_1, \ldots, x_4)$ with $x_0 = x$ and $x_1 = y$. Hence $(d-1)^3$ divides $|GL(n,q)|^2$ and so $d-1$ divides $((q^{n-1}-1)(q^{n-2}-1)\ldots(q-1))^2$. Clearly, $n \geqslant 3$. By ((16), p. 283), $n = 6$ and $q = 2$. But then $3 \Big| |G_1(x,y)|$ by ((14), 1·Hilfssatz). With this contradiction, the proof of the Theorem is complete.

REFERENCES

(1) BÜRKER, M. and KNAPP, W. Zur Vermutung von Sims über primitive Permutationsgruppen II. *Arch. Math.* **27** (1976), 352–359.

(2) DEMPWOLFF, U. A factorization lemma and an application. I and II. *Arch. Math.* **27** (1976), 18–21, 476–479.

(3) GARDINER, A. Arc transitivity in graphs. *Quart. J. Math. Oxford Ser.* (2) **24** (1973), 399–407.

(4) GLAUBERMAN, G. Weakly closed elements of Sylow subgroups. *Math. Z.* **107** (1968), 1–20.

(5) GLAUBERMAN, G. A characteristic subgroup of a p-stable group. *Canad. J. Math.* **20** (1968), 1101-1135.

(6) GLAUBERMAN, G. Failure of factorization in p-solvable groups. *Quart. J. Math. Oxford Ser.* (2) **24** (1973), 71–77.

(7) GORENSTEIN, D. *Finite Groups* (New York, Harper & Row, 1968).

(8) HUPPERT, B. *Endliche Gruppen*, vol. I (Berlin-Heidelberg-New York, Springer–Verlag, 1967).

(9) MCLAUGHLIN, J. Some groups generated by transvections. *Arch. Math.* **18** (1967), 363–368.

(10) MCLAUGHLIN, J. Some subgroups of $SL_n(F_2)$. *Illinois J. Math.* **13** (1969), 108–115.

(11) QUIRIN, W. L. Primitive permutation groups with small orbitals. *Math. Z.* **122** (1971), 267–274.

(12) THOMPSON, J. G. Factorization of p-solvable groups. *Pacific J. Math.* **16** (1966), 371–372.

(13) TUTTE, W. T. A family of cubical graphs. *Proc. Cambridge Philos. Soc.* **43** (1947), 459–474.

(14) WEISS, R. Über symmetrische Graphen und die projektiven Gruppen. *Arch. Math.* **28** (1977), 110–112.

(15) WEISS, R. Symmetrische Graphen mit auflösbaren Stabilisatoren. *J. Algebra* **53** (1978), 412–415.

(16) ZSIGMONDY, K. Zur Theorie der Potenzreste. *Monatsh. Math. u. Phys.* **3** (1892), 265–284.

https://doi.org/10.1017/S030500410005547X Published online by Cambridge University Press