COMPLEX HOMOGENEOUS LINEAR FORMS

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1. Let Z, Q, C denote respectively the ring of rational integers, the field of rational numbers and the field of complex numbers. Minkowski (4) solved the problem of minimizing

$$\max\{|ax+by|, |cx+dy|\} \tag{1·1}$$

for $x, y \in Z(i)$ or $Z(\rho)$, where $a, b, c, d \in C$ have fixed determinant $\Delta \neq 0$. Here $\rho = \exp \frac{2}{3}\pi i$, and $Z(i)$ and $Z(\rho)$ are the rings of integers in $Q(i)$ and $Q(\rho)$ respectively. In fact he found the best possible results

$$\max(|\xi|, |\eta|) \leqslant \left(\frac{\sqrt{2}}{3-\sqrt{3}}|\Delta|\right)^{\frac{1}{2}} \tag{1·2}$$

for $Z(i)$, and

$$\max(|\xi|, |\eta|) \leqslant \sqrt{|\Delta|} \tag{1·3}$$

for $Z(\rho)$, where

$$\xi = ax+by, \quad \eta = cx+dy,$$

while Buchner (1) used Minkowski's method to show that

$$\max(|\xi|, |\eta|) \leqslant \left(\frac{3+\sqrt{6}}{3}\right)^{\frac{1}{4}} |\Delta|^{\frac{1}{2}} \tag{1·4}$$

for $Z(i\sqrt{2})$. Hlawka (3) has also proved (1·2), and Cassels, Ledermann and Mahler (2) have proved both (1·2) and (1·3). In a paper being prepared jointly by H. P. F. Swinnerton-Dyer and the author, general problems of the geometry of numbers in complex space are discussed and a systematic method given for solving the above problem for all complex quadratic fields $Q(\vartheta)$. Here, ϑ is a non-real number satisfying an irreducible quadratic equation with rational coefficients. The above problem is solved in detail for $Q(i\sqrt{5})$, for which

$$\max(|\xi|, |\eta|) \leqslant \left(\frac{2|\Delta|}{\sqrt{15}-3}\right)^{\frac{1}{2}}, \tag{1·5}$$

and the 'critical forms' can be reduced to

$$\xi = i\frac{\sqrt{5}-\sqrt{3}}{2}x + \frac{-3+\sqrt{15}+i(\sqrt{3}-\sqrt{5})}{4}y,$$

$$\eta = x + \frac{-1+i\sqrt{3}}{2}y.$$

The details are complicated by $Q(i\sqrt{5})$ having class number 2. Here we quote the results which indicate the method and derive the best possible result for $Q(i\sqrt{7})$, a field of class-number 1. Writing $\omega = \frac{1}{2}(-1+i\sqrt{7})$, we note that $Z(\omega)$ is the ring of integers of $Q(i\sqrt{7})$ and is a principal ideal ring.

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THEOREM. *Let* $\xi = ax+by$, $\eta = cx+dy$, *where* $a, b, c, d \in C$ *and* $ad-bc = \Delta \neq 0$. *Then there exist* $x, y \in Z(\omega)$, *not both zero, such that*

$$\max(|\xi|, |\eta|) \leqslant \left(\frac{2|\Delta|}{\sqrt{21-3}}\right)^{\frac{1}{2}} = \mu. \tag{1·6}$$

The sign of equality is only necessary for forms which, apart from unimodular factors, can be reduced by transformations with coefficients in $Z(\omega)$ *and determinant* ± 1 *to* $\mu\xi_0, \mu\eta_0$, *where*

$$\left.\begin{aligned} \xi_0 &= x - \frac{1+i\sqrt{3}}{2}y, \\ \eta_0 &= \frac{3-\sqrt{21}+i(\sqrt{7}-\sqrt{3})}{4}x + \frac{\sqrt{3}-\sqrt{7}}{2}iy. \end{aligned}\right\} \tag{1·7}$$

2. The reader is assumed to be familiar with Mahler's general theory of lattices and star-bodies in n-dimensional Euclidean space. By considering a similar theory in complex space, taking as lattices $Z(\omega)$-modules on n free generators, we have shown that star-bodies of finite type have critical lattices. To prove the theorem, we use a finite process for finding all extremal lattices of the star-body $K: \max(|\xi|, |\eta|) \leqslant 1$ in 2-dimensional complex space, showing that the one of least determinant is (1·7), which has determinant $\frac{1}{2}(\sqrt{21}-3)$ in absolute value. The following lemmas, to be used here, are proved in the joint paper with Swinnerton-Dyer.

LEMMA 1. *If* $X = lx+my$, *where* $l, m \in C$, *then any three points in* (x, y)*-space (no two of zero determinant) on* $|X| = 1$ *determine this surface to within two possibilities.*

LEMMA 2. *An extremal* $Q(\vartheta)$*-lattice for*

$$K: \max(|\xi|, |\eta|) \leqslant 1$$

either has three points, of non-zero determinants, on the 'side' $|\xi| = 1$, $|\eta| < 1$, *or it has a point such that* $|\xi| = 1$ *and* $\eta = 0$.

LEMMA 3. *If* Λ *is a* $Q(\vartheta)$*-lattice admissible for* K, *such that the points of* Λ *corresponding to the integer-pairs* (u, v), (u', v') *lie on the boundary of* K, *then*

$$|E|^2 < 16d\pi^{-2}, \tag{2·1}$$

where $E = uv'-u'v$ *and* $-d$ *is the discriminant*† *of the field* $Q(\vartheta)$.

LEMMA 4. *If points* $(p, q), (r, s)$ *of determinant* $E \neq 0$ *lie on the boundary of the convex region*

$$K^*: \quad \phi(x, y) \leqslant 1, \tag{2·2}$$

then the region

$$H: \quad |qx-py| + |sx-ry| \leqslant |E| \tag{2·3}$$

is contained in K^*.

This last lemma is originally due to Minkowski. We quote one more result proved in the joint paper with Swinnerton-Dyer.

LEMMA 5. *If a lattice is admissible for* $K: \max(|\xi|, |\eta|) \leqslant 1$ *and has points* P, Q *on the boundary of* K *which satisfy* $P \equiv Q \pmod 2$, *then the lattice-points* $\frac{1}{2}(P \pm Q)$ *lie on the boundary, giving values* $(1, 0)$ *and* $(0, 1)$ *for* $(|\xi|, |\eta|)$.

† If $1, \theta$ is an integral basis for $Q(\vartheta)$, then $-d = (\theta-\bar{\theta})^2$.

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3. *Proof of the theorem.* The field $Q(i\sqrt{7})$ has discriminant -7, so that (2·1) becomes

$$|E|^2 < 112/\pi^2 < 12.$$

Writing $E = u + v\omega$, we obtain

$$(2u - v)^2 + 7v^2 \leqslant 44,$$

whence the possible values for E are, apart from sign and conjugation,

$$1, 2, 3, \omega, \omega \pm 2, 2\omega \pm 1, 2\omega.$$

Five of these are eliminated by use of Lemma 4. We take $(p, q) = (1, 0)$ and set $\phi(x, y)$ equal to $\max(|\xi|, |\eta|)$ or $|\xi|$ or $|\eta|$, noting that in the first case $K^* = K$. Thus, if the integer-pairs $(1, 0), (R, E)$ give points on the boundary of K, and if the lattice is assumed admissible, then we ought not to be able to find integers (x, y) such that

$$\left|x - \frac{R}{E}y\right| + \left|\frac{y}{E}\right| < 1. \tag{3·1}$$

Thus, if $E = 2\omega - 1$, we can reduce R mod E to one of $0, \pm 1, \pm 2, \pm\omega, \pm\omega^2, \pm\bar{\omega}$. Now $(0, E)$ cannot give a point of an admissible lattice on the boundary, while the other residue classes are eliminated by the fact that (3·1) can be satisfied by $x, y \in Z(\omega)$ and not both zero. Similarly the values $2\omega + 1, 2\omega, \omega - 2$ and 3 for E are eliminated. In the case $E = 2$, the residue classes prime to E are 0 and 1, and hence two lattice points on the boundary have integers $(1, 0)$ and $(1, 2)$. These satisfy the condition $P \equiv Q \pmod 2$ of Lemma 5, and so the points corresponding to $(1, 1), (0, 1)$ give values $(1, 0), (0, 1)$ for $(|\xi|, |\eta|)$. Writing out these equations in matrix form and taking determinants, we deduce that in this case $|\Delta| = 1$. As we shall find an admissible lattice of smaller determinant, we can henceforth assume that no $E = \pm 2$.

For $E = \omega + 2$ we find that R can be taken to be 1, and we have

$$\left|x - \frac{y}{\omega + 2}\right| + \frac{|y|}{2} = 1 \quad \text{for} \quad (x, y) = (0, 1).$$

If $(1, 0), (1, \omega + 2)$ gave points such that for at least one of them $|\xi| < 1$, then by an extension of Lemma 4 with $\phi(x, y) = |\xi|$, we should have $|\xi| < 1$ for $(0, 1)$. Similarly with $|\eta|$, and hence we can suppose that all three integer-pairs give points on $|\xi| = 1$, $|\eta| \leqslant 1$.

The 'anomalous' case of Lemma 2, when there is a point on the boundary of K with one of the forms zero, will now be eliminated. If integer-pairs of determinant E give values $(1, 0)$ and $(\psi, 1)$ for $(|\xi|, |\eta|)$, where we can certainly assume that $\psi < 1$, then $|\Delta E| = 1$. By the above, this cannot arise for $E = \omega + 2$, hence $|E|$ is 1 or $\sqrt{2}$, and so $|\Delta| = 1$ or $\sqrt{\frac{1}{2}}$. As we shall find an admissible lattice of determinant $\frac{1}{2}(\sqrt{21} - 3) < \sqrt{\frac{1}{2}}$, the anomalous case does not give the critical lattices, and hence we can assume that no point of the extremal lattice on the boundary of K has either coordinate zero. By Lemma 2, we can now suppose that each 'side' of K contains at least three lattice-points of non-zero determinants.

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Consider the possible 6×2 integral matrices which arise from three lattice-points on each 'side' of K, and suppose first that all $E = \pm 1$, $\pm\omega$ or $\pm\bar{\omega}$. Determinantal considerations show that some E must be ± 1, and one finds only the cases

$$\begin{array}{cccccccccc} \text{(I)} & 1 & 0 & \text{(II)} & 1 & 0 & \text{(III)} & 1 & 0 \\ & 0 & 1 & & 0 & 1 & & 0 & 1 \\ & 1 & 1 & & -1 & \omega & & -\omega & \bar{\omega} \\ & -1 & \omega & & 1 & 1 & & 1 & 1 \\ & -\omega & \bar{\omega} & & -\omega & \bar{\omega} & & -1 & \omega \\ & \bar{\omega} & -1 & & \bar{\omega} & -1 & & \bar{\omega} & -1 \end{array}$$

where in each case the first three rows are those for which $|\xi| = 1$, $|\eta| < 1$.

In case (I), we put $\xi = ax+by$, $\eta = cx+dy$, and use the three conditions $|\xi| = 1$, $|\eta| < 1$, and the three with $|\eta| = 1$, $|\xi| < 1$, to show that the lattice is that given by (1·7). To check admissibility, we put

$$\left|x - y\frac{1+i\sqrt{3}}{2}\right| \leqslant 1, \quad |cx+dy| \leqslant 1,$$

deduce that $|y|^2 \leqslant 3$, and find on investigation that there are only the above six points on K.

For (II), we find that $\xi = x+(1\pm i)y/\omega$, and the condition $|\xi| < 1$ for $(1, 1)$ shows that the lower sign is needed. The η-points give $c = (-1+i)/\omega$ and $c+d = 1$, and hence we find that

$$|\Delta| = \tfrac{1}{2}(\sqrt{7}-1) > \tfrac{1}{2}(\sqrt{21}-3).$$

For (III), $\xi = x+by$, where $b = -1$ or $(2+\omega)/(\omega-1)$. The first value gives $\xi = 0$ for $(1, 1)$, hence the anomalous case already excluded, and the second value gives $|\xi| > 1$ for $(1, 1)$.

The only case left is when among the six rows provided by Lemma 2 there is an $E = \omega+2$. By earlier remarks we can assume that the three rows with $|\xi| = 1, |\eta| < 1$ are $(1, 0)$, $(1, \omega+2)$, $(0, 1)$, and then $\xi = x+2\omega^{-2}y$. For the three rows with $|\eta| = 1$, $|\xi| < 1$, none can form a determinant equal or conjugate to $\pm(\omega+2)$ with a ξ-row, and hence the E formed from a ξ-row and an η-row are ± 1, $\pm\omega$ or $\pm\bar{\omega}$. These conditions lead to only three rows, namely $(-\bar{\omega}, \omega)$, $(1, -\bar{\omega})$, $(1, 1)$, and the first of these gives $\xi = 0$ and hence the anomalous case.

All cases have been considered, and hence the theorem is proved.

In conclusion I wish to acknowledge my indebtedness to Swinnerton-Dyer, who proved the crucial Lemmas 1 and 2 after we first discussed the idea of extending Mahler's work to complex space.

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