Canad. J. Math. Vol. 73 (1), 2021 pp. 29–62
http://dx.doi.org/10.4153/S0008414X19000464
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Polynomials from Combinatorial K-theory

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Abstract. We introduce two new bases of the ring of polynomials and study their relations to known bases. The first basis is the *quasi-Lascoux* basis, which is simultaneously both a K-theoretic deformation of the quasi-key basis and also a lift of the K-analogue of the quasi-Schur basis from quasi-symmetric polynomials to general polynomials. We give positive expansions of this quasi-Lascoux basis into the glide and Lascoux atom bases, as well as a positive expansion of the Lascoux basis into the quasi-Lascoux basis. As a special case, these expansions give the first proof that the K-analogues of quasi-Schur polynomials expand positively in multifundamental quasi-symmetric polynomials of T. Lam and P. Pylyavskyy.

The second new basis is the *kaon* basis, a K-theoretic deformation of the fundamental particle basis. We give positive expansions of the glide and Lascoux atom bases into this kaon basis.

Throughout, we explore how the relationships among these K-analogues mirror the relationships among their cohomological counterparts. We make several "alternating sum" conjectures that are suggestive of Euler characteristic calculations.

1 Introduction

Let $\mathrm{Poly}_n := \mathbb{Z}[x_1, \ldots, x_n]$ denote the ring of integral polynomials in n commuting variables. Considerations in representation theory and algebraic geometry give rise to a number of interesting and important bases of Poly_n. This paper contributes two new bases and studies their relations to those bases of established importance; we find that our new bases exhibit well-behaved structure and fill natural holes in the previously developed theory. This study is part of a general program to develop a combinatorial theory of Poly_n that mirrors the rich classical theory of symmetric functions.

Foremost among known bases of Poly_n are the celebrated *Schubert polynomials* $\{\mathfrak{S}_a\}$ of A. Lascoux and M.-P. Schützenberger [LS82]. Let $X = \mathrm{Flags}_m(\mathbb{C})$ be the parameter space of complete flags

$$0 = V_0 \subset V_1 \subset \cdots \subset V_m = \mathbb{C}^m$$

of nested vector subspaces of $\mathbb{C}^m$, where $\dim V_i = i$. Denote by B the Borel group of $m \times m$ invertible upper triangular matrices. The standard action of B on $\mathbb{C}^m$ induces an action on X with finitely-many orbits, whose closures are the *Schubert varieties* of X. These subvarieties may be naturally indexed by weak compositions $a = (a_1, \ldots, a_m)$ (*i.e.,* sequences of nonnegative integers) of length m such that $a_i \leq m - i$

Received by the editors September 2, 2018; revised August 11, 2019.
Published online on Cambridge Core September 3, 2019.
Author C.M. was partially supported by a GAANN Fellowship from the Department of Mathematics, University of Illinois at Urbana-Champaign. Author O.P. was partially supported by a Mathematical Sciences Postdoctoral Research Fellowship (#1703696) from the National Science Foundation.
AMS subject classification: **05E05**.
Keywords: Demazure character, Demazure atom, Lascoux polynomial, Lascoux atom, Grothendieck polynomial, quasi-Lascoux polynomial, kaon.

https://doi.org/10.4153/S0008414X19000464 Published online by Cambridge University Press

(for example, by taking a to be the Lehmer code of an associated permutation). The corresponding *Schubert classes* $\{\sigma_a\}$ form a $\mathbb{Z}$-linear basis for the Chow ring $A^\star(X)$ of subvarieties of X modulo rational equivalence. The Schubert polynomials are polynomial representatives for the Schubert classes in the sense that one has (up to truncation)

$$\mathfrak{S}_a \cdot \mathfrak{S}_b = \sum_c C_{a,b}^c \, \mathfrak{S}_c \qquad \text{if and only if} \qquad \sigma_a \cdot \sigma_b = \sum_c C_{a,b}^c \, \sigma_c.$$

Despite the existence of explicit formulas for Schubert polynomials, it remains a major open problem of algebraic combinatorics to give a positive combinatorial formula for the *Schubert structure constants* $C_{a,b}^c \in \mathbb{Z}_{\geq 0}$.

The (type A) *Demazure characters* $\{\mathfrak{D}_a\}$ of M. Demazure [Dem74] form another basis of Poly_n, which is important in representation theory. These are precisely the characters of certain explicitly-defined B-modules [Dem74, RS95]. Remarkably, it was shown in [LS90, RS95] that the Demazure characters *refine* the Schubert polynomials, *i.e.,*

$$\mathfrak{S}_a = \sum_b E_b^a \mathfrak{D}_b$$

for some nonnegative integers E_b^a.

Letting the symmetric group S_n act on Poly_n by permuting variables, the S_n-invariants are the *symmetric polynomials* $\mathrm{Sym}_n \subset \mathrm{Poly}_n$. Another remarkable property of the bases $\{\mathfrak{S}_a\}$ and $\{\mathfrak{D}_a\}$ of Poly_n is that each contains (as a subset) the celebrated *Schur basis* $\{s_\lambda\}$ of Sym_n; in fact,

$$\{\mathfrak{S}_a\} \cap \mathrm{Sym}_n = \{\mathfrak{D}_a\} \cap \mathrm{Sym}_n = \{s_\lambda\}.$$

In this sense, both Schubert polynomials and Demazure characters are *lifts* of the Schur basis to the polynomial ring. The Schur basis, moreover, has well-studied and useful refinements into the *quasi-Schur polynomials* $\{S_\alpha\}$ of [HLMvW11a,HLMvW11b] and further into the *fundamental quasi-symmetric polynomials* $\{F_\alpha\}$ of [Ges84], both of which are bases of the subspace $\mathrm{QSym}_n \subset \mathrm{Poly}_n$ of quasi-symmetric polynomials. (A polynomial $f \in \mathrm{Poly}_n$ is *quasi-symmetric* if it is invariant under exchanging x_i and x_{i+1} in those terms of f where at most one of the two variables appears.)

In general, while a rich combinatorial theory of symmetric and quasi-symmetric polynomials has been and continues to be developed, the analogous theory for the full polynomial ring remains relatively sparse. For example, unlike for general Schubert polynomials, several positive combinatorial formulas (*e.g.,* [LR34, KTW04, Vak06]) are known for the structure constants of Schur polynomials, *i.e.,* the Littlewood–Richardson coefficients. A natural program, championed by A. Lascoux [Las13], is to develop the analogous combinatorial theory of Poly_n by

- lifting known bases and relationships to Poly_n from the better-understood subrings Sym_n and QSym_n; and by
- developing uniform combinatorial models for these lifted bases and for the relations among them, extending models from Sym_n and QSym_n.

The end goal of this program is that this new theory eventually bear dividends on major problems involving polynomials, such as the Schubert problem mentioned above.

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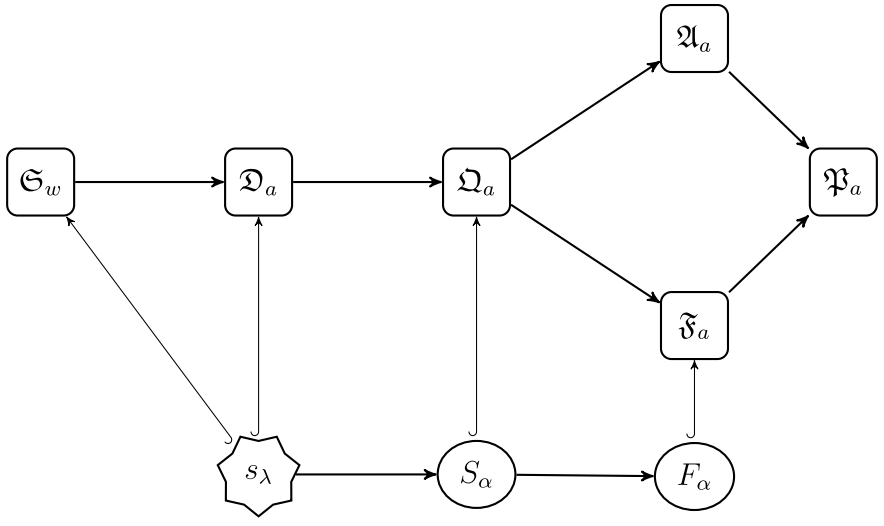


Figure 1: The nine cohomological families of polynomials considered here. Those depicted inside a star shape are bases of Sym_n, those inside an ellipse are bases of QSym_n, and those inside a rectangle are bases of Poly_n. The thinner hooked arrows denote that the basis at the tail is a subset of the basis at the head. The thicker arrows denote that the basis at the head refines the basis at the tail.

Recent work in this area has provided lifts to Poly_n of the quasi-Schur and fundamental bases of QSym_n: respectively, the *quasi-key polynomials* $\{\mathfrak{Q}_a\}$ of [AS18b] and the *(fundamental) slide polynomials* $\{\mathfrak{F}_a\}$ of [AS17]. (Note that despite the name, quasi-key polynomials are not quasi-symmetric. The name instead reflects that they stand in a similar relation to key polynomials as quasi-Schur polynomials to do Schur polynomials.) These families provide further refinements of Schubert polynomials: each Demazure character is a nonnegative combination of quasi-keys [AS18b, Theorem 3.7], each of which is, in turn, a nonnegative combination of slides [AS18b, Theorem 3.4]. The slide basis, moreover, like the Schubert basis, has nonnegative structure constants; in fact, unlike the Schubert basis, one even has an analogue of the Littlewood–Richardson rule for multiplying slide polynomials [AS17, Theorem 5.11].

A more classical approach to the study of Demazure characters is to consider their refinement, not into slides, but rather into the *Demazure atom* basis $\{\mathfrak{A}_a\}$ of [LS90] (see also, [Mas09]). While the Demazure atoms refine the quasi-keys [Sea20, Theorem 3.4], just as slides do, the Demazure atoms have no known direct relation to the slide basis. A common refinement of the Demazure atoms and the slides is provided by the *(fundamental) particle* basis $\{\mathfrak{P}_a\}$ of [Sea20]. The relations among these nine families of polynomials, which we call "cohomological", are illustrated in Figure 1. (Despite the terminology, many of these families do not currently have an interpretation in terms of the cohomology of any space; indeed, for many of them, it also remains open to give a representation-theoretic interpretation.)

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In this work, we are interested in the *K-theoretic analogues* of the bases in Figure 1. A major theme of the modern Schubert calculus is the investigation of the geometry of $X = \mathrm{Flags}_n$ (and other generalized flag varieties) via richer complex oriented cohomology theories. In the most general such theories, there is ambiguity in the appropriate definition of Schubert classes, as the analogues of the usual push-pull operators fail to satisfy the appropriate braid relations. (For further discussion and partial progress on these problems, see, *e.g.*, [GR13, CZZ15, LZ17].)

It turns out that this definitional problem can be avoided by working in the connective K-theory (or a specialization thereof) of X [BE90]; hence, we restrict ourselves to this context. Complex oriented cohomology theories are determined by their formal group laws, describing how to express the Chern class of a tensor product of two line bundles in terms of the original two Chern classes. For connective K-theory, this formal group law is

$$c_1(L \otimes M) = c_1(L) + c_1(M) + \beta c_1(L)c_1(M), \tag{1.1}$$

where β is a formal parameter and L, M are any complex line bundles on X. Hence, the ordinary cohomology ring is recovered by specializing $\beta = 0$, and the ordinary K-theory ring is recovered by specializing β to any element of $\mathbb{C}^\star$.

In the connective K-theory of X, polynomial representatives for a Schubert basis are given by the β-Grothendieck polynomials $\{\overline{\mathfrak{S}}_a^{(\beta)}\}$ of S. Fomin and A. Kirillov [FK94] (see, [Hud14]). These polynomials form an inhomogeneous basis of $\mathrm{Poly}_n[\beta]$, where β is the formal parameter from equation (1.1). Specializing at $\beta = 0$, one recovers the Schubert basis $\{\mathfrak{S}_a\} = \{\overline{\mathfrak{S}}_a^{(0)}\}$ of Poly_n. The usual Grothendieck polynomials of A. Lascoux and M.-P. Schützenberger [LS82] are realized at $\beta = -1$. (To help keep track of the relations between bases, we deviate from established practice by denoting the connective K-analogue of each basis of Figure 1 by merely attaching an "overbar" to the notation for that basis. Moreover, for simplicity, from now on we routinely drop "(β)" from the notation for connective K-analogues whenever possible.)

Intersecting $\{\overline{\mathfrak{S}}_a\}$ with $\mathrm{Sym}_n[\beta]$ yields the basis $\{\overline{s}_\lambda\}$ of *symmetric Grothendieck polynomials*. These represent connective K-theory Schubert classes on Grassmannians. A number of Littlewood–Richardson rules for $\{\overline{s}_\lambda\}$ are now known (*e.g.*, [Vak06, TY09, PY17]), following the first found by A. Buch [Buc02]. Like Schur polynomials, symmetric Grothendieck polynomials have quasi-symmetric refinements; each $\overline{s}_\lambda$ expands positively in the quasi-Grothendieck basis $\{\overline{S}_\alpha\}$ of $\mathrm{QSym}_n[\beta]$. This basis, introduced in [Mon16], is the connective K-analogue of the quasi-Schur basis of QSym_n.

Our first new result is that the quasi-Grothendieck basis refines further into the basis of multifundamental quasi-symmetric polynomials $\{\overline{F}_\alpha\}$ of [LP07, PS19], the connective K-analogue of Gessel's fundamental basis of QSym_n.

Theorem 1.1 *Each quasi-Grothendieck polynomial $\overline{S}_\alpha \in \mathrm{QSym}[\beta]$ is a positive sum of multifundamental quasi-symmetric polynomials. That is,*

$$\overline{S}_\alpha = \sum_\gamma J_\gamma^\alpha \, \overline{F}_\gamma,$$

where $J_\gamma^\alpha \in \mathbb{Z}_{\geq 0}[\beta]$ is a positive polynomial in β.

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The multifundamental basis $\{\overline{F}_\alpha\}$ of $\mathrm{QSym}_n[\beta]$ lifts to the *glide basis* $\{\overline{\mathfrak{F}}_a\}$ of $\mathrm{Poly}_n[\beta]$ [PS19], a β-deformation of the slide polynomials. An analogous deformation of the Demazure characters has been studied in [Las01, RY15, Kir16, Mon16, MPS18]. We call these the *Lascoux polynomials* $\{\overline{\mathfrak{D}}_a\}$ in honor of A. Lascoux, who essentially introduced them. They can be approached via the *Lascoux atom* basis $\{\overline{\mathfrak{A}}_a\}$ of $\mathrm{Poly}_n[\beta]$ [Mon16], a β-deformation of the Demazure atoms. Finding a positive formula for the decomposition of Grothendieck polynomials $\{\overline{\mathfrak{S}}_a\}$ into Lascoux polynomials $\{\overline{\mathfrak{D}}_a\}$ analogous to that for the decomposition of Schubert polynomials $\{\mathfrak{S}_a\}$ into Demazure characters $\{\mathfrak{D}_a\}$ is an open problem. (There is an unpublished conjecture for this decomposition by V. Reiner and A. Yong; see [RY15] for discussion.)

Our next major result is to introduce an appropriate β-deformation of the fundamental particles. The *kaon* basis $\{\overline{\mathfrak{P}}_a\}$ of $\mathrm{Poly}_n[\beta]$ yields a common refinement of the glide and Lascoux atom bases; we give explicit positive formulas for these refinements.

Theorem 1.2 *The set $\{\overline{\mathfrak{P}}_a\}$ of kaons is a basis of $\mathrm{Poly}_n[\beta]$. The kaons deform the fundamental particles, in that specializing $\overline{\mathfrak{P}}_a$ at $\beta = 0$ yields the particle $\mathfrak{P}_a$. The kaons are a common refinement of the glide polynomials and of the Lascoux atoms; that is,*

$$\overline{\mathfrak{F}}_a = \sum_b P_b^a \, \overline{\mathfrak{P}}_b \qquad \text{and} \qquad \overline{\mathfrak{A}}_a = \sum_b Q_b^a \, \overline{\mathfrak{P}}_b,$$

where $P_b^a, Q_b^a \in \mathbb{Z}_{\geq 0}[b]$ are positive polynomials in β.

Finally, our last major result is to introduce the new basis $\{\overline{\Omega}_a\}$ of *quasi-Lascoux polynomials*, simultaneously lifting the quasi-Grothendieck basis from $\mathrm{QSym}_n[\beta]$ to $\mathrm{Poly}_n[\beta]$ and giving a β-deformation of the quasi-key polynomials. (Note that, like the quasi-key polynomials, quasi-Lascoux polynomials are not quasi-symmetric.) The quasi-Lascoux polynomials yield a common coarsening of the glide and Lascoux atom bases. We give explicit positive formulas for refining Lascoux polynomials in quasi-Lascoux polynomials and for refining quasi-Lascoux polynomials in both glides and Lascoux atoms.

Theorem 1.3 *The set $\{\overline{\Omega}_a\}$ of quasi-Lascoux polynomials is a basis of $\mathrm{Poly}_n[\beta]$. This basis lifts the quasi-Grothendieck basis of $\mathrm{QSym}_n[\beta]$ in that*

$$\{\overline{\Omega}_a\} \cap \mathrm{QSym}_n[\beta] = \{\overline{S}_\alpha\}.$$

The quasi-Lascoux polynomials deform the quasi-keys, in that specializing $\overline{\Omega}_a$ at $\beta = 0$ yields the quasi-key Ω_a. Finally, the quasi-Lascoux polynomials are a refinement of the Lascoux polynomials and are further refined by the glide polynomials and separately by the Lascoux atoms. That is,

$$\overline{\mathfrak{D}}_a = \sum_b L_b^a \, \overline{\Omega}_b, \qquad \overline{\Omega}_a = \sum_b M_b^a \, \overline{\mathfrak{F}}_b, \quad \text{and} \quad \overline{\Omega}_a = \sum_b N_b^a \, \overline{\mathfrak{A}}_b,$$

where each of $L_b^a, M_b^a, N_b^a \in \mathbb{Z}_{\geq 0}[\beta]$ is a positive polynomial in β.

The relations among these nine families of K-theoretic polynomials are illustrated in Figure 2; compare to the relations among their $\beta = 0$ analogues, as illustrated in Figure 1.

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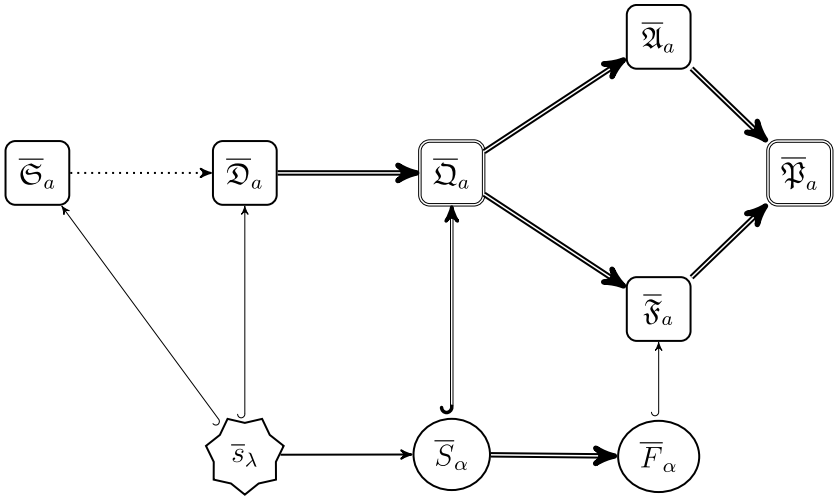


Figure 2: The K-theoretic analogues of the nine cohomological families of polynomials of Figure 1. As in Figure 1, families depicted inside a star are bases of Sym_n, those inside an ellipse are bases of QSym_n, and those inside a rectangle are bases of Poly_n. Those families that are original to this paper are marked with doubled borders. The hooked arrows denote that the basis at the tail is a subset of the basis at the head. The unhooked arrows denote that the basis at the head refines the basis at the tail. Those arrows that are original to this paper are doubled. The dotted arrow is conjectural; see [RY15].

Except for the β-Grothendieck polynomials $\{\overline{\mathfrak{S}}_a\}$ and their symmetric subset $\{\overline{s}_\lambda\}$, the geometric significance of these K-analogues is currently obscure. While, for example, the glide polynomials seem useful in the study of β-Grothendieck polynomials (and thereby of the connective K-theory of Flags_n), it is currently unknown how to interpret any single glide polynomial $\overline{\mathfrak{F}}_a$ as representing a geometric object or datum. We conclude with some conjectures that suggest geometric meaning for these polynomials. While it is possible that these conjectures might be proved by entirely combinatorial means (*e.g.*, sign-reversing involutions), they seem to have the flavor of Euler characteristic calculations. Ideally, we desire a proof of these conjectures where the relevant polynomials are given appropriate geometric interpretations, so that the coefficients in question become the Euler characteristics of some explicit objects. We note that these conjectures are fundamentally K-theoretic, having no analogue in the cohomological setting.

For weak compositions a and b, let $M_b^a(\beta)$ denote the coefficient of $\overline{\mathfrak{F}}_b$ in the glide expansion of $\overline{\mathfrak{Q}}_a$ and let $Q_b^a(\beta)$ denote the coefficient of $\overline{\mathfrak{P}}_b$ in the kaon expansion of the Lascoux atom $\overline{\mathfrak{A}}_a$. Note that $M_b^a(\beta)$ and $Q_b^a(\beta)$ are nonnegative monomials in the single variable β.

Conjecture 1.4 *Let a be a weak composition. Then we have*

$$\sum_b M_b^a(-1) \in \{0,1\} \quad \textit{and} \quad \sum_b Q_b^a(-1) \in \{0,1\},$$

where both sums are over all weak compositions b.

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For example, for $a = (0, 6, 6, 2)$, we have

$$\sum_b M_b^a(\beta) = 16\beta^3 + 75\beta^2 + 94\beta + 36,$$
$$\sum_b Q_b^a(\beta) = 16\beta^3 + 66\beta^2 + 80\beta + 31.$$

In both cases, substituting $\beta = -1$ yields 1, as predicted. We have verified Conjecture 1.4 by computer for all a with at most 3 zeros and $|a| \leq 7$.

This paper is organized as follows. Section 2 recalls the necessary combinatorics of the bases studied in previous works. Section 3 introduces the kaon basis and proves Theorem 1.2, giving the key properties of this basis. Similarly, Section 4 introduces the quasi-Lascoux basis and establishes its key properties by proving Theorems 1.1 and 1.3. We also suggest there (Conjecture 4.25) a remarkable positivity phenomenon for products of Lascoux polynomials.

2 Definitions and Preliminaries

2.1 Glide Polynomials and Fundamental Slide Polynomials

Given a weak composition a, the *positive part* of a is the (strong) composition a^+ obtained by deleting all zero terms from a. For example, $0102^+ = 12$.

Given weak compositions a and b of length n, we say that b *dominates* a, denoted by $b \geq a$, if

$$b_1 + \cdots + b_i \geq a_1 + \cdots + a_i$$

for all $i = 1, \ldots, n$. For example, $0120 \geq 0111$. Note that this partial ordering on weak compositions extends the usual dominance order on partitions.

In [PS19], a *weak komposition* is defined to be a weak composition where the positive integers may be colored arbitrarily black or gray. The *excess* $\mathrm{ex}(b)$ of a weak komposition b is the number of gray entries in b.

Definition 2.1 ([PS19, Definition 2.2]) Let a be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. The weak komposition b is a *glide* of a if there exist integers $0 = i_0 < i_1 < \cdots < i_\ell$ such that, for each $1 \leq j \leq \ell$, we have

(G.1) $a_{n_j} = b_{i_{j-1}+1} + \cdots + b_{i_j} - \mathrm{ex}(b_{i_{j-1}+1}, \ldots, b_{i_j})$,
(G.2) $i_j \leq n_j$,
(G.3) the leftmost nonzero entry among $b_{i_{j-1}+1}, \ldots, b_{i_j}$ is black.

Equivalently, a weak komposition b is a glide of the weak composition a if b can be obtained from a by a finite sequence of the following local moves:

(m.1) $0p \Rightarrow p0$, (for $p \in \mathbb{Z}_{>0}$);
(m.2) $0p \Rightarrow qr$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p$);
(m.3) $0p \Rightarrow qr$ (for $p, q, r \in \mathbb{Z}_{>0}$ with $q + r = p + 1$).

Example 2.2 Let $a = (0, 2, 0, 0, 2, 0, 1)$. The weak kompositions $(1, 2, 0, 2, 0, 1, 1)$ and $(2, 1, 2, 1, 1, 1, 0)$ are glides of a.

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Figure 3: Triples for skyline diagrams.

Definition 2.3 ([PS19, Definition 2.5]) For a weak composition a of length n, the *glide polynomial* $\overline{\mathfrak{F}}_a = \overline{\mathfrak{F}}_a(x_1, \ldots, x_n)$ is

$$\overline{\mathfrak{F}}_a = \sum_b \beta^{\mathrm{ex}(b)} x_1^{b_1} \cdots x_n^{b_n},$$

where the sum is over all weak kompositions b that are glides of a.

Example 2.4 We have

$$\begin{aligned}\overline{\mathfrak{F}}_{0201} = {} & \mathbf{x}^{0201} + \mathbf{x}^{1101} + \mathbf{x}^{0210} + \mathbf{x}^{1110} + \mathbf{x}^{2001} + \mathbf{x}^{2010} + \mathbf{x}^{2100} \\ & + \beta\mathbf{x}^{0211} + \beta\mathbf{x}^{1111} + \beta\mathbf{x}^{1201} + \beta\mathbf{x}^{1210} + \beta\mathbf{x}^{2011} + 2\beta\mathbf{x}^{2101} + 2\beta\mathbf{x}^{2110} \\ & + \beta^2\mathbf{x}^{1211} + 2\beta^2\mathbf{x}^{2111},\end{aligned}$$

where $\mathbf{x}^b = x_1^{b_1} \cdots x_n^{b_n}$.

In [PS19, Proposition 2.16], it was observed that the *fundamental slide polynomials* $\mathfrak{F}_a$ of [AS17] are $\beta = 0$ specializations of glide polynomials. We will take this as definitional for fundamental slides.

2.2 Lascoux Atoms and Quasi-Grothendieck Polynomials

The *skyline diagram* $D(a)$ of a weak composition a is the diagram with a_i boxes in row i, left-justified. In our convention, row 1 is the lowest row. (This convention differs from that of [HLMvW11b, Mon16] where the rows are numbered from top to bottom, and from that of [Mas08] where the "rows" are in fact columns and numbered from left to right.) A *triple* of a skyline diagram is a collection of three boxes with two adjacent in a row and either (Type A) the third box is above the right box and the lower row is weakly longer, or (Type B) the third box is below the left box and the higher row is strictly longer.

Given a filling of the skyline diagram with numbers, a triple (of either type) is called an *inversion triple* if either $\gamma < \alpha \leq \beta$ or $\alpha \leq \beta < \gamma$, and a *coinversion triple* if $\alpha \leq \gamma \leq \beta$.

In [Mon16], C. Monical introduced the notion of semistandard set-valued fillings of skyline diagrams in order to define the (combinatorial) Lascoux atoms, which are K-theoretic analogues of the Demazure atoms.

Definition 2.5 A *set-valued filling* of a skyline diagram is an assignment of a nonempty set of positive integers to each box of the diagram. The maximum entry in

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Figure 4: Two elements of $\overline{\mathfrak{A}}\mathsf{SSF}(1,0,3,2)$ and their corresponding monomial weights. The anchor in each box is drawn in bold; the unbolded entries are free.

each box is called the *anchor*, and all other entries are called *free*. A set-valued filling is *semistandard* if

(S.1) entries do not repeat in a column;
(S.2) rows are weakly decreasing where sets $A \geq B$ if $\min A \geq \max B$;
(S.3) every triple of anchors is an inversion triple;
(S.4) each free entry is in the cell of the least anchor in its column such that (S.2) is not violated;
(S.5) anchors in the first column are equal to their row index.

Remark 2.6 Condition (S.5) replaces the equivalent "basement" requirement in [Mon16]. Condition (S.4) above differs slightly from condition (S4) in [Mon16, §1.2], which puts free entries in the lowest possible row such that (S.2) is not violated. These definitions are, however, equivalent in the sense that there is a simple weight-preserving (and moreover column set-preserving) bijection between these two notions of semistandard set-valued skyline fillings via rearranging the free entries appropriately in each column. The convention in Definition 2.5 turns out to be more natural in the context of the operations we wish to perform on these fillings.

Given a set-valued filling F of shape a, we define the *weight* of F to be the weak composition $\mathrm{wt}(F) = (c_1, \ldots, c_n)$, where c_i is the number of i's in F. Furthermore, $|F| = |\mathrm{wt}(F)|$. Likewise, the *excess* of F, denoted $\mathrm{ex}(F)$, is the number of free entries of F, or equivalently, $\mathrm{ex}(F) = |F| - |a|$. Given a weak composition a, let $\overline{\mathfrak{A}}\mathsf{SSF}(a)$ be the set of semistandard set-valued skyline diagrams of shape a. See Figure 4 for examples; the anchor entries are given in bold.

Definition 2.7 ([Mon16]) Given a weak composition a, the *(combinatorial) Lascoux atom* $\overline{\mathfrak{A}}_a$ is

$$\overline{\mathfrak{A}}_a = \sum_{F \in \overline{\mathfrak{A}}\mathsf{SSF}(a)} \beta^{\mathrm{ex}(F)} \mathbf{x}^{\mathrm{wt}(F)}.$$

We will drop the word "combinatorial" in Definition 2.7 from now on, as we will not have cause to consider any of the other conjecturally equivalent definitions of Lascoux atoms.

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Definition 2.8 For a weak composition a, the *Demazure atom* $\mathfrak{A}_a$ is the $\beta = 0$ specialization of the corresponding Lascoux atom. Equivalently, $\mathfrak{A}_a$ is the generating polynomial for semistandard set-valued skyline diagrams of shape a where exactly one number appears in each box.

Demazure atoms were originally defined by A. Lascoux and M.-P. Schützenberger [LS90] in a different way; the equivalence of Definition 2.8 is due to work by S. Mason [Mas09]. Thus, $\overline{\mathfrak{A}}_a$ is an inhomogeneous deformation of $\mathfrak{A}_a$. The Lascoux atoms form a (finite) basis of $\mathrm{Poly}[\beta] = \mathbb{Z}[x_1, x_2, \dots][\beta]$ by [Mon16, Proposition 2.2].

Definition 2.9 ([Mon16, §3]) Given a (strong) composition α, the *quasi-Grothendieck polynomial* $\overline{S}_\alpha$ in n variables is

$$\overline{S}_\alpha(x_1, \dots, x_n) = \sum_{a^+ = \alpha} \overline{\mathfrak{A}}_a,$$

where the sum is over weak compositions of length n. The $\beta = 0$ specialization is the *quasi-Schur polynomial* $S_\alpha(x_1, \dots, x_n)$ of J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg [HLMvW11b].

In [Mon16], it was shown that the quasi-Grothendieck polynomials form another finite basis of $\mathrm{QSym}[\beta]$.

The Lascoux atoms refine the symmetric Grothendieck polynomials.

Theorem 2.10 ([Mon16]) *We have that*

$$\overline{s}_\lambda(x_1, \dots x_n) = \sum_{\mathrm{sort}(a) = \lambda} \overline{\mathfrak{A}}_a,$$

where the sum is over weak compositions of length n and $\mathrm{sort}(a)$ *is the partition formed by sorting the parts of a in weakly decreasing order.*

Combining Theorem 2.10 and Definition 2.9 yields the decomposition of $\overline{s}_\lambda$ into quasi-Grothendieck polynomials.

Corollary 2.11 ([Mon16]) *We have*

$$\overline{s}_\lambda(x_1, \dots, x_n) = \sum_{\mathrm{sort}(\alpha) = \lambda} \overline{S}_\alpha(x_1, \dots, x_n).$$

Setting $\beta = 0$ in Theorem 2.10 and Corollary 2.11 recovers the earlier refinements by J. Haglund, K. Luoto, S. Mason, and S. van Willigenburg [HLMvW11a] of Schur polynomials into Demazure atoms and into quasi-Schur polynomials, respectively.

3 The Mesonic Bases and their Relations

In this section, we introduce a new basis of polynomials, which we call the "kaon" basis. These new polynomials are a simultaneous refinement of both glide polynomials and Lascoux atoms. In Section 4, we will introduce a new "quasi-Lascoux" basis of polynomials, which are a simultaneous coarsening of these two bases. We will, moreover, find that quasi-Lascoux polynomials stand in the same relation to Lascoux

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atoms as glide polynomials do to kaons. See Figure 2 for a visual representation of the relationships among these various bases.

3.1 Kaons

Definition 3.1 Let a be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. The weak komposition b is a *mesonic glide* of a if, for each $1 \le j \le \ell$, we have

(MG.1) $a_{n_j} = b_{n_{j-1}+1} + \cdots + b_{n_j} - \mathrm{ex}(b_{n_{j-1}+1}, \ldots, b_{n_j})$,
(MG.2) the leftmost nonzero entry among $b_{n_{j-1}+1}, \ldots, b_{n_j}$ is black, and
(MG.3) $b_{n_j} \neq 0$.

Equivalently, a weak komposition b is a mesonic glide of a if b can be obtained from a by a finite sequence of the local moves (m.1), (m.2), and (m.3) that never applies (m.1) at positions $n_j - 1$ and n_j for any j.

Observe that, in particular, a mesonic glide is a glide that happens to satisfy additional conditions. In particular, condition (MG.1) strengthens the combination of (G.1) and (G.2), while (MG.2) is essentially the same condition as (G.3); condition (MG.3), however, is entirely new to mesonic glides.

Example 3.2 Let $a = (0, 3, 0, 2)$. Then $b = (2, 1, 1, \overline{2})$ is a mesonic glide of a.

On the other hand, while $b' = (3, 1, 0, \overline{2})$ is also a glide of a, it is not mesonic. To see this fact, observe that $a_{n_1} = a_2 = 3$, while

$$b'_{n_{j-1}+1} + \cdots + b'_{n_j} - \mathrm{ex}(b'_{n_{j-1}+1}, \ldots, b'_{n_j}) = b'_1 + b'_2 - \mathrm{ex}(b'_1, b'_2) = 3 + 1 - 0 = 4,$$

in violation of (MG.1).

The reader may check that both b and b' can be obtained from a by a finite sequence of the local moves (m.1), (m.2), and (m.3). However, the reader may also check that b' cannot be so obtained without applying (m.1) at positions 1 and 2.

Definition 3.3 Let a be a weak composition. The *kaon* $\overline{\mathfrak{P}}_a$ is the following generating function for mesonic glides:

$$\overline{\mathfrak{P}}_a \coloneqq \sum_b \beta^{\mathrm{ex}(b)} x^b,$$

where the sum is over all mesonic glides of a.

Example 3.4 Let $a = (0, 3, 0, 2)$. Then the corresponding kaon is

$$\begin{aligned}\overline{\mathfrak{P}}_a = {} & \mathbf{x}^{0302} + \mathbf{x}^{0311} + \mathbf{x}^{1202} + \mathbf{x}^{1211} + \mathbf{x}^{2102} + \mathbf{x}^{2111} \\ & + \beta\mathbf{x}^{0312} + \beta\mathbf{x}^{0321} + \beta\mathbf{x}^{1212} + \beta\mathbf{x}^{1221} + \beta\mathbf{x}^{1302} + \beta\mathbf{x}^{1311} \\ & + \beta\mathbf{x}^{2112} + \beta\mathbf{x}^{2121} + \beta\mathbf{x}^{2202} + \beta\mathbf{x}^{2211} + \beta\mathbf{x}^{3102} + \beta\mathbf{x}^{3111} \\ & + \beta^2\mathbf{x}^{1312} + \beta^2\mathbf{x}^{1321} + \beta^2\mathbf{x}^{2212} + \beta^2\mathbf{x}^{2221} + \beta^2\mathbf{x}^{3112} + \beta^2\mathbf{x}^{3121}.\end{aligned}$$

The reader may enjoy realizing each exponent vector as a mesonic glide of a.

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Although this example is multiplicity-free, in general, kaons have nontrivial coefficients in their monomial expansions. For example, the kaon $\overline{\mathfrak{P}}_{002}$ contains the monomial $\beta\mathbf{x}^{111}$ with coefficient 2, corresponding to the distinct mesonic glides $(1,1,1)$ and $(1,1,1)$ of $(0,0,2)$.

3.2 Fundamental Properties of Kaons and the Kaon Expansion of Glide Polynomials

Every glide polynomial $\overline{\mathfrak{F}}_a$ is a positive sum of kaons.

Proposition 3.5 *For any weak composition a, we have*

$$\overline{\mathfrak{F}}_a = \sum_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{P}}_b.$$

Proof Let a be a weak composition with nonzero entries in positions $n_1 < \cdots < n_\ell$. Suppose g is a glide of a. Then there are $0 = i_0 < i_1 < \cdots < i_\ell$ satisfying conditions (G.1), (G.2), and (G.3) of Definition 2.1. Then g can be obtained from a via a 2-step process. First, apply (m.1) repeatedly to move each nonzero entry of a from position n_j to position i_j. Call the resulting weak composition b. Note that b satisfies $b \geq a$ and $b^+ = a^+$. Second, apply some sequence of (m.1), (m.2) and (m.3) to obtain the weak komposition g from b. In this second step, note that we never apply (m.1) at positions $i_j - 1$ and i_j for any j.

Hence, every glide g of a is a mesonic glide of a weak composition b with $b \geq a$ and $b^+ = a^+$, and so every term of the left-hand polynomial is a term of the right-hand polynomial.

Conversely, every mesonic glide of such a weak composition b with $b \geq a$ and $b^+ = a^+$ is clearly a glide of a. Thus, to complete the proof, we only need to show that for every glide g of a, there is at most one b with $b^+ = a^+$ such that g is a mesonic glide of b. Let $\alpha = a^+ = (\alpha_1, \ldots, \alpha_\ell)$. Suppose that b, c are weak compositions with $b^+ = c^+ = \alpha$ such that g is a mesonic glide of both b and c. Say b has nonzero entries in positions $s_1 < \cdots < s_\ell$, while c has nonzero entries in positions $t_1 < \cdots < t_\ell$. By the definition of mesonic glide, we know that

$$\alpha_j = b_{s_j} = g_{s_{j-1}+1} + \cdots + g_{s_j} - \mathsf{ex}(g_{s_{j-1}+1}, \ldots, g_{s_j}),$$

the leftmost nonzero entry among $g_{s_{j-1}+1}, \ldots, g_{s_j}$ is black, and $g_{s_j} \neq 0$. In the same way, we have that

$$\alpha_j = c_{t_j} = g_{t_{j-1}+1} + \cdots + g_{t_j} - \mathsf{ex}(g_{t_{j-1}+1}, \ldots, g_{t_j}),$$

the leftmost nonzero entry among $g_{t_{j-1}+1}, \ldots, g_{t_j}$ is black, and $g_{t_j} \neq 0$. If $s_j = t_j$ for all j, then $b = c$, and we are done. Otherwise, there is a least index i such that $s_i \neq t_i$. Without loss of generality, assume $s_i > t_i$. Then

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$$\begin{aligned}\alpha_i &= g_{s_{i-1}+1} + \cdots + g_{s_i} - \mathsf{ex}(g_{s_{i-1}+1}, \ldots, g_{s_i}) \\ &= g_{s_{i-1}+1} + \cdots + g_{t_i} + g_{t_i+1} + \cdots + g_{s_i} - \mathsf{ex}(g_{s_{i-1}+1}, \ldots, g_{t_i}) \\ &\quad - \mathsf{ex}(g_{t_i+1}, \ldots, g_{s_i}) \\ &= g_{t_{i-1}+1} + \cdots + g_{t_i} - \mathsf{ex}(g_{t_{i-1}+1}, \ldots, g_{t_i}) \\ &\quad + g_{t_i+1} + \cdots + g_{s_i} - \mathsf{ex}(g_{t_i+1}, \ldots, g_{s_i}) \\ &= \alpha_i + g_{t_i+1} + \cdots + g_{s_i} - \mathsf{ex}(g_{t_i+1}, \ldots, g_{s_i}),\end{aligned}$$

and so we have

$$0 = g_{t_i+1} + \cdots + g_{s_i} - \mathsf{ex}(g_{t_i+1}, \ldots, g_{s_i}).$$

This is only possible if each of $g_{t_i+1}, \ldots, g_{s_i}$ is either 0 or a gray 1. Since $g_{s_i} \neq 0$, there is at least one gray 1 in this set of entries. However, the first nonzero entry of $g_{t_i+1}, \ldots, g_{t_{i+1}}$ is required to be black, a contradiction. ∎

Theorem 3.6 *The set*

$$\{\beta^k \overline{\mathfrak{P}}_a^{(\beta)} : k \in \mathbb{Z}_{\geq 0} \text{ and } a \text{ is a weak composition of length } n\}$$

is an additive basis of the free $\mathbb{Z}$*-module* $\mathrm{Poly}_n[\beta]$*. Hence, for any fixed* $p \in \mathbb{Z}$*,*

$$\{\overline{\mathfrak{P}}_a^{(p)} : a \text{ is a weak composition of length } n\}$$

is a basis of Poly_n*.*

Proof By Proposition 3.5, every glide polynomial can be written as a positive sum of kaons, and indeed the transition matrix is unitriangular with respect to the lexicographic total order on weak compositions. Hence, the transition matrix is invertible over $\mathbb{Z}$, and the theorem follows from the fact that glide polynomials are an additive basis of $\mathrm{Poly}_n[\beta]$, as shown in [PS19, Theorem 2.6]. ∎

A homogeneous basis $\mathfrak{P}_a$ of Poly_n called *fundamental particles* was introduced in [Sea20]. This basis is a common refinement of fundamental slides and Demazure atoms. We will show that the kaon basis plays the analogous role for glide polynomials and Lascoux atoms.

Proposition 3.7 *The fundamental particles* $\mathfrak{P}_a$ *are the* $\beta = 0$ *specialization of kaons:* $\mathfrak{P}_a = \overline{\mathfrak{P}}_a^{(0)}$.

Proof This is clear from the definitions of the two families of polynomials. ∎

Remark 3.8 Proposition 3.7 motivates our choice of the name "kaon" for these polynomials. In high energy physics, the "K" fundamental particles are the *K-mesons* or *kaons*. By analogy, perhaps, the cohomological specialization $\pi_a = \overline{\mathfrak{P}}_a^{(0)}$ could be named for the lighter analogue of the K-meson, the *π-meson* or *pion*. These mesons play a role in the structural integrity of atomic nuclei; somewhat analogously, we will show momentarily that the kaon polynomials decompose the Lascoux atoms, thereby controlling their structure.

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Remark 3.9 In light of Proposition 3.7, setting $\beta = 0$ in Proposition 3.5 recovers [Sea20, Proposition 4.7] on the expansion of fundamental slide polynomials into fundamental particles. Note that although the K-theoretic deformations of both families of polynomials are significantly larger, the matrix of basis change from Proposition 3.5 is exactly the same as that for fundamental slides into fundamental particles. Taking $p = 0$ in Theorem 3.6 recovers [Sea20, Proposition 4.6] as a special case.

The kaon basis does not have positive structure coefficients. Nonetheless, we conjecture the following.

Conjecture 3.10 *For any weak compositions a and b, the product*

$$\overline{\mathfrak{P}}_a \cdot \overline{\mathfrak{F}}_b$$

of a kaon and a glide polynomial expands positively in the kaon basis.

For example, we have

$$\overline{\mathfrak{P}}_{(2,0,1)} \cdot \overline{\mathfrak{F}}_{(1,0,2)} = \overline{\mathfrak{P}}_{(3,0,3)} + \beta\overline{\mathfrak{P}}_{(3,1,3)} + \beta\overline{\mathfrak{P}}_{(3,2,2)} + \beta^2\overline{\mathfrak{P}}_{(3,2,3)} + \beta^2\overline{\mathfrak{P}}_{(3,3,2)}.$$

We have computationally verified Conjecture 3.10 for all weak compositions a, b with at most 3 zeros and $|a|, |b| \leq 5$. To our knowledge, Conjecture 3.10 is new, even in the special $\beta = 0$ case of the fundamental particle expansion of the product of a fundamental particle by a fundamental slide polynomial.

3.3 The Kaon Expansion of a Lascoux Atom

Definition 3.11 Let a be a weak composition and $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$. We say T is *meson-highest* if, for every integer i appearing in T, either

- the leftmost i is in the leftmost column and is an anchor, or
- there is an $i^{\uparrow}$ in some column weakly to the right of the leftmost i and in a different box, where $i^{\uparrow}$ is the smallest label greater than i appearing in T.

In light of the following Theorem 3.12, we write $\overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)$ for the set of all meson-highest $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$.

Theorem 3.12 *For any weak composition a, we have*

$$\overline{\mathfrak{A}}_a = \sum_{T \in \overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)} \beta^{|T|-|a|}\overline{\mathfrak{P}}_{\mathrm{wt}(T)}.$$

In particular, every Lascoux atom $\overline{\mathfrak{A}}_a$ is a positive sum of kaons.

To prove Theorem 3.12, we must first develop properties of a *destandardization* map on $\overline{\mathfrak{A}}\mathsf{SSF}(a)$. Fix $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$. Consider the least integer i with the property that

- the leftmost i in T is not an anchor in the leftmost column, and
- this leftmost i has no $i^{\uparrow}$ weakly to its right in a different box.

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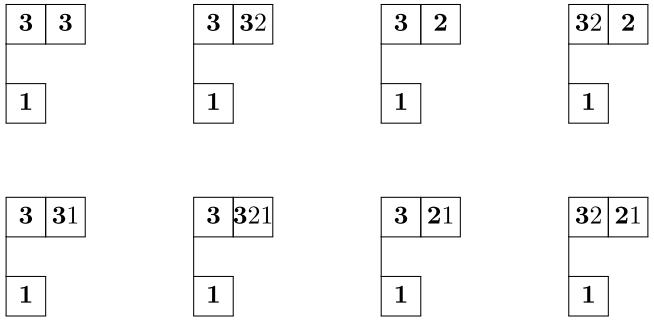


Figure 5: The eight elements of $\overline{\mathfrak{A}}\mathsf{SSF}(102)$. The set $\overline{\mathfrak{A}2\mathfrak{P}}(102)$ consists of two fillings, specifically the leftmost filling in each of the two rows. Each of the eight illustrated fillings destandardizes to the leftmost filling in its row.

Replace every i in T with an $i+1$. (If this results in two instances of $i+1$ in a single box, delete one.) Repeat this replacement process until no further replacements can be made; the final result is the destandardization $\mathrm{dst}(T)$. (This algorithm necessarily terminates, as we only perform replacement on labels i that are strictly less than the maximum entry k of T; this is because k is guaranteed to appear as an anchor in the leftmost column of T.) For an example of these notions, see Figure 5.

Remark 3.13 In fact, it is not hard to see that the order in which we perform replacements does not affect the resulting destandardization. Nonetheless, it is convenient to fix the explicit replacement order chosen above.

Lemma 3.14 *Let a be a weak composition. If $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$, then* $\mathrm{dst}(T) \in \overline{\mathfrak{A}2\mathfrak{P}}(a)$. *Moreover, destandardization is a retraction onto $\overline{\mathfrak{A}2\mathfrak{P}}(a)$, as we have* $\mathrm{dst}(T) = T$ *if and only if $T \in \overline{\mathfrak{A}2\mathfrak{P}}(a) \subseteq \overline{\mathfrak{A}}\mathsf{SSF}(a)$.*

Proof Fix the weak composition a. By definition, if $T \in \overline{\mathfrak{A}2\mathfrak{P}}(a)$, then $\mathrm{dst}(T) = T$. Moreover, if $T \notin \overline{\mathfrak{A}2\mathfrak{P}}(a)$, then by definition $\mathrm{dst}(T) \neq T$. Hence, the third sentence of the lemma is clear.

It remains to establish the second sentence of the lemma. Fix $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$. It is enough to show that $\mathrm{dst}(T) \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$, for then $\mathrm{dst}(T) \in \overline{\mathfrak{A}2\mathfrak{P}}(a)$ follows easily, as the destandardization algorithm does not terminate until the extra conditions defining $\overline{\mathfrak{A}2\mathfrak{P}}(a)$ as a subset of $\overline{\mathfrak{A}}\mathsf{SSF}(a)$ are satisfied. Indeed, since destandardization is defined as a sequence of replacements, it is enough by induction to show that any single such replacement produces an element of $\overline{\mathfrak{A}}\mathsf{SSF}(a)$.

Suppose we apply replacement to the letters i and the result is T'. Then, by assumption, the leftmost $i \in T$

- is not an anchor in row i of the leftmost column of T, and
- does not have an $i^{\uparrow}$ weakly to its right in T and in a different box.

We want to show that T' satisfies conditions (S.1)–(S.5).

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(S.1) If there is no column of T containing both i and $i+1$, then it is clear that T' satisfies (S.1). Hence, suppose column c of T contains both i and $i+1$. Then $i^{\uparrow} = i+1$. Since T has then no $i+1$ weakly to the right of the leftmost i, c must be the column of the leftmost i. Moreover, i and $i+1$ must appear in the same box b of column c in T. Thus, replacement results in two instances of $i+1$ in b, one of which we then delete by construction. Thus, T' has no repeated entries in any column.

(S.2) If row i of T contains an entry i, then by (S.2) and (S.5) for T, row i has i as an anchor in the first column. Thus, in this case, the leftmost i is an anchor in the first column, contradicting our assumptions on the number i.

Therefore by (S.2) and (S.5) for T, every entry i in T is in a row with an index j strictly greater than i. Moreover, for each such $j > i$, we have by (S.2) for T that all labels strictly to the left of the leftmost i in row j are strictly greater than i. Hence, replacing every i in T with $i+1$ preserves the rows being weakly decreasing.

(S.3) To see that no type A coinversion triples appear in T', suppose first that T has a type A inversion triple with $\gamma < \alpha \leq \beta$. This could become a coinversion triple in T' only if $\gamma = i$ and $\alpha = i+1 = i^{\uparrow}$. However, in this case, T has i and $i^{\uparrow}$ in distinct boxes of the same column, contradicting our assumptions on the number i.

Now suppose instead that T has a type A inversion triple with $\alpha \leq \beta < \gamma$. This could become a coinversion triple in T' only if $\gamma = i+1 = i^{\uparrow}$ and $\beta = i$. However, in this case, T has $i+1$ appearing strictly to the right of i, again contradicting our assumptions on the number i.

To see that no type B coinversion triples appear in T', suppose first that T has a type B inversion triple with $\gamma < \alpha \leq \beta$. This could become a coinversion triple in T' only if $\gamma = i$ and $\alpha = i+1 = i^{\uparrow}$. However, then T would have an $i^{\uparrow}$ strictly to the right of an i, contradicting our assumptions on i.

Finally, suppose T has a type B inversion triple with $\alpha \leq \beta < \gamma$. This could become a coinversion triple in T' only if $\gamma = i+1 = i^{\uparrow}$ and $\beta = i$. However, then T would have an i and an $i+1$ in distinct boxes of the same column, again contradicting our assumptions on i.

(S.4) If a free entry i of T becomes a free entry $i+1$ of T' and is not deleted, then its anchor entry j is larger than $i+1$ in both T and T'. In particular, since j was the smallest anchor entry in this column accepting a free entry i in T (by (S.4) for T), j is still the smallest anchor entry accepting a free entry $i+1$ in T'.

If an anchor entry i of T becomes an anchor entry $i+1$ of T', then since any other anchor entry in this column is either greater than $i+1$ or smaller than i, any free entries in the cell of this anchor entry are still with the smallest possible anchor entry. Any other free entries in a column where an anchor entry i becomes an anchor entry $i+1$ are also still with the smallest possible anchor entry, again since other anchor entries in this column are either greater than $i+1$ or smaller than i.

(S.5) By construction, the replacement operation taking T to T' does not affect the anchor entries in the first column. ∎

Lemma 3.15 *Let a be a weak composition and $S \in \overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)$. Then*

$$\overline{\mathfrak{P}}_{\mathrm{wt}(S)} = \sum_{T \in \mathrm{dst}^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^{\mathrm{wt}(T)}.$$

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Proof We establish a weight-preserving bijection between mesonic glides of $\mathrm{wt}(S)$ and fillings $T \in \mathrm{dst}^{-1}(S)$.

Fix $T \in \mathrm{dst}^{-1}(S)$. Define the *colored weight* $\mathrm{kwt}(T)$ of T to be the weak komposition obtained by coloring the $(i+1)$-st entry of $\mathrm{wt}(T)$ gray if any $i+1$ is deleted after replacing every i with $i+1$ during a step of destandardization. (Note that at most one $i+1$ can be so deleted, as otherwise before replacement there would be two boxes, each containing both i and $i+1$; in such a case, there would be an $i+1$ strictly to the right of an i, in violation of the rules for when to perform replacement.)

We claim that $\mathrm{kwt}(T)$ is a mesonic glide of $\mathrm{wt}(S)$. Consider the process of destandardization of T to produce S. Each time we replace every i in T' by $i+1$ to produce T'', we change the i-th and $(i+1)$-st entries of the colored weight by

$$(q, r) \longmapsto (0, q + r - 1),$$

if a duplicate $i+1$ is deleted, or

$$(q, r) \longmapsto (0, q + r),$$

if not. Since these are the inverses of the local move (m.3) in the first case and of either (m.1) or (m.2) in the second case, it follows then that $\mathrm{kwt}(T)$ is a glide of the weak composition $\mathrm{wt}(S)$.

Let $\mathrm{wt}(S)$ have nonzero entries in positions $n_1 < \cdots < n_\ell$. A local change to the colored weight $(q, 0) \mapsto (0, q)$ in positions $n_j - 1$ and n_j for some j would correspond to a step of destandardization replacing every instance of $n_j - 1$ with n_j in a T' that contains no label n_j. Since S is meson-highest, the entries n_j of S satisfy the meson-highest condition. Since T' destandardizes to S, entries $n_j - 1$ of T' therefore also satisfy the meson-highest condition. This contradicts the application of such a destandardization step to T'. Hence, $\mathrm{kwt}(T)$ is a mesonic glide of $\mathrm{wt}(S)$.

For the other direction, let b be a mesonic glide of the weak composition $\mathrm{wt}(S)$. We construct the unique $T \in \mathrm{dst}^{-1}(S)$ such that $\mathrm{kwt}(T) = b$ as follows. Suppose $\mathrm{wt}(S)$ has nonzero entries in positions $n_1 < \cdots < n_\ell$.

Begin with the smallest entries of S, *i.e.,* the $\mathrm{wt}(S)_{n_1}$ instances of the entry n_1 in S. Consider the sequence $s_1 = (1, \ldots, i_1)$ of positions of entries associated (via (G.1)) to n_1 in the mesonic glide b. From this sequence, we construct a string str_1 by appending letters to the empty word as follows. Reading s_1 from left to right, for each $k \in s_1$, consider the entry b_k. If b_k is a black entry, append b_k black k's to the end of the string; if b_k is a gray entry, append one gray k followed by $b_k - 1$ black k's. For example, if $\mathrm{wt}(S) = (0, 0, 0, 0, 4, 3)$ and $b = (1, 0, 2, 0, 2, 3)$, then $i_1 = n_1 = 5$, and $\mathsf{str}_1 = 13355$.

Now replace the instances of n_1 in S with entries of str_1 as follows. In the rightmost box of S containing an n_1, replace that n_1 with the first letter of str_1. If the next letter of str_1 is gray, place it in the same box the previous entry was placed in. Otherwise, place it in the next box of S to the left that contains an n_1, replacing that n_1. Continue in this manner until all letters of str_1 have been placed into S. This procedure is well defined, since by (S.1), no more than one entry n_1 appears in any column of S; and since by (G.1), the number $\mathrm{wt}(S)_{n_1}$ of n_1's in S is exactly the length of the string str_1 minus the number of gray entries, each black entry goes in a different box of S, and each gray entry goes in a box along with a black entry. Repeat this process with $n_2, \ldots, n_\ell$, where s_j is the sequence $(i_{j-1} + 1, \ldots, i_j)$. Since we work with mesonic glides, we will

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have $i_j = n_j$ for all j; we introduce this extra level of generality in order to be able to apply this same procedure for arbitrary glides (where i_j may be smaller than n_j) in the proof of Lemma 4.16 later. (This algorithm is a minor modification of that appearing in [PS19, Proof of Theorem 3.5].)

For example, let S be the leftmost filling in the lower row of Figure 5. We have $\mathrm{wt}(S) = (2,0,2)$. A mesonic glide of $\mathrm{wt}(S)$ is $b = (2,1,2)$. We have $i_1 = n_1 = 1$, $i_2 = n_2 = 3$, $\mathsf{str}_1 = 11$, and $\mathsf{str}_2 = 233$. We construct the filling T from S and b as follows. Replace the two 1s in S with the two 1s from str_1 (which does nothing to S), then replace the two 3s in S with str_2. The 2 and 3 from str_2 are placed in the rightmost box (along with the 1 already there); then the remaining 3 from str_2 is placed in the box to the left. In this way, we obtain T as the second filling from the left in the lower row of Figure 5. Note that indeed T has weight $(2,1,2)$ and destandardizes to S.

By construction, the resulting filling T has weight b and destandardizes to S. We need to show that $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$. To see this, notice that the entries of str_j (which replace the entries n_j in S) are all strictly larger than n_{j-1} and weakly smaller than n_j. This fact implies that all inequalities between entries of boxes in S are preserved, and thus all of (S.1), (S.2), (S.3), and (S.4) are preserved. Finally, since $S \in \overline{\mathfrak{A}2\mathfrak{P}}(a) \subseteq \overline{\mathfrak{A}}\mathsf{SSF}(a)$, the first column of S has an anchor a_i in each nonempty row a_i, and so these (nonzero) a_i's are a subset of the n_i's. Since b is a mesonic glide of $\mathrm{wt}(S)$, by definition the last entry of each str_i is n_i. Therefore, the process of constructing T from b and S ensures that the anchors in the first column are replaced by themselves; *i.e.,* they do not change. Hence, (S.5) is also satisfied, and we have $T \in \overline{\mathfrak{A}}\mathsf{SSF}(a)$.

The uniqueness of T follows from the lack of choice at each step in this process. ■

Proof of Theorem 3.12 For $S \in \overline{\mathfrak{A}2\mathfrak{P}}(a)$, Lemma 3.15 says that

$$\overline{\mathfrak{P}}_{\mathrm{wt}(S)} = \sum_{T \in \mathrm{dst}^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^{\mathrm{wt}(T)}.$$

Therefore,

$$\begin{aligned}\sum_{S \in \overline{\mathfrak{A}2\mathfrak{P}}(a)} \beta^{|S|-|a|} \overline{\mathfrak{P}}_{\mathrm{wt}(S)} &= \sum_{S \in \overline{\mathfrak{A}2\mathfrak{P}}(a)} \beta^{|S|-|a|} \sum_{T \in \mathrm{dst}^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^{\mathrm{wt}(T)} \\ &= \sum_{U \in \overline{\mathfrak{A}}\mathsf{SSF}(a)} \beta^{|U|-|a|} \mathbf{x}^{\mathrm{wt}(U)} = \overline{\mathfrak{A}}_a,\end{aligned}$$

where the second equality is by Lemma 3.14 and the third equality is by definition. ■

Remark 3.16 Setting $\beta = 0$ in Theorem 3.12 recovers the expansion of Demazure atoms into fundamental particles from [Sea20, Theorem 4.17]. In particular, the meson-highest fillings with no free entries are exactly the "particle-highest" fillings of [Sea20].

4 Lascoux and Quasi-Lascoux Polynomials

4.1 Quasi-Lascoux Polynomials

The quasi-key basis of [AS18b] is a common coarsening of the fundamental slide and Demazure atom bases of Poly_n, a refinement of the basis of Demazure characters, and a lifting of the quasi-Schur basis of [HLMvW11a] from QSym_n to Poly_n. We introduce

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a K-theoretic analogue of the quasi-key basis—alternatively, a lifting of the quasi-Grothendieck basis from $\mathrm{QSym}[\beta]$ to $\mathrm{Poly}[\beta]$. See Figure 2 for a visual representation of the relationships among these various bases.

Definition 4.1 Given a weak composition a, the *quasi-Lascoux polynomial* $\overline{\mathfrak{Q}}_a$ is given by

$$\overline{\mathfrak{Q}}_a = \sum_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{A}}_b.$$

Proposition 4.2 *The set*

$$\{\beta^k \overline{\mathfrak{Q}}_a^{(\beta)} : k \in \mathbb{Z}_{\geq 0} \text{ and } a \text{ is a weak composition of length } n\}$$

is an additive basis of the free $\mathbb{Z}$-module $\mathrm{Poly}_n[\beta]$. *Hence, for any fixed $p \in \mathbb{Z}$,*

$$\{\overline{\mathfrak{Q}}_a^{(p)} : a \text{ is a weak composition of length } n\}$$

is a basis of Poly_n.

Proof First, we show that

$$\{\beta^k \overline{\mathfrak{Q}}_a^{(\beta)} : k \in \mathbb{Z}_{\geq 0} \text{ and } a \text{ is a weak composition of length } n\}$$

is a spanning set. A monomial m in $\mathrm{Poly}_n[\beta]$ corresponds to a pair (k, a), where $k \in \mathbb{Z}_{\geq 0}$ records the power of β in m and a is the weak composition of length n recording the degrees of $x_1, x_2, \ldots, x_n$ in m. Write $\mathcal{M}$ for the set of all such pairs.

For a a weak composition of length n, let s_a denote the string $1^{a_1}2^{a_2}\cdots n^{a_n}$. Write $M(a)$ for the largest element of a and write $\ell_+(a)$ for the position of the rightmost nonzero entry of a. Define a total order on $\mathcal{M}$ by $(k, a) \succ (h, b)$ if

- $\ell_+(a) > \ell_+(b)$,
- $\ell_+(a) = \ell_+(b)$ and $M(a) > M(b)$,
- $\ell_+(a) = \ell_+(b)$, $M(a) = M(b)$, and $s_a >_{\mathrm{lex}} s_b$, or
- $a = b$ and $k > h$.

(Here, $>_{\mathrm{lex}}$ denotes lexicographic order on strings of numbers.) For example, we have

$$(2, 23001) \succ (7, 10340) \succ (2, 23010) \succ (2, 32010) \succ (1, 32010).$$

The $\prec$-*leading term* of a polynomial $f \in \mathrm{Poly}_n[\beta]$ is the term of f whose monomial is largest in this order. Now observe that the $\prec$-leading term of $\overline{\mathfrak{Q}}_a$ is $\beta^0 \mathbf{x}^a$. Hence, if the $\prec$-leading term of $f \in \mathrm{Poly}_n[\beta]$ is $c_a \beta^k \mathbf{x}^a$, then the $\prec$-leading term of $f_1 \coloneqq f - c_a \beta^k \overline{\mathfrak{Q}}_a$ is $c_b \beta^h \mathbf{x}^b$ for some $(h, b) \prec (k, a)$. Then $f_2 \coloneqq f - c_b \beta^h \overline{\mathfrak{Q}}_b$ has $\prec$-leading term $c_d \beta^j \mathbf{x}^d$ for some $(j, d) \prec (h, b)$, etc. Since $\prec$ is a well order on $\mathcal{M}$, this process must terminate. Hence, f is a finite $\mathbb{Z}$-linear combination of elements of our putative basis.

Linear independence is immediate from each element of the putative basis having a different leading term. This proves the first sentence of the proposition.

The second sentence of the proposition is immediate from the first. ∎

Proposition 4.3 *The quasi-key polynomials are the $\beta = 0$ specialization of quasi-Lascoux polynomials:* $\overline{\mathfrak{Q}}_a^{(0)} = \mathfrak{Q}_a$.

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Proof In [Sea20], it was proved that

$$\mathfrak{Q}_a = \sum_{\substack{b \geq a \\ b^+ = a^+}} \mathfrak{A}_b.$$

The statement then follows from Definition 4.1 and the fact that $\overline{\mathfrak{A}}_b^{(0)} = \mathfrak{A}_b$. ∎

The following is clear from Definitions 2.9 and 4.1.

Proposition 4.4 *Suppose that the positions of the nonzero entries in the weak composition a form an interval and that a_k is the last nonzero entry of a. Then*

$$\overline{\mathfrak{Q}}_a = \overline{S}_{a^+}(x_1, \ldots, x_k).$$

In particular, every quasi-Grothendieck polynomial is a quasi-Lascoux polynomial.

Moreover, we have the following proposition.

Proposition 4.5 *Let a be a weak composition. Then the stable limit $\lim_{m\to\infty} \overline{\mathfrak{Q}}_{0^m \times a}$ of the quasi-Lascoux polynomial $\overline{\mathfrak{Q}}_a$ is the quasi-Grothendieck function $\overline{S}_{a^+}(x_1, x_2, \ldots)$.*

Proof Let $m > 0$ and consider the polynomial $\overline{\mathfrak{Q}}_{0^m \times a}(x_1, \ldots, x_m)$. Observe that, for any weak composition b, the Lascoux atom $\overline{\mathfrak{A}}_b$ is divisible by x_{b_i} whenever $b_i > 0$. Hence, if $\overline{\mathfrak{A}}_b^{(\beta)}$ appears in the Lascoux atom expansion of $\overline{\mathfrak{Q}}_{0^m \times a}$, then it is annihilated on restriction to m variables unless $\ell_+(b) \leq m$. Thus, by Definitions 2.9 and 4.1,

$$\overline{\mathfrak{Q}}_{0^m \times a}(x_1, \ldots, x_m) = \sum_{\substack{b^+ = a^+ \\ \ell_+(b) \leq m}} \overline{\mathfrak{A}}_b = \overline{S}_{a^+}(x_1, \ldots, x_m).$$

The proposition then follows by letting $m \to \infty$. ∎

Remark 4.6 Setting $\beta = 0$ in Proposition 4.5 gives a new proof of the fact that quasi-key polynomials stabilise to quasi-Schur functions; this was proved using a different method in [AS18b, §4.3].

To give the monomial expansion of a quasi-Lascoux polynomial directly, we define $\overline{\mathfrak{Q}}\mathsf{SSF}(a)$ to be all set-valued skyline fillings of shape a satisfying (S.1)–(S.4), as well as

(S.5′) anchors in the first column are at most their row index and decrease from top to bottom.

We call $\overline{\mathfrak{Q}}\mathsf{SSF}(a)$ the *set-valued quasi-skyline fillings* of shape a. Then we have

Proposition 4.7 *Given a weak composition a, we have*

$$\overline{\mathfrak{Q}}_a = \sum_{S \in \overline{\mathfrak{Q}}\mathsf{SSF}(a)} \beta^{|S|-|a|} \mathbf{x}^{\mathrm{wt}(S)}.$$

Proof There is a weight-preserving bijection

$$\overline{\mathfrak{Q}}\mathsf{SSF}(a) \longleftrightarrow \coprod_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{A}}\mathsf{SSF}(b),$$

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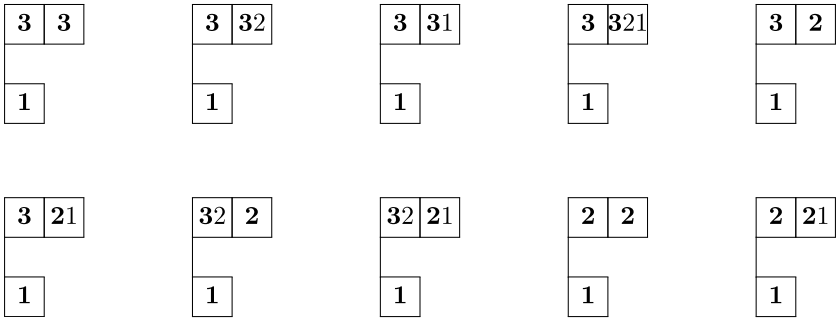


Figure 6: The ten elements of $\overline{\Omega}\mathsf{SSF}(102)$.

where the image of $T \in \overline{\Omega}\mathsf{SSF}(a)$ is obtained by moving each row of T downwards until each anchor in the first column is equal to its row index. This is well defined, since moving rows without changing their relative order does not affect the inversion/coinversion status of any triple. The proposition then follows from Definitions 2.7 and 4.1. ■

Example 4.8 For $a = (1, 0, 2)$, the set $\overline{\Omega}\mathsf{SSF}(a)$ consists of the ten fillings shown in Figure 6. Therefore, we have

$$\overline{\Omega}_{102} = \mathbf{x}^{102} + \beta\mathbf{x}^{112} + \beta\mathbf{x}^{202} + \beta^2\mathbf{x}^{212} + \mathbf{x}^{111} + \beta\mathbf{x}^{211} + \beta\mathbf{x}^{121} + \beta^2\mathbf{x}^{221} + \mathbf{x}^{120} + \beta\mathbf{x}^{220}.$$

4.2 The Glide Expansion of a Quasi-Lascoux Polynomial

Definition 4.9 Let a be a weak composition and let $S \in \overline{\Omega}\mathsf{SSF}(a)$ be a set-valued quasi-skyline filling. We say S is *quasi-Yamanouchi* if, for every integer i appearing in S, either

- the leftmost i is an anchor in row i of the leftmost column, or
- there is an $i+1$ in some column weakly right of the leftmost i and in a different box.

In light of Theorem 4.12, we write $\overline{\Omega}2\overline{\mathfrak{F}}(a)$ for the set of all quasi-Yamanouchi $S \in \overline{\Omega}\mathsf{SSF}(a)$.

Remark 4.10 The quasi-Yamanouchi condition (Definition 4.9) for set-valued quasi-skyline fillings is exactly the meson-highest condition (Definition 3.11) for semistandard set-valued skyline fillings, with $i^{\uparrow}$ replaced by $i+1$.

Example 4.11 The first and third fillings in the top row of Figure 6 are quasi-Yamanouchi. The other fillings in Figure 6 are not.

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Theorem 4.12 *For any weak composition a, we have*

$$\overline{\mathfrak{Q}}_a = \sum_{S \in \overline{\mathfrak{Q}}2\overline{\mathfrak{F}}(a)} \beta^{|S|-|a|} \overline{\mathfrak{F}}_{\mathrm{wt}(S)}.$$

In particular, every quasi-Lascoux polynomial $\overline{\mathfrak{Q}}_a$ is a positive sum of glide polynomials.

To prove Theorem 4.12, we introduce a destandardization map dst_Q on $\overline{\mathfrak{Q}}\mathsf{SSF}(a)$. Fix $T \in \overline{\mathfrak{Q}}\mathsf{SSF}(a)$. Consider the least integer i appearing in T with the property that

- the leftmost i in T is not an anchor in row i of the leftmost column, and
- it has no $i+1$ weakly to its right in a different box.

Replace every i in T with an $i+1$. (If this results in two instances of $i+1$ in a single box, delete one.) Repeat this replacement process until no further replacements can be made; the final result is the destandardization $\mathrm{dst}_Q(T)$.

Remark 4.13 The destandardization map dst_Q is exactly the destandardization map dst of Section 3 with $i+1$ everywhere in place of $i^{\uparrow}$.

Example 4.14 The first, second, fifth, seventh, and ninth fillings of Figure 6 destandardize to the first filling; the remaining fillings destandardize to the third filling.

The following result is entirely analogous to Lemma 3.14.

Lemma 4.15 *Let a be a weak composition. If $T \in \overline{\mathfrak{Q}}\mathsf{SSF}(a)$, then $\mathrm{dst}_Q(T) \in \overline{\mathfrak{Q}}2\overline{\mathfrak{F}}(a)$. Moreover, destandardization is a retraction onto $\overline{\mathfrak{Q}}2\overline{\mathfrak{F}}(a)$, as we have $\mathrm{dst}_Q(T) = T$ if and only if $T \in \overline{\mathfrak{Q}}2\overline{\mathfrak{F}}(a) \subseteq \overline{\mathfrak{Q}}\mathsf{SSF}(a)$.*

Proof This is identical to the proof of Lemma 3.14, with $i+1$ everywhere in place of $i^{\uparrow}$. ∎

Lemma 4.16 *Let a be a weak composition and $S \in \overline{\mathfrak{Q}}2\overline{\mathfrak{F}}(a)$. Then*

$$\overline{\mathfrak{F}}_{\mathrm{wt}(S)} = \sum_{T \in \mathrm{dst}_Q^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^{\mathrm{wt}(T)}.$$

Proof We need to establish a weight-preserving bijection between the glides of the weak composition $\mathrm{wt}(S)$ and the fillings $T \in \mathrm{dst}_Q^{-1}(S)$.

Fix $T \in \mathrm{dst}_Q^{-1}(S)$. Define the colored weight $\mathrm{kwt}(T)$ of T to be the weak komposition obtained by coloring the $(i+1)$st entry of $\mathrm{wt}(T)$ gray if an $i+1$ is deleted after replacing every i with an $i+1$ during a step of destandardization. By the same exact reasoning as in the analogous step of the proof of Lemma 3.15, we have that $\mathrm{kwt}(T)$ is a glide of the weak composition $\mathrm{wt}(S)$.

For the other direction, given a glide b of the weak composition $\mathrm{wt}(S)$, we construct (the unique) $T \in \mathrm{dst}_Q^{-1}(S)$ such that $\mathrm{wt}(T) = b$. This is achieved by the same process, and via the same argument, as in Lemma 3.15. ∎

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Proof of Theorem 4.12 For $S \in \overline{\Omega}2\overline{\mathfrak{F}}(a)$, Lemma 4.16 says that

$$\overline{\mathfrak{F}}_{\mathrm{wt}(S)} = \sum_{T \in \mathrm{dst}_Q^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^{\mathrm{wt}(T)}.$$

Therefore,

$$\begin{aligned}\sum_{S \in \overline{\Omega}2\overline{\mathfrak{F}}(a)} \beta^{|S|-|a|} \overline{\mathfrak{F}}_{\mathrm{wt}(S)} &= \sum_{S \in \overline{\Omega}2\overline{\mathfrak{F}}(a)} \beta^{|S|-|a|} \sum_{T \in \mathrm{dst}_Q^{-1}(S)} \beta^{|T|-|S|} \mathbf{x}^{\mathrm{wt}(T)} \\ &= \sum_{U \in \overline{\Omega}\mathsf{SSF}(a)} \beta^{|U|-|a|} \mathbf{x}^{\mathrm{wt}(U)} = \overline{\Omega}_a,\end{aligned}$$

where the second equality is by Lemma 4.15 and the third equality is Definition 4.1. ■

Remark 4.17 Setting $\beta = 0$ in the statement of Theorem 4.12 yields a positive combinatorial formula for the fundamental slide expansion of a quasi-key polynomial in terms of quasi-Yamanouchi semi-skyline fillings. Such a formula was alluded to in [Sea20], but not stated explicitly.

The following is a K-theoretic analogue of the positive expansion established in [HLMvW11a] of quasi-Schur polynomials into fundamental quasi-symmetric polynomials.

Corollary 4.18 *The quasi-Grothendieck polynomials expand positively in the basis of multi-fundamental quasi-symmetric polynomials.*

Proof By Theorem 4.12, any quasi-Lascoux polynomial expands positively in the glide basis. By Proposition 4.4, the quasi-Grothendieck polynomials are included among the quasi-Lascoux polynomials. The statement then follows from the fact ([PS19, §3.2]) that the quasi-symmetric glide polynomials are the multi-fundamental quasi-symmetric polynomials and form a basis of $\mathrm{QSym}[\beta]$. ■

Remark 4.19 The number of terms in the expansion of Corollary 4.18 generally grows without bound as the number of variables increases. In the limit, one finds that a quasi-Grothendieck function is a positive sum of multi-fundamental quasi-symmetric functions, but that this sum of formal power series has infinitely-many terms.

4.3 Lascoux Polynomials

In this section, we study the combinatorial Lascoux polynomials of [Mon16, §5] and their relations to the other families of polynomials discussed in this paper. Given a skyline diagram, we augment it on the left with an additional column 0, called the *basement*. We write b_i for the entry in row i of the basement. Expanding on Definition 2.5 and Remark 2.6, a set-valued skyline filling with basement is *semistandard* if it (including the basement) satisfies (S.1), (S.2), (S.3), and (S.4). Basement entries do not count towards the weight $\mathrm{wt}(F)$ of a filling F with basement. In diagrams, we shade the boxes of the basement in gray to distinguish them from the ordinary boxes.

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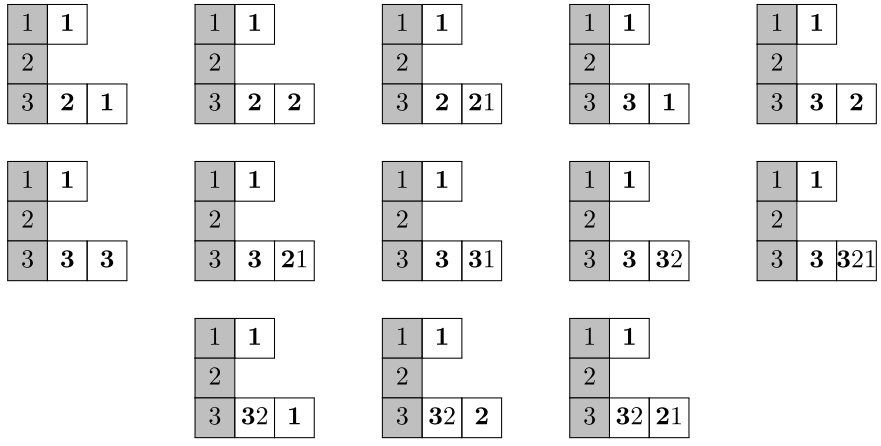


Figure 7: The 13 elements of $\overline{\mathfrak{D}}\mathsf{SSF}(102)$.

For a a weak composition, let $\overleftarrow{a}$ denote the weak composition formed by reversing the order of the parts of a. For example, if $a = (0,1,0,3)$, then $\overleftarrow{a} = (3,0,1,0)$. Let $\overline{\mathfrak{D}}\mathsf{SSF}(a)$ be the set of semistandard set-valued skyline fillings of shape $\overleftarrow{a}$ with basement $b_i = n - i + 1$.

Definition 4.20 ([HLMvW11b, Equation (2.3)], [Mon16, §5]) Let a be a weak composition. The *(combinatorial) Lascoux polynomial* $\overline{\mathfrak{D}}_a$ is given by

$$\overline{\mathfrak{D}}_a = \sum_{F \in \overline{\mathfrak{D}}\mathsf{SSF}(a)} \beta^{\mathrm{ex}(F)} \mathbf{x}^{\mathrm{wt}(F)}.$$

The *Demazure character* is the $\beta = 0$ specialization of the corresponding Lascoux polynomial: $\mathfrak{D}_a = \overline{\mathfrak{D}}_a^{(0)}$.

Demazure characters are, in fact, characters of certain modules with relation to Schubert calculus [Dem74]; no such representation-theoretic realization of Lascoux polynomials is currently known.

Example 4.21 The fillings in Figure 7 show that the monomial expansion of the Lascoux polynomial $\overline{\mathfrak{D}}_{102}$ is

$$\begin{aligned}\overline{\mathfrak{D}}_{102} = {} & x_1^2x_2 + x_1x_2^2 + \beta x_1^2x_2^2 + x_1^2x_3 + x_1x_2x_3 \\ & + x_1x_3^2 + \beta x_1^2x_2x_3 + \beta x_1^2x_3^2 + \beta x_1x_2x_3^2 + \beta^2 x_1^2x_2x_3^2 \\ & + \beta x_1^2x_2x_3 + \beta x_1x_2^2x_3 + \beta^2 x_1^2x_2^2x_3.\end{aligned}$$

The Demazure character $\mathfrak{D}_{102}$ is given by setting $\beta = 0$ in this expression; equivalently, $\mathfrak{D}_{102}$ is the weight generating function for those five fillings in Figure 7 that contain only one number in each box.

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The main result of the remainder of this paper is to show that every Lascoux polynomial $\overline{\mathfrak{D}}_a$ is a positive sum of quasi-Lascoux polynomials. Towards this goal, we first show the weaker result that $\overline{\mathfrak{D}}_a$ is a positive sum of Lascoux atoms.

Given a weak composition a, let $\mathrm{sort}(a)$ be the rearrangement of the parts of a into weakly decreasing order, and let $w(a)$ be the minimal (Coxeter) length permutation sending a to $\mathrm{sort}(a)$.

Theorem 4.22 *For any weak composition a, we have*

$$\overline{\mathfrak{D}}_a = \sum_{\substack{\mathrm{sort}(b)=\mathrm{sort}(a) \\ w(b)\leq w(a)}} \overline{\mathfrak{A}}_b,$$

where $\leq$ denotes the strong Bruhat order on permutations. In particular, every Lascoux polynomial $\overline{\mathfrak{D}}_a$ is a positive sum of Lascoux atoms.

Example 4.23 Let $a = (0,1,0,3)$. Then $\mathrm{sort}(a) = (3,1,0,0)$ and $w(a) = 3241$. Hence,

$$\overline{\mathfrak{D}}_{0103} = \overline{\mathfrak{A}}_{0103} + \overline{\mathfrak{A}}_{1003} + \overline{\mathfrak{A}}_{0130} + \overline{\mathfrak{A}}_{1030} + \overline{\mathfrak{A}}_{1300} + \overline{\mathfrak{A}}_{0301} + \overline{\mathfrak{A}}_{0310} + \overline{\mathfrak{A}}_{3001} + \overline{\mathfrak{A}}_{3010} + \overline{\mathfrak{A}}_{3100}.$$

Specializing Theorem 4.22 at $\beta = 0$ recovers a known formula for the Demazure atom expansion of a Demazure character (see, *e.g.*, [Mas09, HLMvW11b]).

Remark 4.24 In [Las01], A. Lascoux introduced K-theoretic analogues of Demazure characters in terms of divided difference operators. C. Monical conjectured ([Mon16, Conjecture 5.3]) that such an *operator Lascoux polynomial* $\overline{\mathfrak{D}}_a^{\mathrm{op}}$ equals the corresponding combinatorial Lascoux polynomial $\overline{\mathfrak{D}}_a$ of Definition 4.20. Similarly, there is an *operator Lascoux atom* $\overline{\mathfrak{A}}_a^{\mathrm{op}}$ such that conjecturally $\overline{\mathfrak{A}}_a^{\mathrm{op}} = \overline{\mathfrak{A}}_a$ ([Mon16, Conjecture 5.2]). (See, [Mon16, §5] for details.) By [Mon16, Theorem 5.1], these operator Lascoux polynomials expand into operator Lascoux atoms according to the same combinatorial formula as in Theorem 4.22. That is, for any weak composition a, we have

$$\overline{\mathfrak{D}}_a^{\mathrm{op}} = \sum_{\substack{\mathrm{sort}(b)=\mathrm{sort}(a) \\ w(b)\leq w(a)}} \overline{\mathfrak{A}}_b^{\mathrm{op}}.$$

Hence, Theorem 4.22 proves the equivalence of [Mon16, Conjecture 5.2] and [Mon16, Conjecture 5.3].

Before continuing with our proofs of the positive expansions of a Lascoux polynomial in the Lascoux atom and quasi-Lascoux bases, we formulate the following related conjecture, which is in some sense a strengthening of Theorem 4.22.

Conjecture 4.25 *Let a and b be weak compositions. Then $\overline{\mathfrak{D}}_a \cdot \overline{\mathfrak{D}}_b$ is a positive sum of Lascoux atoms.*

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For example,

$$\overline{\mathfrak{D}}_{(0,2)} \cdot \overline{\mathfrak{D}}_{(0,1)} = \overline{\mathfrak{A}}_{(0,3)} + \overline{\mathfrak{A}}_{(1,2)} + 2\beta\overline{\mathfrak{A}}_{(1,3)} + \overline{\mathfrak{A}}_{(2,1)} + \beta\overline{\mathfrak{A}}_{(2,2)} + \beta^2\overline{\mathfrak{A}}_{(2,3)} + \overline{\mathfrak{A}}_{(3,0)} + 2\beta\overline{\mathfrak{A}}_{(3,1)} + \beta^2\overline{\mathfrak{A}}_{(3,2)}.$$

We have checked Conjecture 4.25 by computer for all a, b such that $|a| \leq 5, |b| \leq 5$, and a and b have at most three zeros. Specializing Conjecture 4.25 at $\beta = 0$ recovers a well-known conjecture of V. Reiner and M. Shimozono on products of key polynomials (see [Pun16] for discussion and partial results).

4.4 Proof of Theorem 4.22

First, we need a straightforward operation on weak compositions. Following [AS18b], given a weak composition a, we define a *left swap* to be the exchange of two entries $a_i \leq a_j$ where $i < j$.

Definition 4.26 ([AS18b, §3.2]) Given a weak composition a, let $\mathrm{lswap}(a)$ be the set of weak compositions b that can be obtained from a by a (possibly empty) sequence of left swaps.

Example 4.27 For the weak compositions $a = (0,1,2)$ and $b = (0,3,1)$, we have

$$\mathrm{lswap}(a) = \{(0,1,2), (1,0,2), (1,2,0), (0,2,1), (2,0,1), (2,1,0)\},$$
$$\mathrm{lswap}(b) = \{(0,3,1), (3,0,1), (1,3,0), (3,1,0)\}.$$

The following characterization appears as [Sea20, Lemma 3.1].

Lemma 4.28 *For any weak composition a,*

$$\mathrm{lswap}(a) = \{b : \mathrm{sort}(b) = \mathrm{sort}(a) \text{ and } w(b) \leq w(a)\}. \qquad \blacksquare$$

Hence, to prove Theorem 4.22, it suffices by Lemma 4.28 to construct a weight-preserving bijection

$$\overline{\psi} \colon \overline{\mathfrak{D}}\mathsf{SSF}(a) \longrightarrow \coprod_{b \in \mathrm{lswap}(a)} \overline{\mathfrak{A}}\mathsf{SSF}(b). \tag{4.1}$$

We begin by constructing a column-set preserving (and hence weight-preserving) bijection

$$\psi \colon \mathfrak{D}\mathsf{SSF}(a) \longrightarrow \coprod_{b \in \mathrm{lswap}(a)} \mathfrak{A}\mathsf{SSF}(b) \tag{4.2}$$

for the non-set-valued case. The advantage of equation (4.2) over equation (4.1) is that we know *a priori* that there must exist a weight-preserving bijection between $\mathfrak{D}\mathsf{SSF}(a)$ and $\coprod_{b \in \mathrm{lswap}(a)} \mathfrak{A}\mathsf{SSF}(b)$, since both sets are known to generate the Demazure character $\mathfrak{D}_a$ [HLMvW11b, Mas09]. To our knowledge, an explicit bijection between these sets has, however, not appeared previously in the literature. Our goal is to give an explicit bijection ψ that preserves the column sets of the fillings. Having established ψ with this property, we will find it straightforward to extend ψ to the desired map $\overline{\psi}$ of Equation (4.1), thereby proving the theorem.

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Let $T \in \mathfrak{DSSF}(a)$. Define $\psi(T)$ as follows. Let i_1 be the least entry in the first column of T. For $k > 1$, recursively define i_k to be the greatest entry in column k that is weakly less than i_{k-1}, terminating when there is no such entry. Place the entries $i_1, i_2, \ldots$ in row index i_1 to form the lowest nonempty row of $\psi(T)$, while deleting them from T. Repeat this process on what remains of T to find the next-lowest nonempty row of $\psi(T)$, etc. In [Sea20], this algorithm is referred to as *left row-filling*; by [Sea20, Lemma 5.2, Lemma 5.3], $\psi(T) \in \mathfrak{ASSF}(b)$ for some b. (Strictly speaking, the domain of the map defined in [Sea20] consists of reverse semistandard Young tableaux rather than the fillings of $\mathfrak{DSSF}(a)$. However, as the map operates at the level of column sets (the positions of boxes in a column are irrelevant) and as the column sets of $T \in \mathfrak{DSSF}(a)$ can be reordered to create a reverse semistandard Young tableau, this distinction is insignificant.)

Example 4.29 Suppose that $a = (1, 2, 0, 3, 3)$. Then we have, for example,

$\mathfrak{DSSF}(1,2,0,3,3) \ni$

1	1		
2	2	1	
3			
4	3	3	2
5	5	4	3

$\xrightarrow{\psi}$

5	4	2	
3	3	3	
2			
1	1		

$\in \mathfrak{ASSF}(2,1,3,0,3).$

Observe that $b = (2, 1, 3, 0, 3) \in \mathrm{lswap}(a)$, as desired.

It is clear that b always satisfies $\mathrm{sort}(a) = \mathrm{sort}(b)$. We need to establish the stronger property that $b \in \mathrm{lswap}(a)$. To this end, we make use of the fact (observed in [AS18a]) that the fillings in $\mathfrak{DSSF}(a)$ are exactly the same objects (flipped upside-down) as the *semistandard key tableaux* of [Ass18], and hence are in straightforward bijection with *Kohnert diagrams*.

A *Kohnert move* [Koh91] on a finite set of boxes in $\mathbb{N} \times \mathbb{N}$ (realized as the lattice points of the first quadrant of the plane) moves the rightmost box in some row down to the highest empty space below it in the same column. The *Kohnert diagrams* of $D(a)$ are the box diagrams that can be obtained from $D(a)$ by a (possibly empty) sequence of Kohnert moves. A semistandard key tableau records a particular choice of Kohnert moves to achieve a given Kohnert diagram; the number in each box of a semistandard key tableau represents the row that box has moved to in the corresponding Kohnert diagram.

Example 4.30 Figure 8 shows the semistandard key tableau obtained by flipping the filling $T \in \mathfrak{DSSF}(a)$ of Example 4.29 upside-down, together with the corresponding Kohnert diagram. We shade the boxes of the Kohnert diagram in red to distinguish them from empty space.

This reinterpretation in terms of Kohnert diagrams is useful because it facilitates a diagrammatic understanding of the lswap operation.

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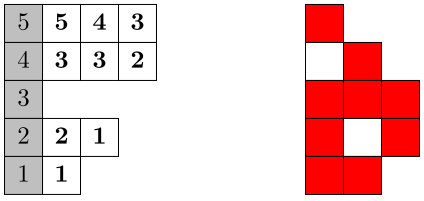


Figure 8: The semistandard key tableau equivalent to the filling $T \in \mathfrak{D}\mathsf{SSF}(a)$ of Example 4.29, together with its corresponding Kohnert diagram. (colour online)

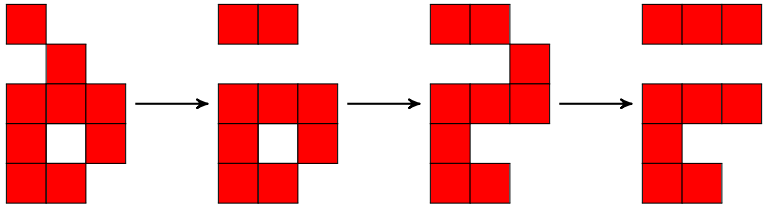


Figure 9: The iterative computation of the nearest skyline diagram of the Kohnert diagram from Figure 8. (colour online)

Lemma 4.31 *Let a and b be weak compositions. Then $b \in \mathrm{lswap}(a)$ if and only if $D(b)$ is a Kohnert diagram of $D(a)$.*

Proof Suppose $b \in \mathrm{lswap}(a)$. Then b is obtained from a by a sequence of swaps that move larger entries leftwards. Any such swap can be achieved by Kohnert moves: given $D(a)$, to swap a row of length r with a lower row of length r', where $0 \leq r' \leq r$, move the rightmost $r - r'$ boxes of the higher row down to the lower row, in order from right to left. Clearly, these moves are all valid Kohnert moves, and so $D(b)$ can be realized as a Kohnert diagram of $D(a)$.

Conversely, suppose $D(b)$ is a Kohnert diagram of $D(a)$. Since Kohnert moves keep each box in its column, we have $\mathrm{sort}(b) = \mathrm{sort}(a)$. It remains to observe that the only way one can rearrange rows of a skyline diagram via Kohnert moves is to move the "overhang" of a longer row down to join onto a (possibly empty) shorter row. But this operation corresponds to an application of the lswap operation, so $b \in \mathrm{lswap}(a)$. ∎

Definition 4.32 Suppose that a is a weak composition and K is a Kohnert diagram of $D(a)$. Define the *nearest skyline* diagram of K to be the skyline diagram $K^{\uparrow}$ obtained as follows: Reading rows of K from left to right, top to bottom, find the first box $\mathfrak{b}$ of K with an empty space immediately to its left; move $\mathfrak{b}$ upwards to the first available space above it. Repeat this operation until all boxes are in left-justified rows, *i.e.,* until a skyline diagram is obtained.

An example of the computation of a nearest skyline diagram appears in Figure 9.

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Lemma 4.33 *If a is a weak composition and K is a Kohnert diagram of $D(a)$, then $K^{\uparrow}$ is also a Kohnert diagram of $D(a)$.*

Proof Since K is a Kohnert diagram of $D(a)$, $D(a)$ can be obtained by a sequence of reverse Kohnert moves on K. Let $\mathfrak{b}$ be the box moved by an iteration of the "nearest skyline" algorithm of Definition 4.32. Since all boxes strictly higher than $\mathfrak{b}$ are by definition in left-justified rows, $\mathfrak{b}$ does not land left of another box in the same row. So Definition 4.32 uses valid reverse Kohnert moves. We will show these reverse Kohnert moves on K (or other sequences of reverse Kohnert moves resulting in the same diagram) are all necessary to obtain *any* skyline diagram of which K is a Kohnert diagram, hence are necessary to obtain $D(a)$.

In order for a skyline diagram to be obtained, since a reverse Kohnert move can never cause a box to land the left of another box in the same row, any box of K that has an empty space to its left in its row must at some stage move upwards. Moreover, a reverse Kohnert move on any box does not cause the left-justification status of other boxes in the diagram to change: those boxes that were in left-justified rows still are, and those boxes that had an empty space to their left still do.

Now observe that when the algorithm raises a box $\mathfrak{b}$, it moves boxes within the column of $\mathfrak{b}$ by the minimal amount possible in order to raise $\mathfrak{b}$. In the case $\mathfrak{b}$ jumps over other boxes, it is of course possible the same diagram may be obtained by moving the boxes above $\mathfrak{b}$ first and then moving $\mathfrak{b}$ upwards by a smaller distance. But this gives rise to the same diagram: the entire interval of boxes weakly above $\mathfrak{b}$ moves up one space. Since Kohnert diagrams do not distinguish between individual boxes, this move is thus the minimal move on a Kohnert diagram that raises $\mathfrak{b}$.

So the algorithm moves only the boxes that have to be moved upwards to obtain a skyline diagram, it moves boxes upwards by the minimal distance needed to achieve this, and the set of boxes of K that need to be moved upwards is independent of the order in which these boxes are moved. So all these reverse Kohnert moves (or equivalent sequences of reverse Kohnert moves resulting in the same diagrams) are necessary in order to obtain $D(a)$, and thus $K^{\uparrow}$ is a Kohnert diagram of $D(a)$. ∎

We can reinterpret ψ to act on Kohnert diagrams. Let K be a Kohnert diagram of $D(a)$. We first decompose K into *threads* $\theta_1, \theta_2, \ldots$. Let $\mathfrak{b}^1$ be the lowest box in column 1 of K. Then for $k > 1$, recursively define $\mathfrak{b}^k$ to be the highest box in column k of K that is weakly lower than $\mathfrak{b}^{k-1}$. Let $\theta_1 = \{\mathfrak{b}^1, \mathfrak{b}^2, \ldots\}$. After deleting θ_1, repeat this process on the remainder of K to obtain θ_2, etc. Now define $\psi(K)$ to be the skyline filling given by placing the row indices of the boxes in thread θ_k into row k in weakly decreasing order.

Example 4.34 The left of Figure 10 shows the Kohnert diagram K of Figure 8 with threads indicated both by labels in the boxes and also by coloring. On the right of Figure 10 is the corresponding filling $\psi(K) \in \mathfrak{ASSF}(2,1,3,0,3)$. Observe that the filling $\psi(K)$ of Figure 10 coincides with the filling obtained in Example 4.29.

The following lemma justifies the mild abuse of notation in using the same symbol ψ both for a map on skyline fillings and for a map on Kohnert diagrams.

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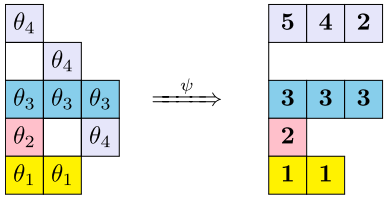


Figure 10: The threading of the Kohnert diagram K of Figure 8 and the corresponding filling $\psi(K) \in \mathfrak{ASSF}(2,1,3,0,3)$. Here, we have labeled the boxes of K by the threads of which they are part. The color-coding of the boxes of K is redundant with this labeling by threads, and the color-coding of the boxes of $\psi(K)$ matches the color-coding of K. (Colour online)

Lemma 4.35 *Let $T \in \mathfrak{DSSF}(a)$ and let K be the corresponding Kohnert diagram of $D(a)$. Then $\psi(T) = \psi(K)$.*

Proof Observe that the definition of $\psi(T)$ only depends on the sets of labels appearing in each column of T. This set of labels coincides with the set of row indices of the boxes in the corresponding column of K. It is then straightforward from unwinding the definitions that $\psi(T) = \psi(K)$, as desired. ∎

Lemma 4.36 *The threading of a Kohnert diagram K is preserved under the "nearest skyline" algorithm of Definition 4.32; that is, whenever the algorithm moves a box upwards, the set of boxes in each thread and the location of the leftmost box in each thread both remain unchanged.*

Proof Any box in the first column of K has no empty space to its left and thus is never moved by the "nearest skyline" algorithm. Hence, the location of the leftmost box in each thread does not change during construction of $K^{\uparrow}$.

To see that threads retain the same set of boxes, it is enough by induction to consider a single application of the "box raising" operation. Suppose this operation takes K to K^+ by acting on the box $\mathfrak{b}^c$ in column c. Suppose $\mathfrak{b}^c$ is in the thread $\theta_i = \{\mathfrak{b}^1, \mathfrak{b}^2, \ldots\}$ in K, where $\mathfrak{b}^k$ is in column k. Certainly, $c > 1$, so $\mathfrak{b}^{c-1}$ exists.

First, consider the case that $\mathfrak{b}^c$ moves up one space; *i.e.,* K has no box immediately above $\mathfrak{b}^c$ in its column. By definition, K has no box immediately to the left of $\mathfrak{b}^c$ in its row, so $\mathfrak{b}^{c-1}$ is strictly above $\mathfrak{b}^c$ in K. Thus, $\mathfrak{b}^c$ still lies in the same thread as $\mathfrak{b}^{c-1}$ in K^+. By definition, $\mathfrak{b}^c$ is not left of any box in its row in K^+, so K^+ has $\mathfrak{b}^c$ in the same thread as $\mathfrak{b}^{c+1}$, if such a box exists. Thus, in this case the threading of K coincides with the threading of K^+.

Now suppose $\mathfrak{b}^c$ "jumps" over at least one box, *i.e., K* has a box immediately above $\mathfrak{b}^c$ in its column. Since K has all boxes above $\mathfrak{b}^c$ in left-justified rows, $\mathfrak{b}^{c-1}$ is necessarily the rightmost box of the lowest row above $\mathfrak{b}^c$ that ends in column $c-1$. Since the rows that $\mathfrak{b}^c$ jumps over have length at least c, $\mathfrak{b}^c$ remains threaded with $\mathfrak{b}^{c-1}$ in K^+. By definition, $\mathfrak{b}^c$ is not left of any box in its row in K^+. In the rows that $\mathfrak{b}^c$ jumps over, observe that both K and K^+ have each box threaded with all the boxes to its left in its row. In particular, K^+ does not thread $\mathfrak{b}^c$ with a box in column $c+1$ from a row that

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has been jumped over. So $\mathfrak{b}^c$ remains threaded with $\mathfrak{b}^{c+1}$ in K^+, if such a box exists. Thus, the threadings of K and K^+ coincide. ∎

Lemma 4.37 *Let K be a Kohnert diagram. Then the shape of $\psi(K)$ is the nearest skyline diagram $K^\uparrow$.*

Proof By Lemma 4.36, the threading of K is the same as the threading of $K^\uparrow$. Hence, the shape of $\psi(K)$ equals the shape of $\psi(K^\uparrow)$. Since $K^\uparrow$ is a skyline diagram, it is clear from the definitions that the threads of $K^\uparrow$ are just its rows. Hence, the shape of $\psi(K^\uparrow)$ is the skyline diagram $K^\uparrow$ itself. Thus, the shape of $\psi(K)$ is $K^\uparrow$. ∎

Lemma 4.38 *Let $T, U \in \mathfrak{D}\mathsf{SSF}(a)$ be distinct. Then there is a column in which T and U do not have the same set of labels.*

Proof Let K_T and K_U be the Kohnert diagrams corresponding to T and U, respectively. If, in each column, T and U have the same set of labels, then, in each column, K_T and K_U have boxes in the same positions. Hence, $K_T = K_U$, and so $T = U$. ∎

Theorem 4.39 *The map*

$$\psi\colon \mathfrak{D}\mathsf{SSF}(a) \to \coprod_{b\in \mathrm{lswap}(a)} \mathfrak{A}\mathsf{SSF}(b)$$

is well defined and is a column-set-preserving (and thus weight-preserving) bijection.

Proof Let $T \in \mathfrak{D}\mathsf{SSF}(a)$. We know from [Sea20] that $\psi(T) \in \mathfrak{A}\mathsf{SSF}(b)$ for some b. Let K be the Kohnert diagram corresponding to T and suppose the nearest skyline diagram $K^\uparrow$ has shape d. By definition, K is a Kohnert diagram of $D(a)$. Hence by Lemma 4.33, we have that $D(d)$ is a Kohnert diagram of $D(a)$. Thus by Lemma 4.31, $d \in \mathrm{lswap}(a)$. By Lemma 4.37, $\psi(K)$ has shape d. Thus, by Lemma 4.35, $\psi(T)$ has shape d, and so $\psi(T) \in \mathfrak{A}\mathsf{SSF}(d) \subseteq \coprod_{b\in \mathrm{lswap}(a)} \mathfrak{A}\mathsf{SSF}(b)$.

By definition, ψ preserves column sets. Since no two elements of $\mathfrak{D}\mathsf{SSF}(a)$ have identical column sets by Lemma 4.38, this implies ψ is injective. Since $\mathfrak{D}\mathsf{SSF}(a)$ and $\coprod_{b\in \mathrm{lswap}(a)} \mathfrak{A}\mathsf{SSF}(b)$ both generate the Demazure character $\mathfrak{D}_a$ [HLMvW11b], [Mas09], they are equinumerous (and finite). Hence ψ is a bijection. ∎

Lemma 4.40 *The bijection ψ extends to a column-set-preserving (and thus weight-preserving) bijection*

$$\overline{\psi}\colon \overline{\mathfrak{D}}\mathsf{SSF}(a) \to \coprod_{b\in \mathrm{lswap}(a)} \overline{\mathfrak{A}}\mathsf{SSF}(b),$$

as follows: For $\overline{T} \in \overline{\mathfrak{D}}\mathsf{SSF}(a)$, let T be the filling obtained by deleting the free entries of $\overline{T}$. Then, $\overline{\psi}(\overline{T})$ is given by placing each free entry in $\overline{T}$ with the smallest possible anchor in the corresponding column of $\psi(T)$, subject to the decreasingness condition (S.2).

Proof Let $T \in \mathfrak{D}\mathsf{SSF}(a)$ and let $S = \psi(T) \in \mathfrak{A}\mathsf{SSF}(b)$. The only thing that needs to be checked is that, for any valid assignment of free entries to columns of T, there is a corresponding valid assignment of the same free entries to the same columns of S, and vice versa.

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Suppose for a contradiction that this is false for some $\overline{T} \in \overline{\mathfrak{D}}\mathsf{SSF}(a)$, whose anchor entries form the filling T. Let k be the greatest free entry of column c of $\overline{T}$ that cannot be added as a free entry in column c of S without violating the decreasingness condition (S.2). Then, for every entry m greater than k in column c of S, the entry immediately right in column $c+1$ of S is strictly greater than k. However, since ψ preserves column sets, the entries of each column of S are a permutation of the entries of the corresponding column of T. Thus, T also has at least as many entries that are strictly greater than k in column $c+1$, as it has entries that are that are strictly greater than k in column c. Hence, by (S.2) for T, every entry in column $c+1$ of T that is strictly greater than k must be immediately right of an entry that is strictly greater than k in column c. Thus, k cannot be added as a free entry in column c of T, contradicting the existence of $\overline{T}$. The argument for the other direction is identical. ■

This completes the proof of Theorem 4.22.

4.5 The Quasi-Lascoux Expansion of a Lascoux Polynomial

The Lascoux polynomials expand positively in the basis of quasi-Lascoux polynomials. Following [AS18b], define $\mathrm{Qlswap}(a)$ to be all $b \in \mathrm{lswap}(a)$ such that if $c \in \mathrm{lswap}(a)$ and $b^+ = c^+$, then $c \geq b$.

Example 4.41 (*cf.* Example 4.27) For the weak compositions $a = (0,1,2)$ and $b = (0,3,1)$, we have

$$\mathrm{Qlswap}(a) = \{(0,1,2),(0,2,1)\} \quad \text{and} \quad \mathrm{Qlswap}(b) = \{(0,3,1),(1,3,0)\}.$$

Theorem 4.42 *For any weak composition a, we have*

$$\overline{\mathfrak{D}}_a = \sum_{b \in \mathrm{Qlswap}(a)} \overline{\mathfrak{Q}}_b.$$

In particular, every Lascoux polynomial $\overline{\mathfrak{D}}_a$ is a positive sum of quasi-Lascoux polynomials.

Proof Suppose that $b \in \mathrm{Qlswap}(a)$ and that c is another weak composition with $c \geq b$ and $c^+ = b^+$. Then clearly $c \in \mathrm{lswap}(a)$. By the definitions of lswap and Qlswap, every $c \in \mathrm{lswap}(a)$ is either in $\mathrm{Qlswap}(a)$ or else dominates some $b \in \mathrm{Qlswap}(a)$ with $c^+ = b^+$. This establishes the second equality in the following:

$$\overline{\mathfrak{D}}_a = \sum_{c \in \mathrm{lswap}(a)} \overline{\mathfrak{A}}_c = \sum_{b \in \mathrm{Qlswap}(a)} \sum_{\substack{c \geq b \\ c^+ = b^+}} \overline{\mathfrak{A}}_c = \sum_{b \in \mathrm{Qlswap}(a)} \overline{\mathfrak{Q}}_b,$$

where the first equality is by combining Theorem 4.22 and Lemma 4.28, and the third equality is by Definition 4.1. ■

Remark 4.43 The $\beta = 0$ specialization of Theorem 4.42 recovers the expansion of Demazure characters in the quasi-key basis [AS18b, Theorem 3.7].

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Acknowledgements OP is grateful for educational conversations with Allen Knutson, Emily Sergel, and Ardea Thurston-Shaine.

We are grateful to the anonymous referees for their helpful suggestions on improving the exposition.

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