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The anisotropic harmonic oscillator group

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1. *The oscillator group.* The one-dimensional oscillator group was studied by Streater (7) when he was comparing different methods of calculating representations. It is interesting to generalize this work and consider the group corresponding to the n-dimensional anisotropic oscillator with frequencies ω_j. In fact, the ideas can be generalized to even larger groups and, in the process of considering their representations, several points emerge connected with the harmonic oscillator degeneracies.

Definition 1·1. The oscillator group, G, is R^{2n+2}; i.e. $g(a, \mathbf{u}, t)$ with $a,\ t \in R$ and $\mathbf{u} \in R^{2n}$ such that

$$g(a'', \mathbf{u}'', t'') = g(a, \mathbf{u}, t).g(a', \mathbf{u}', t'),$$

where

$$\left.\begin{aligned} t'' &= t + t', \\ \mathbf{u}'' &= \mathbf{u} + R(t)\mathbf{u}', \\ a'' &= a + a' + \tfrac{1}{2}(\mathbf{u}, J.R(t).\mathbf{u}'). \end{aligned}\right\} \tag{1}$$

$R(t)$ is the $2n \times 2n$ rotation matrix with zeros everywhere except in the 2×2 blocks $R_j(t)$ down the diagonal and

$$R_j(t) = \begin{pmatrix} \cos \omega_j.t & -\sin \omega_j.t \\ \sin \omega_j.t & \cos \omega_j.t \end{pmatrix};$$

and

$$(J\mathbf{u})_{2_j} = -u_{2_{j-1}}, \quad (J\mathbf{u})_{2_{j-1}} = u_{2_j}.$$

Sometimes it is more convenient to use the complex form of the group G. In this case the elements are $g(a, \mathbf{z}, t)$ with $\mathbf{z} \in C^n$ and the multiplication defined on $R \times C^n \times R$ by

$$(a, \mathbf{z}, t)(a', \mathbf{z}', t') = (a + a' + \tfrac{1}{2}\operatorname{Im}(\mathbf{z}, E(t)\mathbf{z}'), \mathbf{z} + E(t)\mathbf{z}', t + t') \tag{2}$$

where

$$\{E(t)\mathbf{z}\}_j = e^{i\omega_j.t}.z_j.$$

It is easy to see that the multiplication laws defined in (1) and (2) are isomorphic and that they define a group. Further, the analytic and topological structure of these sets gives G all the nice properties of a lie group and so it is possible to study the lie algebra derived from G.

Let the basis elements of this lie algebra be I, H, P_j, Q_j, where

$$I = \left[\frac{d}{da}.g(-a, 0, 0)\right]_{a=0}, \quad H = \left[\frac{d}{dt}.g(0, 0, t)\right]_{t=0},$$

$$P_j = \left[\frac{d}{du_{2_j}}.g(0, \mathbf{u}, 0)\right]_{\mathbf{u}=0}, \quad Q_j = \left[\frac{d}{du_{2_{j-1}}}.g(0, \mathbf{u}, 0)\right]_{\mathbf{u}=0},$$

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then the structure constants are found from the commutation relations between these elements. These relations can be calculated using the formula

$$[a_x, a_y] = \left(\frac{d}{dy}.\left[\frac{d}{dx}g(x).g(y).g(x)^{-1}g(y)^{-1}\right]_{x=0}\right)_{y=0},$$

where a_x, a_y are the elements of the algebra derived from the one parameter subgroups $g(x)$ and $g(y)$ of G. It turns out that I commutes with H, P_j, Q_j and that

$$[H, P_j] = -\omega_j Q_j, \quad [H, Q_j] = \omega_j . P_j, \quad [P_j, Q_j] = I.$$

But these are precisely the commutation relations that define the n-dimensional harmonic oscillator with frequencies ω_j; and so G arises naturally from the harmonic oscillator with hamiltonian $H(\boldsymbol{\omega})$.

THEOREM 1·2. *The oscillator group is a lie group and gives rise to a lie algebra which is isomorphic to the algebra generated by the anisotropic harmonic oscillator and the associated commutation relations.*

Clearly all the elements of G with $t = 0$ form a closed subgroup of G. This subgroup will be denoted by $W(n)$, and when studying $W(n)$ it is better to use a different parametrization of G. Starting from the complex form of G, let the element $(a, \mathbf{z}, 0)$ be represented by $(b, \mathbf{r}, \mathbf{s})$ where $\mathbf{r}, \mathbf{s} \in R^n$ and $\mathbf{r}$ = Real part of $\mathbf{z}$; $i\mathbf{s}$ = Imaginary part of $\mathbf{z}$; and $b = a - \frac{1}{2}(\mathbf{r}, \mathbf{s})$.

In these coordinates

$$(b, \mathbf{r}, \mathbf{s}).(b, \mathbf{r}', \mathbf{s}') = (b + b' - (\mathbf{r}, \mathbf{s}), \mathbf{r} + \mathbf{r}', \mathbf{s} + \mathbf{s}'), \tag{3}$$

so that $W(n)$ is recognized as the n-dimensional Weyl group, which represents the canonical commutation relations.

Now $W(n)$ is normal in G and the $(0, 0, t)$ are a subgroup of G, isomorphic to R, which acts on $W(n)$. In fact it is easy to check that G is the semi-direct product of $W(n)$ and R, and it is well known that $W(n)$ is the semi-direct product of R^{n+1} and R^n.

THEOREM 1·3. $G = W(n) \circledS R = (R^{n+1} \circledS R^n) \circledS R$.

2. *Extensions of G.* The way the parameter t enters into the expression (1) suggests that G is a particular case of some larger class of groups, which can be dealt with more generally. So consider the multiplication

$$(a, \mathbf{u}, A)(a', \mathbf{u}', A') = (a + a' + \tfrac{1}{2}B(\mathbf{u}, A\mathbf{u}'), \mathbf{u} + A\mathbf{u}', AA'), \tag{4}$$

where $\mathbf{u}, \mathbf{u}' \in R^{2n}$ and $B(\ ,\ )$ is an alternating bilinear form on $R^{2n} \times R^{2n}$. If A and A' are maps on R^{2n} this is a generalization of (1), but it does not define a group unless each A belongs to a group contained in $Gl(2n, R)$.

The multiplication is only associative for those A which leave the bilinear form invariant and so the largest group is all the $(a, \mathbf{u}, A)$ with A an invertible linear transformation on R^{2n} which leaves $B(\ ,\ )$ invariant. But $B(\mathbf{u}, \mathbf{u}') = (\mathbf{u}, B\mathbf{u}')$ where B is skew adjoint and using the spectral theorem any skew adjoint operator can be written $B = U^T.D.J.U$ where U is orthogonal and D is a positive diagonal matrix which commutes with J.

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If $B(\,,\,)$ is non-degenerate, the diagonal elements of D are strictly greater than zero and so consider the map

$$(a, \mathbf{u}, A) \to (a, D^{\frac{1}{2}}.U\mathbf{u},\ D^{\frac{1}{2}}.U.A.U^TD^{-\frac{1}{2}}).$$

It can be checked that this is an isomorphism between (4) and

$$(a, \mathbf{u}, A)(a', \mathbf{u}', A') = (a+a'+\tfrac{1}{2}(\mathbf{a}, JA\mathbf{u}'), \mathbf{u}+A\mathbf{u}', AA'), \tag{5}$$

which defines a group provided that all A satisfy

$$A^T.J.A. = J. \tag{6}$$

The subgroup of $Gl(2n, R)$ which satisfies (6) is the real symplectic group $Sp(n, R)$ and it is easy to check that (5) and (6) together define a semi-direct product of $W(n)$ and $Sp(n, R)$, which will be denoted by G^s.

THEOREM 2·1. *If $B(\,,\,)$ is non-degenerate, the most general groups defined by* (4) *are isomorphic to the group G^s given by* (5); *i.e.* $G^s = W(n) \circledS \mathrm{Sp}(n, R)$.

Each $R(t)$ defined in (1) belongs to $Sp(n, R)$ and so G is a subgroup of G^s. The $R(t)$ also commute with J and belong to the orthogonal group $O(2n)$, so consider the subgroup of $O(2n)$, denoted by $O(2n)^J$, which commutes with J. As this is also a subgroup of $Sp(n, R)$ the semi-direct product of $W(n)$ and $O(2n)^J$ can be formed using (5).

DEFINITION 2·2(*a*). $G^u = W(n) \circledS O(2n)^J$. (Clearly G is a subgroup of G^u, and G^u a subgroup of G^s.)

In some of the calculations, it is better to use the complex forms of G^s and G^u. The generalization of (2) is

$$(a, \mathbf{z}, A)(a', \mathbf{z}', A') = (a+a'+\tfrac{1}{2}\mathrm{Im}(\mathbf{z}, A\mathbf{z}'), \mathbf{z}+A\mathbf{z}', AA') \tag{7}$$

and this defines a group provided that A acts on C^n and

$$A(\mathbf{z}+\mathbf{z}') = A\mathbf{z}+A\mathbf{z}';\quad \mathrm{Im}(A\mathbf{z}, A\mathbf{z}') = \mathrm{Im}(\mathbf{z}, \mathbf{z}') \tag{8}$$

for all $\mathbf{z}, \mathbf{z}' \in C^n$. Now Itzykson (3), showed that (8) implies that $A\mathbf{z} = \bar{c}.\mathbf{z} - d.\bar{\mathbf{z}}$ where c and d are complex matrices and $c^*.c - d^*.d = I$, $c^t.d = d^t.c$. Furthermore, if

$$A^{\sim} = \left(\begin{array}{c|c} c_1-d_1 & c_2-d_2 \\ \hline -c_2-d_2 & c_1+d_1 \end{array}\right),$$

where $c = c_1 + i.c_2$ and $d = d_1 + i.d_2$ it can be shown that $A^{\sim}$ belongs to a real form of $Sp(n, R)$ and that $A \to A^{\sim}$ is an isomorphism. Hence, (8) defines the complex form of the real symplectic group and so (7) represents the complex version of $W(n) \circledS Sp(n, R)$.

Note that A does not belong to $Gl(n, C)$ unless $d = 0$, but in this case A also belongs to $U(n)$ and so (8) is trivially satisfied. Further, when d is zero $A^{\sim}$ commutes with J, and conversely, so that $U(n)$ is the complex version of $O(2n)^J$. Hence the complex version of G^u is the semi-direct product of $W(n)$ and $U(n)$ as described by (7).

DEFINITION 2·2(*b*). $G^u = W(n) \circledS U(n)$.

3. *Representation theory.* Having defined the groups G, G^u and G^s as semi-direct products it is appropriate to try and apply the results of Mackey (5, 6) to calculating their representations. These results can be summarized as follows:

If G is the semi-direct product of H and B, if the dual of H, $H^{\sim}$, is type 1, and if there

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exists a borel subset O_H of $H^\sim$ which meets every orbit of $H^\sim$ under G exactly once, then,

(*a*) For every $x \in H^\sim$ there is a $w(f)^{-1}$ representation, $M^x(g)$ of H_x, the little group of H with respect to x, such that $M^x \downarrow H = x$ and $M^x(g).x.M^x(g)^{-1} = g.x$ for all g in H_x. (f is the projection from H_x to $X_x = H_x/H$ and w is a multiplier of X_x.)

(*b*) Every irreducible representation of G is unitarily equivalent to some $U^{x,y} = (M^x \otimes y)\uparrow G$, where x is an element of O_H and y is an irreducible w-representation of X_x and $U^{x,y}$ is the same type as y.

If x is a representation of H the induced representations can be evaluated on $\mathscr{L}^2(G/H, \mathscr{H}(x))$ by the formula,

$$(x \uparrow G)(g)f(\mathbf{v}) = x(c(\mathbf{v}).g.c(\mathbf{v}.g)^{-1}).f(\mathbf{v}.g) \tag{9}$$

where c is a cross-section from G/H to G and $\mathbf{v} \in G/H$.

Thus the first step in calculating the representations of G, G^u and G^s is to examine the representations of $W(n)$ and to calculate their orbits under the action of G, etc.

(As the groups involved are all solvable, the development of Mackey's theory by Auslander and Moore (1) could have been used, but we have preferred to work with the more general results.)

4. *Representations of* $W(n)$. As $W(n)$ is the semi-direct product of R^{n+1} and R^n, its representations can be calculated by the method just mentioned. In fact these representations are well known and can be divided into two sets of equivalence classes W' and W^0.

DEFINITION 4·1. W' is the set of all one dimensional representations $x^{\mathbf{m}_1, \mathbf{m}_2}$ of $W(n)$

$$x^{\mathbf{m}_1, \mathbf{m}_2}(b, \mathbf{r}, \mathbf{s}) = E\{(\mathbf{r}, \mathbf{m}_1) + (\mathbf{s}, \mathbf{m}_2)\}, \tag{10}$$

where $\mathbf{m}_1; \mathbf{m}_2 \in R^n$.

DEFINITION 4·2. W^0 is the set of all infinite dimensional representations x^k of $W(n)$, realized on $\mathscr{L}^2(R^n, C)$ by

$$x^k(b, \mathbf{r}, \mathbf{s})f(\mathbf{v}) = E^k\{b - (\mathbf{r}, \mathbf{v})\}f(\mathbf{v} + \mathbf{s}), \tag{11}$$

where $\mathbf{v} \in R^n$ and $0 < k \in R$.

Note that $E^k(b) = e^{2\pi i k b}$ and $E(b) = E^1(b)$.

All the representations in these two sets are type 1 and so the dual of $W(n)$ is type 1.

Unfortunately, it is difficult to calculate the orbits and little groups from (11) because of the lack of symmetry between the parameters $\mathbf{r}$ and $\mathbf{s}$. It would be more convenient if these representations used the complex form of the group. Itzykson (3) achieved this in the case $k = 1$ by using the space $\mathscr{F}^n$, introduced by Bargmann (2), and the unitary operator $\mathscr{F}$ linking this space and $\mathscr{L}^2(R^n, C)$. So consider the operator F_k where

$$F_k = \mathscr{F}.S_k$$

and S_k is an automorphism of $\mathscr{L}^2(R^n, C)$ given by

$$(S_k.f)(\mathbf{x}) = |K|^{\frac{1}{2}}.f(K\mathbf{x})$$

where $(K\mathbf{x})_j = (2\pi k)^{-\frac{1}{2}}.x_j$, and so $|K| = (2\pi k)^{-\frac{1}{2}n}$.

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F_k is unitary as both S_k and $\mathscr{F}$ are unitary and so the representations x^k can be evaluated on $\mathscr{F}^n$ by calculating the action of $F_k . x^k . F_k^{-1}$ on $F(\mathbf{z})$, $\mathbf{z} \in C^n$. If $k' = \sqrt{\pi} k$, $a = b + \frac{1}{2}(\mathbf{r}, \mathbf{s})$ and $\mathbf{w} = \mathbf{r} + i . \mathbf{s}$, then

$$\{F_k . x^k(b, \mathbf{r}, \mathbf{s}) F_k^{-1}\} F(\mathbf{z}) = E(ka) . e^{-\frac{1}{2}k'^2(\mathbf{w},\mathbf{w})} . e^{-ik'(\mathbf{w},\mathbf{z})} . F(\mathbf{z} - ik'\mathbf{w}).$$

This can be slightly simplified by introducing the unitary operator I,

$$(I . F)(\mathbf{z}) = F(-i\mathbf{z})$$

and, writing $I . F_k . x^k(b, \mathbf{r}, \mathbf{s}) . F_k^{-1} . I^{-1}$ as $x^k(a, \mathbf{w})$, the representations of W^0 are realized by

$$\{x^k(a, \mathbf{w}) . F\}(\mathbf{z}) = E(ka) . e^{-\frac{1}{2}k'^2(\mathbf{w},\mathbf{w})} . e^{-k'(\mathbf{w},\mathbf{z})} . F(\mathbf{z} + k' . \mathbf{w}). \tag{12}$$

Similarly the representations of W' can be taken either in the form (10) or as

$$x^{\mathbf{m}}(a, \mathbf{w}) = E\{\text{Real}(\mathbf{m}, \mathbf{w})\}, \tag{13}$$

where $\mathbf{m} = \mathbf{m}_1 + i . \mathbf{m}_2 \in C^n$.

If the theory of Auslander and Moore (1) had been used, $W(n)$ would have been the appropriate nilpotent group, and the representations would have been calculated by Kirillov's method (4), which leads more directly to Bargmann's space.

5. *Representations of G.* As G is a semi-direct product of $W(n)$ and R, the representations of G can be calculated using the method outlined in section 3, provided G satisfies all the conditions mentioned. It has been noted that $W(n)$ is type 1 so the next step is to examine the orbits of the dual of $W(n)$ under the action of G.

From (2) it can be seen that for all $(a, \mathbf{w})$ in $W(n)$ and any $g = (a', \mathbf{z}, t)$ of G,

$$g . (a, \mathbf{w}) . g^{-1} = (a', \mathbf{z}, 0)\,(a, E(t)\mathbf{w}) . (a', \mathbf{z}, 0)^{-1}$$

so that for any representation x of $W(n)$

$$g . x(a, \mathbf{w}) = x(a', \mathbf{z}) . x(a, E(t)\mathbf{w}) . x(a', \mathbf{z})^{-1}. \tag{14}$$

Clearly this action of G cannot take an infinite dimensional representation into a one dimensional representation or vice versa, so the sets W' and W^0 are invariant under G and can be taken separately.

Consider first W^0 and the action of $x^k(a, E(t)\mathbf{w})$ given by (12). Remembering that $E(t)$ is unitary and that $E^*(t) = E(-t)$ this becomes:

$$x^k(a, E(t)\mathbf{w}) . F(\mathbf{z}) = E(k . a) . e^{-\frac{1}{2}k'^2(\mathbf{w},\mathbf{w})} . e^{-k'(\mathbf{w}, E(-t)\mathbf{z})} . F(\mathbf{z} + k' E(t)\mathbf{w}).$$

Now define the automorphism M_t on Bargmann's space by

$$(M_t . F)(\mathbf{z}) = F(E(t) . \mathbf{z})$$

and it is clear that

$$x^k(a, E(t)\mathbf{w}) = M_{-t} . x^k(a, \mathbf{w}) . M_t. \tag{15}$$

Substituting this expression in (14),

$$g . x^k = M^k(g) . x^k . M^k(g)^{-1} \tag{16}$$

where, for all $g = (a, \mathbf{z}, t)$ in G,

$$M^k(g) = x^k(a, \mathbf{z}) . M_{-t}. \tag{17}$$

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LEMMA 5·1. *If* $M^k(g) = x^k(a, \mathbf{z})\, E\{-t' . H(\boldsymbol{\omega})\}$, *where* $t = 2\pi t'$ *and* $H(\boldsymbol{\omega})$ *is an anisotropic harmonic oscillator hamiltonian, then*:

(i) $M^k(g)$ *is an ordinary representation of* G.

(ii) $M^k \downarrow W(n) = x^k$.

(iii) $g . x^k = M^k(g) . x^k . M^k(g)^{-1}$.

Proof. This result follows easily when it has been shown that $M_t = E\{t' . H(\boldsymbol{\omega})\}$, where $H(\boldsymbol{\omega})$ represents the hamiltonian of the oscillator with frequencies ω_j. To prove this, consider the eigenfunctions $u_{\mathbf{m}}(\mathbf{z})$ of $H(\boldsymbol{\omega})$ which form a complete orthonormal basis of $\mathscr{F}^n$. The eigenvalue of $u_{\mathbf{m}}(\mathbf{z})$ is $(\boldsymbol{\omega}, \mathbf{m})$ and so

$$e^{it . H(\omega)} . u_{\mathbf{m}}(\mathbf{z}) = e^{it(\omega, \mathbf{m})} . u_{\mathbf{m}}(\mathbf{z}).$$

Now $e^{it(\omega, \mathbf{m})} = \prod_j (e^{it\omega_j})^{m_j}$ and $u_{\mathbf{m}} = \prod_j z_j^{m_j}$, so that

$$e^{itH(\omega)} u_{\mathbf{m}}(\mathbf{z}) = \prod (e^{it\omega_j} . z_j)^{m_j} = u_{\mathbf{m}}(E(t)\mathbf{z})$$

and as this is true for all the basis it implies that

$$e^{it . H(\omega)} F(\mathbf{z}) = F(E(t)\mathbf{z}),$$

which proves that $M_t = E\{t' . H(\boldsymbol{\omega})\}$.

Hence the $M^k(g)$ defined in 5·1 is identical with the $M^k(g)$ of (17). It is easy to check that M_t is unitary and as x^k is obviously unitary both (i) and (ii) follow from (16), (17); and (iii) is just a restatement of (16).

As $M^k(g)$ is unitary, it is clear from (16) that x^k and $g . x^k$ are unitarily equivalent for all g in G so that the little group of $W(n)$ with respect to x^k is the whole of G, and the orbit of x^k under G is just the point x^k itself, up to unitary equivalence. Hence the orbits of W^0 under G are the individual elements of W^0 and so W^0 meets each of the orbits of the infinite dimensional representations under G exactly once.

Now consider W' and the action of g on $x^{\mathbf{m}}$. As $x^{\mathbf{m}}$ is one dimensional (14) implies that

$$g . x^{\mathbf{m}}(a, \mathbf{z}) = x^{\mathbf{m}}(a, E(t)\mathbf{z}),$$

so that using (13) and $E(-t) = E^*(t)$

$$g . x^{\mathbf{m}}(a, \mathbf{z}) = E\{\text{Real}\,(E(-t)\,\mathbf{m}, \mathbf{z})\}$$

i.e.
$$g . x^{\mathbf{m}} = x^{E(-t) . \mathbf{m}}. \tag{18}$$

Thus the action of G on W' is equivalent to the action of R on C^n given by

$$t : \mathbf{m} \to E(-t) . \mathbf{m}.$$

This action leaves the length of each coordinate of $\mathbf{m}$ fixed, while rotating m_j through an angle $t . \omega_j$ about the origin. Thus, if none of the coordinates are zero the orbits are equivalent to paths traced out on the n-dimensional torus $T^n_{2\pi}$. In fact each orbit can be characterized by the n lengths of each coordinate of $\mathbf{m}$ and by a point on the torus representing the starting position.

In order to apply the method of section 3, there must be a Borel subset of C^n such that all the orbits starting in this subset are distinct and that together they cover the whole of C^n. But, this is impossible if some of the frequencies are irrationally related because

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the paths on $T^n_{2\pi}$ are not closed and none of them completely cover the torus, although some paths are dense in $T^n_{2\pi}$. Hence if any one of the frequencies is irrationally related to the others, there is no borel subset of W' which meets each orbit of W' under G exactly once. Thus the representation theory previously described does not apply and G may possess an irregular pattern of bizarre representations.

To avoid further complications, it will now be assumed that the frequencies ω_j of the oscillator are rationally related and in this case it is no loss in generality to assume that they are integers with H.C.F. equal to 1 and L.C.M. equal to M.

When the frequencies are integers the orbits will return to their starting points when $t = 2\pi$ and the closed path will be retraced every time t goes from $2\pi n$ to $2\pi(n+1)$. Such a closed path does not cover the whole of the torus and so the problem is to find a minimum subset B such that the union of all paths starting in B is a covering.

To start with assume that each coordinate of $\mathbf{m}$ is non-zero and let $a_j = \arg m_j$, so that the argument of the jth component of each point on the orbit is $a_j - t.\omega_j$. Writing a_j and ω_j as n-dimensional vectors, B must be a subset of the torus such that every $\mathbf{a}$ in $T^n_{2\pi}$ can be uniquely written as

$$\mathbf{a} \equiv \mathbf{a}_0 - t.\boldsymbol{\omega} \bmod 2\pi \tag{19}$$

with $\mathbf{a}_0$ in B and t in $[0, 2\pi)$.

Let B be the cartesian product of n subsets B_j, where $B_j = [0, 2\pi b)_j$. The problem is now linear so divide through by 2π and let $x_j = (2\pi)^{-1}.a_j$, and solving (19) is now equivalent to finding b_j, such that for all $\mathbf{x}$ in T^n there is a unique solution of the n equations

$$X_j \equiv x_j + t.\omega_j \bmod 1 \tag{20}$$

with $X_j \in [0, b_j)$ and $t \in [0, 1)$.

LEMMA 5·2. *Assuming that $\omega_{j-1} \geqslant \omega_j$ for all j, let h_j be the* H.C.F. *of ω_j and h_{j-1}, with h_1 equal to ω_1. If $b_1 = 0$ and $b_j = h_j.(h_{j-1})^{-1}$, then for any set of n real numbers $x_j \in [0, 1)$ there exists a unique solution of the congruence relations* (20) *with X_j in $[0, b_j)$ and t in* $[0, 1)$.

Proof. Note that h_j is the H.C.F. of $\omega_1, \ldots, \omega_j$ and that b_j^{-1} is an integer for $j > 1$. A solution of (20) can now be constructed inductively. For $k < n$, assume there is a solution of the first k relations given by $X_1, \ldots, X_k, t_k$, with t_k perhaps lying outside $[0, 1)$.

Let $y = x_{k+1} + \omega_{k+1}.t_k \bmod 1$ and use the Euclidean algorithm to write

$$y = r.b_{k+1} + X_{k+1},$$

where r is a positive integer less than b_{k+1}^{-1} and $0 \leqslant X_{k+1} < b_{k+1}$. As h_{k+1} is the H.C.F. of h_k and ω_{k+1}, there are integers p and q such that $p.h_k + q.\omega_{k+1} = h_{k+1}$. Let $s = -r.q.h_k^{-1}$, then $s.\omega_j$ is an integer for all $j \leqslant k$, so put $t_{k+1} = t_k + s$ and t_{k+1} still satisfies the first k equations. The definitions of X_{k+1}, r, and s imply that

$$X_{k+1} \equiv x_{k+1} + \omega_{k+1}.t_{k+1} \bmod 1$$

and so $X_1, \ldots, X_{k+1}, t_{k+1}$ is a solution of the first $k+1$ congruences.

The induction is started by putting $t_1 = 1 - x_1.\omega_1^{-1}$, and the final step is to put $t \equiv t_n \bmod 1$ where $t \in [0, 1)$, so that a complete solution of (20) has been constructed.

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To show that this solution is unique, it is sufficient to prove that if, for all j,

$$\omega_j . t \equiv y_j \bmod 1 \tag{21}$$

with $|y_j|$ in $[0, b_j)$ and t in $[0, 1)$; then every y_j is zero and so is t. As y_1 is obviously zero it is again best to use induction.

Assume that $y_j = 0$ for all $j \leqslant k$ so that $\omega_j . t \equiv 0 \bmod 1$. As h_k is the H.C.F. of the first k ω_j, there are integers p_j such that

$$\sum_{j=1}^{k} p_j . \omega_j = h_k$$

and hence $t . h_k \equiv 0 \bmod 1$ so that $t = r . h_k^{-1}$ where r is an integer. This means that $t . \omega_{k+1} = r . \omega_{k+1} . h_k^{-1} = s . h_{k+1} . h_k^{-1} = s . b_{k+1}$ where s is an integer, and so from (21) y_{k+1} is zero. Proceeding in this way, y_j is zero for all j and as the H.C.F. of ω_j is 1 this implies t is zero, which completes the proof of the lemma.

Now define S_j to be the subset of C containing all $r . e^{ia_j}$ with $r > 0$ and $a_j \in B_j$ and consider the subset S' of W', where $x^{\mathbf{m}}$ belongs to S' if and only if each m_j is in S_j. The preceding work shows that all the orbits of W' under G, which start in S' are distinct, and further that almost every representation in W' lies on one of these orbits. Each S_j is a borel subset of C and so S' is a borel subset of W' which meets almost every orbit of W' under G exactly once.

The only representations excluded from the previous statement are those with some m_j equal to zero, because so far it has been assumed that every coordinate of $\mathbf{m}$ is non-zero. When every m_j is zero, $x^{\mathbf{m}}$ is the identity representation and so is its own orbit. In between the two extremes, there will be $2^n - 2$ different sorts of cases with some, but not all, $m_j = 0$.

Suppose $k < n$ of the coordinates are zero, then these will be left unaffected by the action of G while the others are rotated through the appropriate angle. This is similar to the $n - k$ dimensional case with all the coordinates non-zero, except that the H.C.F. of the frequencies involved could be some integer p greater than one. However, Lemma 5·2 can be adapted to this situation by restricting t to lie in $[0, p^{-1})$ and then all the comments about the initial case apply. Thus there are $2^n - 2$ borel subsets S^k of W' which meet the appropriate orbits of W' under G exactly once, so let S^2 be union of these sets.

Define $S = S' \cup S^2 \cup \{0\}$, then S is a borel subset of W'. Finally, let $O_W = W^0 \cup S$, then O_W is a borel subset of the dual of $W(n)$ and combining the various statements about its constituents, it is clear that O_W meets every orbit of the dual of $W(n)$ under the action of G, exactly once. This means that G satisfies the conditions laid down in section 3, whenever the frequencies are rationally related. It was noted earlier that if any two of the frequencies are irrationally related the conditions are not satisfied.

THEOREM 5·3. *If the frequencies of the harmonic oscillator $H(\boldsymbol{\omega})$ are rationally related, then every irreducible representation of G, the corresponding oscillator group, is unitarily equivalent to a representation in one of the following series*:

(i) *Principal series $U^{k,q}$.*

If $k, q \in R$ and $0 \neq k$, with $t = 2\pi t'$,

$$U^{k,q}(a, \mathbf{z}, t) = E(qt) . x^k(a, \mathbf{z}) . E\{-t' . H(\boldsymbol{\omega})\}.$$

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(ii) *Secondary series (non-zero)* $U^{\mathbf{m},q}$.

If $\mathbf{m} \in S'$, $q \in [0, 1)$

$$U^{\mathbf{m},q}(a, \mathbf{z}, t)f(s) = E\{q.[s+t] + Re(\mathbf{m}, E(s).\mathbf{z})\}f(\overline{s+t}),$$

where $s \in [0, 2\pi)$ *and* $\overline{s+t} = s+t-[s+t] \in [0, 2\pi)$.

(iii) *Secondary series (with zeros)* $U^{\mathbf{m},q}$.

Similar to (ii) *with* $\mathbf{m}$ *in* S^2 *and* q *in* $[0, p)$ *where* p *is an appropriate integer.*

(iv) *Trivial series* U^q.

If $q \in R$

$$U^q(a, \mathbf{z}, t) = E(q.t).$$

Proof. As the frequencies are rationally related, G satisfies the conditions of section 3 and so it is sufficient to consider each element of O_W in turn. As O_W consists of $W^0, S^1, S^2.\{0\}$, taking each of these subsets one by one gives rise to the four series of representations.

First consider W^0, and recall that the little group of x^k is G itself so that the little co-group of each x^k is $R = G/W(n)$. Further, by Lemma 5·1, $M^k(g)$ is an ordinary representation of G which intertwines $g.x^k$ and x^k, so that every representation of G arising from x^k is of the form $U^{k,\nu} = (M^k \otimes y)\uparrow G$, where $y(t)$ is an irreducible representation of R and the inducing is trivial. But $y(t) = E(q.t)$ so that $U^{k,\nu}(g) = E(q.t)\,M^k(g)$ and so as both q and k range over $R(k \neq 0)$ these exhaust all the representations which can be generated from W^0 and so define the principal series.

Secondly, consider the representations $x^{\mathbf{m}}$ in S'. S' was chosen so that $g.x^{\mathbf{m}}$ and $x^{\mathbf{m}}$ are equivalent if and only if $g = (a, \mathbf{z}, 2\pi p)$, where p is an integer, and so the little group is $W(n) \circledS 2\pi Z$. As $x^{\mathbf{m}}$ is one dimensional $M^x = x^{\mathbf{m}}(a, \mathbf{z})$ is an appropriate representation of the little group, and the representations of Z are $y(p) = E(p.q)$ with q in $[0, 1)$. Hence every irreducible representation of G arising from S' is given by $U^{\mathbf{m},q}$ where

$$U^{\mathbf{m},q} = \{x^{\mathbf{m}}(a, \mathbf{z}).E(p.q)\}\uparrow G.$$

Using the multiplier definition of the induced representation $U^{\mathbf{m},q}$ is defined on the space of square integrable functions on the circle and the simplest choice of cross section leads to the formula in the theorem.

The representations $x^{\mathbf{m}}$ is S^2 give rise to a very similar series of representations. The only difference is, that as the H.C.F. of the frequencies corresponding to the non-zero m_j may be some integer p, the little group may be $W(n) \circledS 2\pi p^{-1}.Z$, and in this case the range of q is increased to $[0, p)$ while the range of s is reduced to $[0, 2\pi p^{-1})$.

Finally, the little group of $W(n)$ with respect to the identity representation x^0 is clearly the whole of G and so all the representations arising from the set $\{0\}$ are of the form $U^q = \{I \otimes y(t)\} \uparrow G = E(qt)$ with q in R.

There are two points to be noted about this result. First, the principal series explicitly contains the harmonic oscillator hamiltonian and so can be defined on any space on which x^k and the oscillator problem are realized.

Secondly, this result effectively distinguishes between the oscillator with different sorts of frequencies. If any pair of frequencies is irrationally related, there will be some intransitive orbits which are dense in transitive ones and so Theorem 5·3 is true if and

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only if the frequencies are rationally related. Further the difference between the isotropic and the rationally related case is reflected in the size of the secondary series. It is interesting to note that n-dimensional oscillator has degeneracies connected with $SU(n)$ if and only if the frequencies are rationally related, which suggest there may be some connexion between the degeneracies of $H(\boldsymbol{\omega})$ and the representations of G.

6. *Representations of G^u.* The representations of $G^u = W(n) \circledS U(n)$ can be found by extending the work on G, and in fact the calculations are somewhat simpler.

From (7), it can be seen that for all $(a, \mathbf{w})$ in $W(n)$, for any $g = (a', \mathbf{z}, U)$ in G^u, and for any representation x of $W(n)$,

$$g.x(a, \mathbf{w}) = x(a', \mathbf{z}).x(a, U\mathbf{w}).x(a', \mathbf{z})^{-1}. \tag{22}$$

Thus, W' and W^0 are still invariant under G^u and can be taken separately.

Generalizing M_t, let $M(U)$ be the automorphism of Bargmann's space given by

$$(M(U).F)(\mathbf{z}) = F(U^{-1}.\mathbf{z}), \tag{23}$$

and using (12) and the fact that U is unitary, it can be checked that

$$x^k(a, U.\mathbf{w}) = M(U).x^k(a, \mathbf{w}).M(U)^{-1}$$

for all x^k in W^0. This leads to a result parallel to 5·1 and which can be proved by a trivial generalization of the proof of 5·1.

LEMMA 6·1. *For $k \neq 0$, let $M^k(g) = x^k(a, \mathbf{z}).M(U)$ where $M(U)$ is defined in* (23) *and then*

(i) *$M^k(g)$ is a unitary representation of G^u.*

(ii) $M^k(g)\downarrow W(n) = x^k$.

(iii) $g.x^k = M^k(g).x^k.M^k(g)^{-1}$.

This result implies that the little group of $W(n)$ with respect to x^k is now the whole G^u and that the orbit x^k under G^u is again just x^k, up to unitary equivalence. So, as before, W^0 meets each of the orbits of infinite dimensional representations of $W(n)$ under the action of G^u exactly once.

Turning to W', (22) means that the action of g on $x^{\mathbf{m}}$ is

$$g.x^{\mathbf{m}} = x^{U^{-1}.\mathbf{m}}$$

and so the action of G^u on W' is equivalent to the action $\mathbf{m} \to U^{-1}.\mathbf{m}$ of $U(n)$ on C^n. The orbits under this action are the spheres of fixed radius centred on the origin. Now $\mathbf{k} = (k, 0, \ldots, 0)$ lies in the orbit of radius $k > 0$ and so consider the subset S of W' consisting of all the representations $x^{\mathbf{k}}$, where $\mathbf{k} = (k, 0, 0, \ldots, 0)$ and $k > 0$.

S is a borel subset of W' as R^+ is a borel subset of C^n, and S meets each orbit of W' under G^u exactly once.

THEOREM 6·2. *Every irreducible representation of G^u is unitarily equivalent to a representation in one of the following series*:

(i) *Principal series $U^{k,\nu}$*

where $0 \neq k \in R$, y is an irreducible representation of $U(n)$, and

$$U^{k,\nu}(a, \mathbf{z}, U) = M^k(a, \mathbf{z}, U) \otimes y(U).$$

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(ii) *Secondary series* $U^{k,y}$
where $0 < k \in R$, y *is an irreducible representation of* $U(n-1)$

$$U^{k,y} = y^k \uparrow G^u$$

and $y^k(a, \mathbf{z}, U) = E(k.\text{Real}\, z_1).y(U)$ *is a representation of* $W(n) \circledS U(n-1)$.

(iii) *Trivial series* U^y
where y *is a representation of* $U(n)$ *and*

$$U^y(a, \mathbf{z}, U) = y(U).$$

Proof. Let $O_W = W^0 \cup S$, then it is clear that O_W is a borel subset of the dual of $W(n)$, and that it meets each orbit of this dual under the action of G^u exactly once. Thus the representations of G^u are found by considering each of the members of W^0 and S in turn.

As the little group of each x^k in W^0 is G^u itself, the little co-group is $U(n) = G^u/W(n)$. Further, Lemma 6·1 shows that $M^k(g)$ intertwines $g.x^k$ and so every representation of G^u arising from x^k is of the form $U^{k,y} = (M^k \otimes y) \uparrow G^u$, where y is a representation of $U(n)$. The inducing is trivial and so the principal series accounts for all th representations arising from W^0.

The subset S of W' contains the identity representation which gives rise to the trivial series as the little group is again the whole of G^u and $(I \otimes y) \uparrow G^u = y$.

The remaining part of S consists of all the x^k with $k > 0$. Now $g = (a, \mathbf{z}, U)$ will belong to the little group of $x^{\mathbf{k}}$ if and only if $g.x^{\mathbf{k}} = x^{\mathbf{k}}$ and this is equivalent to

$$\text{Real}\,(\mathbf{k}, U\mathbf{w}) = \text{Real}\,(\mathbf{k}, \mathbf{w})$$

for all $\mathbf{w}$ in C^n. This implies $(\mathbf{k}, U\mathbf{w}) = (\mathbf{k}, \mathbf{w})$ and as U is in $U(n)$, the condition is satisfied if and only if U leaves $\mathbf{k}$ fixed, i.e. $U \in U(n-1)$. Hence the little group of each $x^{\mathbf{k}}$ is $W(n) \circledS U(n-1)$ and the appropriate intertwining operator is

$$M^{\mathbf{k}}(g) = x^{\mathbf{k}}(a, \mathbf{z}) = E(\text{Real}\,(\mathbf{k}, \mathbf{z})) = E(k.\text{Real}\, z_1),$$

so that, for any representation y of $U(n-1)$, $M^k \otimes y = y^k(a, \mathbf{z}, U)$. Thus all the representations of G^u arising from $x^{\mathbf{k}}$ form the secondary series given by $y^k \uparrow G^u$; $0 < k \in R$, y a representation of $U(n-1)$.

There is an important corollary to this theorem obtained by taking y to be the identity representation in the principal series. $M^k(g)$ is now an irreducible representation of G^u and can be restricted to the subgroup $SU(n)$ of G^u, to produce the representation $M(U)$ which was defined in (23). Bargmann showed that this representation splits up into the direct sum of all the irreducible symmetric representations, $D^m(U)$ of $SU(n)$ with m a positive integer, i.e. $U^{k,I} \downarrow SU(n) = \bigoplus_{m=0}^{\infty} D^m$.

Now there is a 1–1 correspondence between the representations D^m and the degeneracy spaces of the n-dimensional isotropic oscillator. Thus when the irreducible representation $U^{k,I}$ of G^u is restricted to $SU(n)$, it decomposes into irreducibles corresponding to the degeneracies of the oscillator. In other words, G^u is a non-invariance group of the harmonic oscillator.

COROLLARY 6·3. *G^u is a non-invariance group of the isotropic harmonic oscillator.*

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7. *Representations of G^s.* The representations of $G^s = W(n) \circledS Sp(n, R)$ are similar to those of G^u, but the calculations are much more difficult.

Clearly the action of G^s on the representations of $W(n)$ is again given by

$$g.x(a, \mathbf{w}) = x(a', \mathbf{z}).x(a, A\mathbf{w}).x(a', \mathbf{z})^{-1} \tag{24}$$

where $A \in Sp(n, R)$.

Taking the representations x^k of W^0 first, and using (12),

$$x^k(a, A\mathbf{w}).F(\mathbf{z}) = E(ka)e^{-\frac{1}{2}k^2(A\mathbf{w}, A\mathbf{w})}.e^{-k'(A\mathbf{w}, \mathbf{z})}.F(\mathbf{z} + k'A\mathbf{w}).$$

Consider the operator $M(A)$ defined on $\mathscr{F}^n$ by

$$M(A)F(\mathbf{z}) = |\det A|^{-\frac{1}{2}}.e^{\frac{1}{2}\{(\mathbf{z}, \mathbf{z}) - (A^{-1}\mathbf{z}, A^{-1}\mathbf{z})\}}.F(A^{-1}\mathbf{z}). \tag{25}$$

This is obviously a generalization of the $M(U)$ defined in the previous section and using $\mathrm{Im}(\mathbf{w}, A^{-1}\mathbf{z}) = \mathrm{Im}(A\mathbf{w}, \mathbf{z})$ in some rather involved manipulations, it can be shown that

$$x^k(a, A\mathbf{w}) = M(A).x^k(a, \mathbf{w}).M(A)^{-1}.$$

Result 7·1. *For $k \neq 0$ let $M^k(g) = x^k(a, \mathbf{z}).M(A)$ where $M(A)$ is defined in* (28), *and then*:

(i) *$M^k(g)$ is a unitary representation of G^s.*

(ii) $M^k \downarrow W(n) = x^k$.

(iii) $g.x^k = M^k(g).x^k.M^k(g)^{-1}$.

It is straightforward to check these properties, and the result implies that the little group of $W(n)$ with respect to x^k is again the whole of the larger group, namely G^s, and as before W^0 meets each of the orbits of the infinite dimensional representations under the action of G^s exactly once.

The action of G^s on W' is more complicated than previously, as A is not unitary. But $U(n)$ is a subgroup of $Sp(n, R)$ and as the subset S of W' mentioned in the previous section meets each orbit of W' under G^u exactly once, it is sufficient to consider the action of G^s on S. Let $g = (a', \mathbf{w}, A)$ then

$$g.x^{\mathbf{k}}(a, \mathbf{z}) = E(\mathrm{Real}(\mathbf{k}, A\mathbf{z})) = E\{k.\mathrm{Real}(A\mathbf{z})_1\} \tag{26}$$

and so for $k > 0$ consider the operator A_k defined on C^n by

$$A_k.\mathbf{z} = (k^{-1}.x_1 + i.ky_1, z_2, z_3, \ldots, z_n),$$

where $\mathbf{z} = (x_1 + i.y_1, z_2, \ldots, z_n)$.

A_k preserves the additive structure of C^n and it can be checked that

$$\mathrm{Im}(A_k\mathbf{z}', A_k\mathbf{z}) = \mathrm{Im}(\mathbf{z}', \mathbf{z})$$

for all $\mathbf{z}, \mathbf{z}' \in C^n$ so that A_k is a member of $Sp(n, R)$. Furthermore $\mathrm{Real}(A_k\mathbf{z})_1 = k^{-1}.x_1$ so that, if $g^k = (a', \mathbf{w}, A_k)$,

$$g^k.x^{\mathbf{k}}(a, \mathbf{z}) = E(x_1).$$

Hence every $x^{\mathbf{k}}$ $(k > 0)$ lies in the orbit of x^*

$$x^*(a, \mathbf{z}) = E(x_1)$$

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and combining this with the earlier remarks, the orbit of x^* under G^s includes every representation of W' except the identity representation. But x^* and the identity are borel subsets of W' and together they must meet each orbit of W' under G^s exactly once.

THEOREM 7·2. *Every irreducible representation of G^s is unitarily equivalent to a representation in one of the following series*:

(i) *Principal series $U^{k,y}$ where $0 \neq k \in R$ and y is an irreducible of $Sp(n, R)$ and*

$$U^{k,y}(a, \mathbf{z}, A) = M^k(a, \mathbf{z}, A) \otimes y(A).$$

(ii) *Secondary series V^y where y is an irreducible representation of*

$$G^s(n-1) = W(n-1) \circledS Sp(n-1, R)$$
$$V^y = y^* \uparrow G^s$$

and $y^*(a, \mathbf{z}, A') = E(x_1).y(A')$ is a representation of $W(n) \circledS G^s(n-1)$.

(iii) *Trivial series U^y where y is a representation of $Sp(n, R)$ and*

$$U^y(a, \mathbf{z}, A) = y(A).$$

Proof. This time, let $O_W = W^0 \cup \{x^*\} \cup \{I\}$ so that O_W is again a borel subset of the dual of $W(n)$ which meets each orbit of this dual under the action of G^s exactly once.

The representations of G^s are found by considering W^0, I and x^* in turn. In fact the derivations of the principal and trivial series are exactly the same as outlined for G^u in Theorem 6·2, which leaves x^* for special consideration.

From (26), it can be shown that $(a, \mathbf{z}', A)$ belongs to the little group of x^* provided that $\text{Real}\,(A\mathbf{z}_1) = \text{Real}\, z_1$ for all $\mathbf{z}$ in C^n. If $\mathbf{k} = (1, 0, 0, \ldots, 0)$, every $\mathbf{z}$ can be written $\mathbf{z} = c.\mathbf{k} + d.i\mathbf{k} + \mathbf{w}$ where c and d are real, and $\mathbf{w}$ is in the C^{n-1} subspace perpendicular to $\mathbf{k}$. The condition that the real part of the first components of $A\mathbf{z}$ and $\mathbf{z}$ are equal, implies that the real part of the first components of $Ai\mathbf{k}$ and $A\mathbf{w}$ are zero, and that $A\mathbf{k} = \mathbf{k} + b.i\mathbf{k} + \mathbf{v}$ where $\mathbf{v} \perp \mathbf{k}$. The fact that A belongs to $Sp(n, R)$, then implies that

$$A.\mathbf{z} = c.\mathbf{k} + \{d + c.b - \text{Im}\,(\mathbf{v}, A'\mathbf{w})\}.i\mathbf{k} + c\mathbf{v} + A'\mathbf{w} \tag{27}$$

and $A' \in Sp(n-1, R)$ where A' is the restriction of A to the subspace perpendicular to $\mathbf{k}$.

Thus $A = A(b, \mathbf{v}, A')$ with b real, $\mathbf{v} \in C^{n-1}$ and $A' \in Sp(n-1, R)$. The composition for two such A can be calculated from (27) by applying them one after another, and it turns out that the set of all such $A(b, \mathbf{v}, A')$ is isomorphic to the $n-1$ dimensional version of G^s, which is written $G^s(n-1)$. As the little group of x^* consists of all $(a, \mathbf{z}, A)$ with $A \in G^s(n-1)$, it is isomorphic to $W(n) \circledS G^s(n-1)$.

Clearly $M(a, \mathbf{z}, A) = x^*(a, \mathbf{z})$ is the appropriate intertwining representation of the little group and so all the representations of G^s arising from x^* are of the form $(x^* \otimes y) \uparrow G^s$, where y is a representation of $G^s(n-1)$. These simplify to the representations listed in the theorem as the secondary series.

As with G^u, there is an interesting corollary to this theorem. The proof is exactly the same as that for 6·3.

COROLLARY 7·3. *G^s is a non-invariance group of the isotropic harmonic oscillator.*

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