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# Affable equivalence relations and orbit structure of Cantor dynamical systems

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*Abstract*. We prove several new results about $AF$-equivalence relations and relate these to Cantor minimal systems (i.e. to minimal $Z$-actions). The results we obtain turn out to be crucial for the study of the topological orbit structure of more general countable group actions (as homeomorphisms) on Cantor sets, which will be the topic of a forthcoming paper. In all this, Bratteli diagrams and their dynamical interpretation are indispensable tools.

## 1. *Introduction*

In this paper we prove some new results about $AF$-equivalence relations (cf. Definition 3.7) that, besides being of interest in their own right, turn out to be powerful new tools for the study of the topological orbit structure of countable group actions as homeomorphisms on Cantor sets. In a forthcoming paper we will apply these new techniques to prove that certain minimal and free $\mathbb{Z}^2$-actions on Cantor sets are (topologically) orbit equivalent to Cantor minimal systems $(X, T)$, or, equivalently, to minimal $\mathbb{Z}$ actions. The strategy is to prove that the equivalence relation $R_{\mathbb{Z}^2}$ associated to the given $\mathbb{Z}^2$-action is *affable* ('$AF$-able'), i.e. may be given an $AF$-equivalence structure. To prove the affability of $R_{\mathbb{Z}^2}$ we need a delicate 'glueing' procedure, the technical part of which is stated in this paper as Lemma 4.15, the 'key lemma'. We demonstrate the power of Lemma 4.15 by establishing results that are highly non-trivial, concerning the intimate link that exists between minimal $AF$-equivalence relations and Cantor minimal systems (Theorems 4.16 and 4.18; compare also Theorem 4.8).

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Our ultimate goal is to attack the following question (which is an analogue in the topological dynamical setting of the celebrated Connes–Feldman–Weiss result in the measure-theoretic setting [**2**]).

> Let $G$ be a countable, amenable group acting minimally (i.e. every $G$-orbit is dense) and freely (i.e. $gx = x$ for some $x \in X$ implies that $g$ is the identity element of $G$) as homeomorphisms on the Cantor set $X$. Then $(X, G)$ is topologically orbit equivalent to a Cantor minimal system $(Y, T)$, i.e. there exists a homeomorphism $F : X \to Y$ mapping $G$-orbits onto $T$-orbits.

By Theorems 4.8 and 4.16 this is equivalent to showing that the equivalence relation $R_G$ associated to $G$ (cf. Example 2.7(i)) is affable. Previously, this is known to be true for locally finite groups $G$ (in fact, in this case we do not need to require free action, only that $\mathrm{fix}(g) = \{x \in X \mid gx = x\}$ is a clopen set for each $g \in G$) and for $\mathbb{Z}^n$-actions that split as Cartesian products, and also for the case that $G = \mathbb{Z} \times H$, where $H$ is a finite cyclic group. These facts can be deduced from results contained in [**6, 9, 11**]. It is also noteworthy that in the (standard) Borel setting the analogous question has an affirmative answer for certain group actions, including $\mathbb{Z}^n$-actions [**8**, Theorem 1.16].

We need the key concept of a Bratteli diagram, both ordered and unordered, and we refer to [**6**] and [**7**] for details and proofs of basic results (cf. also Example 2.7(ii)). We state two results that we need in the following, concerning the interplay between Bratteli diagrams and Cantor minimal systems.

PROPOSITION 1.1. [**7**, §3] *Let $(V, E)$ be a simple (standard) Bratteli diagram and let $X = X_{(V,E)}$ be the Cantor set consisting of the (infinite) paths associated to $(V, E)$. Let $x, y \in X$ be two paths that are not cofinal. There exists a Cantor minimal system $(X, T)$ such that $T$ preserves cofinality, except that $Tx = y$.*

*Proof.* Without loss of generality we may assume (by appropriately telescoping the original diagram) that, for every $n = 0, 1, 2, \ldots$, there is at least one edge between every vertex $v$ at level $n$ and every vertex $w$ at level $n+1$ of $(V, E)$. Furthermore, we may assume that the $n$th edge of $x$ is distinct from the $n$th edge of $y$. It is an easy observation that $(V, E)$ may be given a proper ordering (also called simple ordering) such that $x$ becomes the unique max path and $y$ becomes the unique min path. The associated Bratteli–Vershik system $(X, T)$ has the desired properties. □

THEOREM 1.2. [**6**, Lemma 5.1] *Let $(X, T)$ be Cantor minimal system and let $Y$ be a closed (non-empty) subset of $X$ that meets each $T$-orbit at most once. There exists an ordered Bratteli diagram $B_Y = (V, E, \geq)$, where $(V, E)$ is a simple (standard) Bratteli diagram, such that the associated Bratteli–Vershik system $(X_{(V,E)}, T_{B_Y})$ is conjugate to $(X, T)$. The conjugating map $F : X \to X_{(V,E)}$ maps $Y$ onto the set of maximal paths and $T(Y)$ onto the set of minimal paths. Furthermore, if $y \in Y$, the backward orbit of $y$, $\{T^n y \mid n \leq 0\}$, is mapped onto the set of paths cofinal with $F(y)$, while the forward orbit $\{T^n y \mid n \geq 1\}$ is mapped onto the set of paths cofinal with $F(Ty)$. Any $T$-orbit, $\{T^n x | n \in \mathbb{Z}\}$, that does not meet $Y$ is mapped onto the set of paths cofinal with $F(x)$.*

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COROLLARY 1.3. [**7**, Theorem 4.7] *If* $Y = \{y\}$, *then* $B_{\{y\}} = (X, V, \geq)$ *is a properly ordered (also called simply ordered) Bratteli diagram, with* $F(y)$ *equal to the unique max path and* $F(Ty)$ *equal to the unique min path.*

## 2. *Étale equivalence relations*

Let $X$ be a Hausdorff locally compact, second countable (hence metrizable) space. For the most part we consider the case when $X$ is zero-dimensional, i.e. $X$ has a countable basis of closed and open (clopen) sets. (This is equivalent to $X$ being totally disconnected.) Of particular importance is the case when $X$ is a *Cantor set*, i.e. $X$ is a totally disconnected compact metric space with no isolated points—it is a well known fact (going back to Cantor and Hausdorff) that all such spaces are homeomorphic.

We consider countable equivalence relations $R$ on $X$, i.e. $R \subset X \times X$ is an equivalence relation so that each equivalence class $[x]_R = \{y \in X \mid (x, y) \in R\}$ is countable (or finite) for each $x$ in $X$. $R$ has a natural (principal) *groupoid* structure, with *unit space* equal to the diagonal set $\Delta = \{(x, x) \mid x \in X\}$, which we may identify with $X$. Specifically, if $(x, y)$, $(y, z) \in R$, then the product of this *composable* pair is defined as

$$(x, y)(y, z) = (x, z),$$

and the inverse of $(x, y) \in R$ is $(x, y)^{-1} = (y, x)$. The unit space of $R$ is by definition the set consisting of products of elements of $R$ with their inverses and so equals $\Delta$. Assume $R$ is given a Hausdorff locally compact, second countable (hence metrizable) topology $\mathcal{T}$, so that the product of composable pairs (with the topology inherited from the product topology on $R \times R$) is continuous. Also, the inverse map on $R$ is a homeomorphism. With this structure $(R, \mathcal{T})$ is a *locally compact (principal) groupoid*, cf. [**14**].

The range map $r : R \to X$ and the source map $s : R \to X$ are defined by $r((x, y)) = x$ and $s((x, y)) = y$, respectively, where $(x, y) \in R$—both maps being surjective.

*Definition 2.1.* (Étale equivalence relation) The locally compact groupoid $(R, \mathcal{T})$, where $R$ is a countable equivalence relation on the locally compact metric space $X$, is *étale* if $r : R \to X$ is a local homeomorphism, i.e. for every $(x, y) \in R$ there exists an open neighborhood $U^{(x,y)} \in \mathcal{T}$ of $(x, y)$ so that $r(U^{(x,y)})$ is open in $X$ and $r : U^{(x,y)} \to r(U^{(x,y)})$ is a homeomorphism. In particular, $r$ is an open map. If $X$ is zero dimensional, we may clearly choose $U^{(x,y)}$ to be a clopen set.

We call $(R, \mathcal{T})$ an *étale equivalence relation* on $X$ and occasionally refer to the local homeomorphism condition as the *étaleness condition*.

*Remark 2.2.* The definition we have given of étaleness—which is the most convenient to use for our objects of study—is equivalent to the various definitions of an étale (or $r$-discrete) locally compact groupoid (applied to our setting) that can be found in the literature. See, for example, [**14**, Definition 2.6 and Proposition 2.8] and [**12**, Definitions 2.2.1 and 2.2.3]—the existence of a (essential) unique Haar system consisting of counting measures follows from our definition, cf. [**12**, Proposition 2.2.5]. Furthermore, we can prove that the diagonal $\Delta = \Delta_X = \{(x, x) \mid x \in X\}$ is a clopen subset of $R$ [**14**, Proposition 2.8]. Also, $\Delta$ is homeomorphic to $X$ and so we are justified in identifying $\Delta$ with $X$.

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We observe that $s$ is a local homeomorphism, since $s((x, y)) = r((x, y)^{-1})$. It is easily deduced that $r^{-1}(x) = \{(x, y) \in R\}$, as well as $s^{-1}(x) = \{(y, x) \in R\}$, are (countable) discrete topological spaces in the relative topology for each $x \in X$. Clearly $R$ can be written as a union of graphs of local homeomorphisms of the form $s \circ r^{-1}$.

Note that the topology $\mathcal{T}$ on $R$ ($\subset X \times X$) is rarely the topology $\mathcal{T}_{\text{rel}}$ inherited from the product topology on $X \times X$. Necessarily $\mathcal{T}$ is finer than $\mathcal{T}_{\text{rel}}$. For details on topological groupoids, in particular locally compact and étale ($r$-discrete) groupoids, and the associated $C^*$-algebras, we refer the reader to [**11–13**].

The following proposition is the analogue in our setting of Theorem 1 of [**4**], where countable (standard) Borel equivalence relations were studied. Our proof mimics the proof in [**4**].

PROPOSITION 2.3. *Let $(R, \mathcal{T})$ be an étale equivalence relation on the zero-dimensional space $X$. There exists a countable group $G$ of homeomorphisms of $X$ so that $R = R_G$, where $R_G = \{(x, gx) \mid x \in X, g \in G\}$.*

*Proof.* Let $\{C_k\}_{k=1}^{\infty}$ be a clopen partition of $R \setminus \Delta$, where $\Delta$ is the diagonal $\{(x, x) \mid x \in X\}$, so that for each $k$, the maps $r$ and $s$ are homeomorphisms from $C_k$ onto the clopen sets $r(C_k) \subset X$ and $s(C_k) \subset X$, respectively. We refine the partition $\{C_k\}$ so that $r(C_k) \cap s(C_k) = \emptyset$ for each $k$. In fact, this may be achieved as follows. Let $\{I_i \times J_i\}_{i=1}^{\infty}$ be a clopen covering of $(X \times X) \setminus \Delta$ so that $I_i \cap J_i = \emptyset$ for every $i$, and each $I_i$ and $J_i$ are clopen. Define $D_k^i = C_k \cap (I_i \times J_i)$. Then $\{D_k^i\}_{i,k=1}^{\infty}$ is a clopen partition of $R \setminus \Delta$, so that $r$ and $s$ are homeomorphisms of $D_k^i$ onto the clopen sets $r(D_k^i) \subset X$ and $s(D_k^i) \subset X$, respectively, and $r(D_k^i) \cap s(D_k^i) = \emptyset$ for all $i, k$. We relabel the non-empty sets in $\{D_k^i\}$ and so get a sequence $\{E_i\}_{i=1}^{\infty}$, which is a clopen refinement of $\{C_k\}_{k=1}^{\infty}$, with the property that $r(E_i) \cap s(E_i) = \emptyset$ for every $i$. For each $i$ we define the continuous function

$$g_i(x) = \begin{cases} y(= s_i r_i^{-1}(x)) & \text{if } (x, y) \in E_i, \\ y(= r_i s_i^{-1}(x)) & \text{if } (y, x) \in E_i, \\ x & \text{otherwise,} \end{cases}$$

where $r_i$ and $s_i$ denote the restrictions to $E_i$ of $r$ and $s$, respectively. Observe that $g_i^2 = \text{id}$ and so $g_i$ is a homeomorphism. The graph $\Gamma(g_i)$ of $g_i$ is easily seen to be $E_i \cup \theta(E_i) \cup (\Delta \cap (F_i \times F_i))$, where $\theta : X \times X \to X \times X$ denotes the flip map $(x, y) \to (y, x)$ and $F_i = X \setminus (r(E_i) \cup s(E_i))$. Hence $\Gamma(g_i) \subset R$. Let $G$ be the (countable) group generated by the $g_i$. Clearly $R_G \subset R$. On the other hand, $\bigcup_{i=1}^{\infty} \Gamma(g_i) \supset R \setminus \Delta$, since $\{E_i\}$ is a covering of $R \setminus \Delta$. Since clearly $\Delta \subset R_G$, we conclude that $R = R_G$. □

*Open problem.* Assume that for every $x \in X$, the equivalence class $[x]_R$ is dense in $X$, where $(R, \mathcal{T})$ is as in Proposition 2.3. Is it possible to choose $G$ so that $R = R_G$ and $G$ acts freely on $X$ (i.e. $gy = y$ for some $y \in X$, $g \in G$, implies that $g$ is an identity element)?

The analogous question has a negative answer in the setting of countable (standard) Borel equivalence relations [**1**]. Furthermore, in the ergodic, measure-preserving case there are examples of countable equivalence relations that cannot be generated by an essentially free action of a countable group [**5**, Theorem $D$].

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Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ be two étale equivalence relations on $X_1$ and $X_2$, respectively. There is an obvious notion of isomorphism, namely a homeomorphism of $(R_1, \mathcal{T}_1)$ onto $(R_2, \mathcal{T}_2)$ respecting the groupoid operations. Since the unit spaces $\Delta_i = \{(x, x) \mid x \in X_i\}$ —which we identify with $X_i$—are equal to $\{aa^{-1} \mid a \in R_i\}$, $i = 1, 2$, the definition of isomorphism may be given as follows.

*Definition 2.4.* (Isomorphism and orbit equivalence) Let $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ be two étale equivalence relations on $X_1$ and $X_2$, respectively. $(R_1, \mathcal{T}_1)$ is *isomorphic* to $(R_2, \mathcal{T}_2)$—we use the notation $(R_1, \mathcal{T}_1) \cong (R_2, \mathcal{T}_2)$—if there exists a homeomorphism $F : X_1 \to X_2$ so that the following occur:

(i) $(x, y) \in R_1 \iff (F(x), F(y)) \in R_2$;

(ii) $F \times F : (R_1, \mathcal{T}_1) \to (R_2, \mathcal{T}_2)$ is a homeomorphism, where $F \times F((x, y)) = (F(x), F(y))$, $(x, y) \in R_1$. We say $F$ *implements* an isomorphism between $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$.

We say that $(R_1, \mathcal{T}_1)$, or $R_1$, is *orbit equivalent* to $(R_2, \mathcal{T}_2)$, or to $R_2$, if (i) is satisfied and we call $F$ an *orbit map* in this case.

*Remark 2.5.* Observe that $(R_1, \mathcal{T}_1)$ is orbit equivalent to $(R_2, \mathcal{T}_2)$ via the orbit map $F : X_1 \to X_2$ if and only if $F([x]_{R_1}) = [F(x)]_{R_2}$ for each $x \in X_1$. So $F$ maps equivalence classes onto equivalence classes.

Note that if $R_i = R_{G_i}$ for some countable group $G_i$, $i = 1, 2$, then the equivalence classes coincide with $G_i$-orbits and so the term orbit equivalence is appropriate (cf. Proposition 2.3).

There is a notion of invariant probability measure associated to an étale groupoid $(R, \mathcal{T})$. It suffices to say here that the probability measure $\mu$ on $X$ is $(R, \mathcal{T})$-invariant iff $\mu$ is $G$-invariant, where $G$ is as in Proposition 2.3. It is straightforward to show that if $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ on $X_1$ and $X_2$, respectively, are orbit equivalent via the orbit map $F : X_1 \to X_2$, then $F$ maps the set of $(R_1, \mathcal{T}_1)$-invariant probability measures injectively onto the set of $(R_2, \mathcal{T}_2)$-invariant probability measures.

*Remark 2.6.* It is very important to be aware of the fact that a countable equivalence relation $R$ on $X$ may be given distinct non-isomorphic topologies $\mathcal{T}_1$ and $\mathcal{T}_2$, so that $(R, \mathcal{T}_1)$ and $(R, \mathcal{T}_2)$ are étale equivalence relations. In fact, we may give examples of non-isomorphic $(R, \mathcal{T}_1)$ and $(R, \mathcal{T}_2)$ where every equivalence class is dense. Specifically, $(R, \mathcal{T}_1)$ may be chosen to be the étale equivalence relation associated to a Cantor minimal system $(X, T)$, while $(R, \mathcal{T}_2)$ is the cofinal relation associated to a (simple) standard Bratteli diagram, appropriately topologized—see the description given in the two examples below (compare with §4 and [**6**, Theorem 2.3]).

The above fact contrasts with the situation in the countable (standard) Borel equivalence relation setting, where the Borel structure is uniquely determined by $R(\subset X \times X)$.

*Examples 2.7.* (Two étale equivalence relations)

(i) Let $G$ be a countable discrete group acting freely as homeomorphisms on the locally compact metric space $X$. Let

$$R_G = \{(x, gx) \mid x \in X, g \in G\} \subset X \times X,$$

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i.e. the $R_G$-equivalence classes are simply the $G$-orbits. Topologize $R_G$ by transferring the product topology on $X \times G$ to $R_G$ via the bijection $(x, g) \to (x, gx)$. (This is a bijection since $G$ acts freely on $X$.) Then it is easily verified that $R_G$ becomes an étale equivalence relation. (If $G$ do not act freely, we get a bijection between $R_G$ and a closed subset of $X \times G \times X$ by the map $(x, gx) \to (x, g, gx)$ and we transfer the product topology on $X \times G \times X$ to $R_G$.)

We are especially interested in the situation when $G$ acts *minimally* (and freely) on the Cantor set $X$, i.e. each orbit $Gx = \{gx \mid g \in G\}$ is dense in $X$ for every $x$ in $X$. In particular, when $G = \mathbb{Z}$, we let $(X, T)$, where $T$ is the (necessarily minimal) homeomorphism corresponding to $1 \in \mathbb{Z}$, denote the associated *Cantor* (dynamical) system. We use the term '$T$-orbit' instead of '$\mathbb{Z}$-orbit' in this case.

Let $(X, T)$ and $(Y, S)$ be two Cantor minimal systems, and denote the associated étale equivalence relations by $R(X, T)$ and $R(Y, S)$, respectively. We claim that $R(X, T) \cong R(Y, S)$ if and only if $(X, T)$ is flip conjugate to $(Y, S)$ (i.e. $(X, T)$ is conjugate to either $(Y, S)$ or $(Y, S^{-1})$). In fact, one direction is obvious. Conversely, assume $R(X, T)$ is isomorphic to $R(Y, S)$ via the implementing map $F : X \to Y$. Let $(x_i)$ be a sequence in $X$ so that $x_i \to x$. Then $(x_i, Tx_i) \to (x, Tx)$ in $R(X, T)$. Now $F \times F((x, Tx)) = (Fx, S^m(Fx))$, $F \times F((x_i, Tx_i)) = (Fx_i, S^{m_i}(Fx_i))$, for some $m, m_i \in \mathbb{Z}$. Since $F \times F((x_i, Tx_i)) \to F \times F((x, Tx))$ in $R(Y, S)$, this implies that $m_i = m$ for all but finitely many $i$. By a theorem of Boyle (cf. Theorem 1.4. of [**6**]), this implies that $(X, T)$ and $(Y, S)$ are flip conjugate.

(ii) We begin with a special infinite directed graph $(V, E)$, called a *Bratteli diagram*, which consists of a vertex set $V$ and an edge set $E$, where $V$ and $E$ can be written as a countable disjoint union of non-empty finite sets:

$$V = V_0 \cup V_1 \cup V_2 \cup \cdots \quad \text{and} \quad E = E_0 \cup E_1 \cup E_2 \cup \cdots$$

with the following property. An edge $e$ in $E_n$ goes from a vertex in $V_{n-1}$ to one in $V_n$, which we denote by $i(e)$ and $f(e)$, respectively. We call $i$ the *source map* and $f$ the *range map*. We require that there are no *sinks*, i.e. $i^{-1}(v) \neq \emptyset$ for all $v \in V$.

It is convenient to give a diagrammatic presentation of a Bratteli diagram with $V_n$ the vertices at level $n$ and $E_n$ the edges (downward directed) between $V_{n-1}$ and $V_n$, see Figure 1.

We let $X = X_{(V,E)}$ denote the space of infinite paths in the diagram beginning at some *source* $v \in V$, i.e. $f^{-1}(v) = \emptyset$. Say $v \in V_n$ is a source and let

$$X_v = \{(e_{n+1}, e_{n+2}, \ldots) \mid i(e_{n+1}) = v, i(e_{k+1}) = f(e_k), k > n\}$$

which is given the relative topology of the product space $\prod_{k>n} E_k$ and is, therefore, compact, metrizable and zero dimensional. We let $X$ be the disjoint union of the $X_v$ with the topological sum topology. Then $X$ is locally compact, metrizable and zero dimensional, which has a basis consisting of clopen *cylinder* sets, i.e. sets of the form $U_{(e_{n+1},\ldots,e_m)} = \{(f_{n+1}, f_{n+2}, \ldots) \in X \mid f_{n+1} = e_{n+1}, \ldots, f_m = e_m; i(e_{n+1}) \in V_n$ is a source$\}$. The equivalence $R$ on $X$ shall be *cofinal* or *tail equivalence*: two paths are

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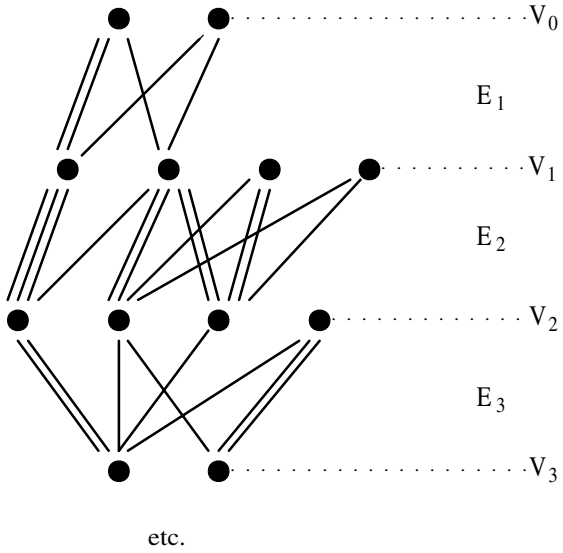



FIGURE 1. Bratteli diagram.

equivalent if they agree from some level on. For $N = 0, 1, 2, \ldots$, let

$$R_N = \{((e_{m+1}, e_{m+2}, \ldots), (e'_{n+1}, e'_{n+2}, \ldots)) \in X \times X \mid m, n \leq N \text{ and } e_k = e'_k \text{ for all } k > N\}.$$

Give $R_N$ the relative topology $\mathcal{T}_N$ of $X \times X$. Then $R_N$ is compact and is an open subset of $R_{N+1}$ for all $N$. Let $R = \bigcup_{N=0}^{\infty} R_N$ and give $R$ the inductive limit topology $\mathcal{T}$, so that a set $U$ is in $\mathcal{T}$ if and only if $U \cap R_N$ is in $\mathcal{T}_N$ for each $N$. This means that a sequence $\{(x_n, y_n)\}$ in $R$ converges to $(x, y)$ in $R$ if and only if $\{x_n\}$ converges to $x$, $\{y_n\}$ converges to $y$ (in $X$) and, for some $N$, $(x_n, y_n)$ is in $R_N$ for all but finitely many $n$.

It is now a simple task to verify that $(R, \mathcal{T})$ is an étale equivalence relation. In §3 (Theorem 3.9) we prove that this Bratteli diagram example is the prototype of an $AF$-equivalence relation—this is defined in Definition 3.7. Therefore we will denote $(R, \mathcal{T})$ by $AF(V, E)$.

There is an obvious (countable) locally finite group $G$ of homeomorphisms of $X$ so that $R = R_G = \{(x, gx) \mid x \in X, g \in G\}$, where the fixed point set, $\text{fix}(g) = \{x \in X \mid g(x) = x\}$, is a clopen subset of $X$ for every $g \in G$ (cf. Proposition 2.3). In fact, $G = \bigcup_{n=0}^{\infty} G_n$, where $\{id\} = G_0 \subset G_1 \subset G_2 \subset \cdots$ is an increasing sequence of finite groups with $G_n = \bigoplus_{v \in V_n} G_v$. Here $G_v$ is the group consisting of those homeomorphisms $g$ of $X = X_{(V,E)}$, such that $g(x) = x$ for those paths $x \in X$ that do not pass through $v$, and otherwise $g(x)$ is obtained by permuting the initial segments (above level $n$) of the various $x$'s passing through $v$, leaving the tails unchanged. We omit the details. Conversely, let $G$ be a locally finite group acting as homeomorphisms on a zero-dimensional space $X$, such that the fixed point set of each $g \in G$ is clopen. Then one can construct a Bratteli diagram $(V, E)$ so that $X$ may be identified with $X_{(V,E)}$, and $R_G$ coincides with the cofinal equivalence relation, cf. [**9**] and [**11**].

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If the Bratteli diagram $(V, E)$ has only one source $v_0 \in V$—which necessarily entails that $V_0 = \{v_0\}$—we call $(V, E)$ a *standard* Bratteli diagram. The path space $X_{(V,E)}$ associated to a standard Bratteli diagram $(V, E)$ is compact. We observe that if $(V', E')$ is a *telescope* of $(V, E)$, i.e. $(V', E')$ is obtained from $(V, E)$ by telescoping $(V, E)$ to certain levels $0 < n_1 < n_2 < n_3 < \cdots$, then $AF(V, E) \cong AF(V', E')$. In fact, there is a natural homeomorphism $\alpha : X_{(V,E)} \to X_{(V',E')}$ and $\alpha$ clearly implements the isomorphism, according to the description we have given of convergence in $AF(V, E)$, respectively $AF(V', E')$.

The standard Bratteli diagram $(V, E)$ is *simple* if for each $n$ there is an $m > n$ so that by telescoping the diagram between levels $n$ and $m$, every vertex $v$ in $V_n$ is connected to every vertex $w$ in $V_m$. It is a simple observation that $(V, E)$ is simple if and only if every $AF(V, E)$-equivalence class is dense in $X_{(V,E)}$.

A *dimension group*, denoted $K_0(V, E)$, which is simple if and only if $(V, E)$ is simple is associated to the Bratteli diagram $(V, E)$. (We refer the reader to [**3**] and [**6**, §3] for more details on this.)

## 3. *AF-equivalence relations*

We recall some terminology that we shall use. Let $R$ be an equivalence relation on $X$ and let $A \subset X$. We denote by $R|_A$ the *restriction* of $R$ to $A$; that is, $R|_A = R \cap (A \times A)$. We say that $A$ is *R-invariant* if $(x, y) \in R$ and $x \in A$, implies $y \in A$. In other words, every $R$-equivalence class that meets $A$ lies entirely inside $A$.

If $R'$ is another equivalence relation on $X$, we say that $R'$ is a *subequivalence relation* of $R$ if $R' \subset R$.

*Definition 3.1.* (Compact étale equivalence relation) Let $(R, \mathcal{T})$ be an étale equivalence relation on the locally compact space $X$, and let $\Delta = \Delta_X \subset R$ be the diagonal in $X \times X$ (i.e., the unit space of $R$). We say that $(R, \mathcal{T})$ is a *compact étale equivalence relation* (CEER) if $R \setminus \Delta$ is a compact subset of $R$. If $X$ itself is compact this is equivalent to saying that $R$ is compact, since $\Delta$ is then compact.

PROPOSITION 3.2. *Let $(R, \mathcal{T})$ be a CEER on $X$. Then:*

(i) *$\mathcal{T}$ is the relative topology from $X \times X$;*

(ii) *$R$ is a closed subset of $X \times X$ (with the product topology) and the quotient topology of the quotient space $X/R$ is Hausdorff;*

(iii) *$R$ is uniformly finite; that is, there is a natural number $N$ such that the number $\#([x]_R)$ of elements in each equivalence class $[x]_R$ is at most $N$.*

*Proof.*

(i) The (identity) map $r \times s : R \setminus \Delta \to X \times X$ is continuous and injective, and so is a homeomorphism onto its image, since $R \setminus \Delta$ is compact. Now $\Delta$ is a clopen subset of $R$ and its relative topology with respect to $\mathcal{T}$ and the product topology of $X \times X$ coincide (Remark 2.2). Hence the assertion follows.

(ii) Clearly $R$ is a closed subset of $X \times X$ by (i). Now the quotient map $q : X \to X/R$ is open since $r$ and $s$ are open maps. In fact, if $U$ is an open subset of $X$, the set $q(U) = s(r^{-1}(U))$ is open in $X/R$. By [**10**, Theorem 11], $X/R$ is Hausdorff.

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(iii) Since $R \setminus \Delta$ is compact, it can be covered by finitely many (open) sets $U$ such that the restriction of the map $r$ to $U$ is one-to-one. This obviously implies that $R$ is uniformly finite. □

*Remark 3.3.* It is not true that an étale equivalence relation satisfying (ii) and (iii) of Proposition 3.2 is CEER, even when $X$ is compact. In fact, let $X$ be the unit interval $[0, 1]$ and let the graph of $R$ in $X \times X$ be the union of the diagonal $\Delta$ and the graph of the function $f(t) = 1 - t$. The equivalence classes have cardinality two, except the equivalence class of $\frac{1}{2}$, which has cardinality one. Clearly $R$ is a closed (hence compact) subset of $X \times X$. It is easy to see that $R$ may be given a (non-compact) topology $\mathcal{T}$ so that $(R, \mathcal{T})$ becomes an étale equivalence relation on $X$. However, $\mathcal{T}$ is not the relative topology of $X \times X$ and so is not CEER (we can construct a similar example with $X$ equal to the Cantor set).

We can show that an étale equivalence relation $(R, \mathcal{T})$ on $X$ satisfying (i) of Proposition 3.2 is characterized by the property that the map $r$ is a local homeomorphism of $R$, when the latter is given the relative topology of $X \times X$.

The *disjoint union* of a finite set of étale equivalence relations is defined in the obvious way. Let $(R_i, \mathcal{T}_i)$ be an étale equivalence relation on $X_i$ for $i = 1, 2, \ldots, k$, where the $X_i$ are disjoint. Let $X = \bigsqcup_{i=1}^{k} X_i$ be the disjoint union and let $R = \bigsqcup_{i=1}^{k} R_i$ be the equivalence relation on $X$ defined in the obvious way. Let $\mathcal{T} = \bigsqcup_{i=1}^{k} \mathcal{T}_i$ be the disjoint union topology (also called the sum topology). Then $(R, \mathcal{T})$—the *disjoint union* of $\{(R_i, \mathcal{T}_i)\}_{i=1}^{k}$—is an étale equivalence relation on $X$.

The *product* of two étale equivalence relations is also defined in the obvious way. Let $(R_i, \mathcal{T}_i)$ be an étale equivalence relation on $X_i$, for $i = 1, 2$. The *product* $(R, \mathcal{T})$ of $(R_1, \mathcal{T}_1)$ and $(R_2, \mathcal{T}_2)$ is an étale equivalence relation on $X = X_1 \times X_2$ in an obvious way, where $R = R_1 \times R_2$ is given the product topology.

The following lemma, besides giving the structure of CEERs on zero-dimensional spaces, is used in connection with the construction of various Bratteli diagrams that we associate to AF-equivalence relations. We also point out the relevance of Remark 3.6 in this regard.

LEMMA 3.4. *Let $(R, \mathcal{T})$ be a CEER (see Definition 3.1) on $X$, where $X$ is a zero-dimensional space. Then $(R, \mathcal{T})$ is isomorphic to a finite disjoint union of CEERs $\{(R_i, \mathcal{T}_i)\}_{i=1}^{k}$ of type $m_1, \ldots, m_k$, respectively, where $X = \bigsqcup_{i=1}^{k} X_i$ and $R_i$ is an equivalence relation on $X_i$ of type $m_i$ for $i = 1, \ldots, k$. Specifically, $X_i$ is (homeomorphic to) $Y_i \times \{1, 2, \ldots, m_i\}$ for some natural number $m_i$, where $R_i$ is the product of the trivial equivalence relation on $\{1, 2, \ldots, m_i\}$ (all points are equivalent) with the cotrivial equivalence relation (the identity relation) on $Y_i$. Furthermore, if $X$ is non-compact, then $Y_k$ is the only non-compact set of the $Y_i$, and $m_k = 1$.*

*Conversely, an étale equivalence relation of the type described is CEER.*

*The family of sets*

$$\mathcal{O} = \{Y_i \times \{j\} \mid 1 \leq j \leq m_i \text{ for } i = 1, 2, \ldots, k\}$$

*is a finite clopen partition of $X$. If $\mathcal{P}$ is an initially given finite clopen partition of $X$, we may choose the $X_i$ so that $\mathcal{O}$ is finer than $\mathcal{P}$. Furthermore, $\mathcal{O}$ gives rise to a clopen partition*

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*$\mathcal{O}'$ of $R$ in a natural way, namely $\mathcal{O}'$ consists of the graphs of the local homeomorphisms $\gamma_{lm}^{(i)} : Y_i \times \{l\} \to Y_i \times \{m\}$ where*

$$\gamma_{lm}^{(i)}((y, l)) = (y, m), \quad 1 \leq l,\ m \leq m_i \text{ and } i = 1, \ldots, k.$$

*(So the maps $\gamma_{lm}^{(i)}$ are of the form $s \circ r^{-1}$, appropriately restricted.) If $\mathcal{P}'$ is an initially given finite clopen partition of $R$, we may choose the $X_i$ so that $\mathcal{O}'$ is finer than $\mathcal{P}'$.*

*Proof.* Let $X_1 = r(R \setminus \Delta)$. Then $X_1$ is a compact, clopen subset of $X$ that is $R$-invariant. Now the restriction $R|_{X \setminus X_1}$ of $R$ to the invariant set $X \setminus X_1$ coincides with $\Delta|_{X \setminus X_1}$. Hence $(R, \mathcal{T})$ is the disjoint union of $R|_{X_1}$ and $\Delta|_{X \setminus X_1}$ (with the relative topologies) and so we may assume at the outset that $X$ is compact and hence *a fortiori* $(R, \mathcal{T})$ is compact. For $x \in X$, let $r^{-1}(x) = \{(x, y_1), \ldots, (x, y_m)\}$; in other words, $[x]_R = \{y_1, \ldots, y_m\}$. (We may assume that $y_1 = x$.) We claim that we can find a clopen neighbourhood $U^{(y_i, y_j)}$ of $(y_i, y_j) \in R$ for every $i, j$, so that the restrictions of $r$ and $s$, respectively, to $U^{(y_i, y_j)}$ are homeomorphisms onto their clopen images. Furthermore, $r(U^{(y_i, y_j)}) = U^{y_i}$ is independent of $j$ and is a clopen neighbourhood of $y_i$, and the sets $U^{y_1}, \ldots, U^{y_m}$ are disjoint. Likewise, $s(U^{(y_i, y_j)})$ equals $U^{y_j}$ and so is independent of $i$. In fact, by the étaleness condition it follows easily that there exist disjoint clopen neighborhoods $U^{(y_1, y_j)}$, for $j = 1, \ldots, m$, such that both $r$ and $s$, restricted to $U^{(y_1, y_j)}$, are homeomorphisms onto their respective clopen images, with $r(U^{(y_1, y_j)}) = U^{y_1}$ for every $j$. Set $U^{y_j} = s(U^{(y_1, y_j)})$. By appropriately restricting the $r$ and $s$ maps we construct from this homeomorphisms $\gamma_{ij} : U^{y_i} \to U^{y_j}$ for each $i, j$, such that the graph of $\gamma_{ij}$ is $U^{(y_i, y_j)}$. Set $R^x = \bigcup_{i,j=1}^m U^{(y_i, y_j)}$ and $V^x = \bigcup_{i=1}^m U^{y_i}$. (Recall that $U^{y_1} = U^x$.) Then $R^x$ ($\subset R$) is a CEER on the clopen set $V^x$ ($\subset X$), which is easily seen to be isomorphic to the product of the cotrivial and the trivial equivalence relation on $U^x \times \{1, \ldots, m\}$. Now $V^x$ need not be $R$-invariant and so $R^x$ may not be equal to $R$ restricted to $V^x$. In order to obtain $R$-invariant sets $V^x$, we show first that the map $F\colon X \to \mathbb{N}$, defined by $x \mapsto \#[x]_R$, is continuous. Now $F$ is easily seen to be lower semi-continuous (we only need the étaleness condition—and not the compactness—of $(R, \mathcal{T})$ to show this), and so we must prove that $F$ is upper semi-continuous. Let $x_n \to n$. We must show that $F(x) \geq \limsup_n F(x_n)$. Clearly we may assume that $F(x_n) = \#[x_n]_R = t$ for all $n$ (cf. Proposition 3.2(iii)) and so we must show that $F(x) = \#[x]_R \geq t$. Let $r^{-1}(x_n) = \{(x_n, y_n^1), \ldots, (x_n, y_n^t)\}$, where $y_n^1, \ldots, y_n^t$ are $t$ distinct points. By the compactness of $(R, \mathcal{T})$, there exists a subsequence of $\{(x_n, y_n^1)\}$ converging to some $(x, y^1) \in R$. By relabelling, we again denote this subsequence by $\{(x_n, y_n^1)\}_{n=1}^\infty$. Using the compactness of $(R, \mathcal{T})$ once more, we can find a subsequence of $\{(x_n, y_n^2)\}_{n=1}^\infty$ converging to $(x, y^2) \in R$. By relabelling, and doing this argument $t$ times, we get $t$ sequences $\{(x_n, y_n^i)\}_{n=1}^\infty$, so that $(x_n, y_n^i) \to (x, y^i)$ as $n \to \infty$, $i = 1, \ldots, t$. For $i \in \{1, \ldots, t\}$, let $U_i \in \mathcal{T}$ be a clopen neighbourhood of $(x, y^i)$ such that $r\colon U \to r(U)$ is a homeomorphism. Since $r$ is a one-to-one map, we get that $(x_n, y_n^j) \notin U$ for large $n$, when $j \neq i$. This clearly implies that $y^j \neq y^i$. Hence $\#[x]_R \geq t$ and we are done.

Let $k_1 > k_2 > \cdots > k_l \geq 1$ be the (distinct) cardinalities of the $R$-equivalence classes. The sets $X_i = \{x \in X \mid \#[x]_R = k_i\}$, $i = 1, \ldots, l$, are obviously $R$-invariant and by the continuity of $x \mapsto \#[x]_R$ these sets are clopen. Hence, each of the (disjoint) compact

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subsets $r^{-1}(X_i)$ of $R$, $i = 1, \ldots, l$, may be covered by a finite number of sets of the form $R^x$, as described above. The associated subsets $V^x$ of X are $R$-invariant, so we have that $R^x = R|_{V^x}$ $(=R \cap (V^x \times V^x))$. We may clearly choose the various $U^x$ so small that each $U^{y_i}$ that occurs in $V^x = \bigcup_{i=1}^m U^{y_i}$ is contained in some element of $\mathcal{P}$, where we use the notation above. Similarly we may choose $U^x$ so small, that each graph $U^{(y_i, y_j)}$ (associated to the local homomorphism $\gamma_{ij} \colon U^{y_i} \to U^{y_j}$) that occurs in $R^x = \bigcup_{i,j=1}^m U^{(y_i, y_j)}$ is contained in some element of $\mathcal{P}'$. For $i \in \{1, \ldots, l\}$, let $r^{-1}(X_i) = R^{x_1^i} \cup \cdots \cup R^{x_{n_i}^i}$, $X_i = V^{x_1^i} \cup \cdots \cup V^{x_{n_i}^i}$, where $x_1^i, \ldots, x_{n_i}^i \in X_i$. By considering $R^{x_s^i} \setminus \bigcup_{j<s} R^{x_j^i}$, respectively $V^{x_s^i} \setminus \bigcup_{j<s} V^{x_j^i}$, instead of $R^{x_s^i}$, respectively $V^{x_s^i}$, $s = 1, \ldots, n_i$, we may assume that the sets $R^{x_1^i}, \ldots, R^{x_{n_i}^i}$, respectively $V^{x_1^i}, \ldots, V^{x_{n_i}^i}$, are pairwise disjoint.

Combining the above, we get the desired decomposition of $(R, \mathcal{T})$, such that the associated partition $\mathcal{O}'$ of $R$, respectively $\mathcal{O}$ of $X$, is finer than $\mathcal{P}'$, respectively $\mathcal{P}$. □

*Comment*. Let $R$ be an equivalence on a zero-dimensional space $X$. Then $R$ (given the relative topology from $X \times X$) is CEER iff:

(i) the quotient space $X/R$ is Hausdorff;

(ii) the quotient map $\pi : X \to X/R$ is a local homeomorphism, which is one-to-one outside a compact $R$-invariant set.

This makes it clear that the definition of CEER does not involve the topology of $R$ and it shows that $\pi$ is a covering map. (We are indebted to the referee for this comment.)

COROLLARY 3.5. *Let $(R, \mathcal{T})$ be a CEER on the zero-dimensional space $X$. If $\mathcal{P}$ and $\mathcal{P}'$ are finite clopen partitions of $X$ and $R$, respectively, there exist clopen partitions $\mathcal{O}$ and $\mathcal{O}'$ of $X$ and $R$, respectively, as described in Lemma 3.4, so that $\mathcal{O}$ is finer than $\mathcal{P}$ and $\mathcal{O}'$ is finer than $\mathcal{P}'$. In fact, $\mathcal{O} = \Delta \cap \mathcal{O}'$, where $\Delta = \Delta_X$ is the diagonal in $X \times X$. (We make the obvious identification between $X$ and $\Delta$.)*

*Remark 3.6.* It is instructive to draw a picture to illustrate the content of the above corollary: $X$ can be composed into $n$ (disjoint) compact, clopen towers—the $k$th tower being $V^{x_k}$ of height $\#([x_k]_R)$—with clopen floors, and possibly one non-compact tower of height one; see Figure 2. (Some of the towers may have the same height.) The equivalence classes of $R$ are formed by the sets of points lying vertically above or below one another in each tower. The clopen partition $\mathcal{O}$ of $X$ is the set consisting of the floors of the various towers. The partition $\mathcal{O}'$ of $R$ can also be easily described in terms of the towers in Figure 2. In fact, for each tower there is a local homeomorphism of the form $s \circ r^{-1}$(appropriately restricted), between every pair of floors. The graphs of these maps make up $\mathcal{O}'$. With this picture it is obvious how we may identify $\mathcal{O}$ with $\mathcal{O}' \cap \Delta_X$, where $\Delta_X$ is the diagonal in $X \times X$. In Figure 2 we have also shown the 'contribution' of one of the towers, say $V^{x_i}$ of height three (i.e. $\#([x_i]_R) = 3$), to the partition $\mathcal{O}'$ of $R$—we have drawn the graphs of the local homeomorphism associated to $V^{x_i}$ in boldface.

Note that $\mathcal{O}'$ is a very special partition of $R$. In fact, $\mathcal{O}'$ has a natural (abstract) principal groupoid structure, with unit space identifiable with $\mathcal{O}$, which we now describe. If we define $U \cdot V$ for two subsets $U, V$ of $R$ to be

$$U \cdot V = \{(x, z) | (x, y) \in U, (y, z) \in V \text{ for some } y \in X\}$$

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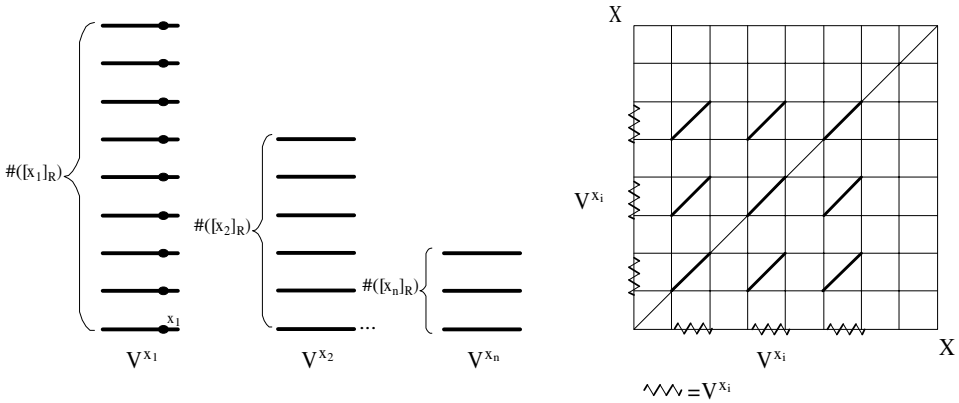



FIGURE 2. Illustration of the towers (see Corollary 3.5).

then we can list the properties of $\mathcal{O}'$ as follows.

(i) $\mathcal{O}'$ is a finite clopen partition of $R$ finer than $\{\Delta, R \setminus \Delta\}$.

(ii) For all $U \in \mathcal{O}'$, the maps $r, s : U \to X$ are local homeomorphisms and if $U \in \mathcal{O}' \cap (R \setminus \Delta)$, then $r(U) \cap s(U) = \emptyset$.

(iii) For all $U, V \in \mathcal{O}'$, we have $U \cdot V = \emptyset$ or $U \cdot V \in \mathcal{O}'$. Also, $U^{-1}(= \{(y, x) | (x, y) \in U\})$ is in $\mathcal{O}'$ for every $U$ in $\mathcal{O}'$.

(iv) With $\mathcal{O}'^{(2)} = \{(U, V) | U, V \in \mathcal{O}', U \cdot V \neq \emptyset\}$, define $(U, V) \in \mathcal{O}'^{(2)} \longrightarrow U \cdot V \in \mathcal{O}'$. Then $\mathcal{O}'$ becomes a principal groupoid with unit space equal to $\{U \in \mathcal{O}' | U \subset \Delta\}$, which clearly may be identified with $\mathcal{O}$.

We call $\mathcal{O}'$ a *groupoid partition* of $(R, \mathcal{T})$, (or of $R$). Note that if we define the equivalence relation $\sim_{\mathcal{O}'}$ on $\mathcal{O}$ by $A \sim_{\mathcal{O}'} B$ if there exists $U \in \mathcal{O}'$ so that $U^{-1} \cdot U = A$, $U \cdot U^{-1} = B$, then the equivalence classes, denoted $[\cdot]_{\mathcal{O}'}$, are exactly the towers in Figure 2.

The CEERs are the building blocks with which we define *AF-equivalence relations*. (*AF* stands for 'approximately finite dimensional' and we refer to [**6**] and [**14**] for further explanation of the terminology.)

*Definition 3.7.* (*AF*-equivalence relation) Let $\{(R_n, \mathcal{T}_n)\}_{n=1}^{\infty}$ be a sequence of CEERs on a zero-dimensional (second countable, locally compact Hausdorff) space $X$, so that $R_n$ is an open subequivalence relation of $R_{n+1}$, i.e. $R_n \subset R_{n+1}$ and $R_n \in \mathcal{T}_{n+1}$ for every $n$. (Note that this implies that $R_n$ is a clopen subset of $R_{n+1}$, since $R_n \setminus \Delta$ is compact.) Let $(R, \mathcal{T})$ be the *inductive limit* of $\{(R_n, \mathcal{T}_n)\}$ with the *inductive limit topology* $\mathcal{T}$, i.e. $R = \bigcup_{n=1}^{\infty} R_n$ and $U \in \mathcal{T}$ if and only if $U \cap R_n \in \mathcal{T}_n$ for every $n$. We say that $(R, \mathcal{T})$ is an *AF-equivalence relation* on $X$ and we use the notation $(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$.

*Comment.* In Definition 3.7, the condition that $R_n$ is open in $R_{n+1}$ is superfluous, because it is automatically satisfied. In fact, consider the quotient maps $\pi_n : X \to X/R_n$, $\pi_{n+1} : X \to X/R_{n+1}$ and $\pi_{n+1,n} : X/R_n \to X/R_{n+1}$. Let $S$ be the equivalence relation on $X/R_n$ defined by $\pi_{n+1,n}$. We observe that the diagonal is open in $S$. By considering their inverse images by $\pi_n \times \pi_n$, we get the assertion. Definition 3.7 of an *AF*-equivalence relation coincides with the definition of an *AF*-equivalence relation given in [**15**, Definition 3.1]. (We are indebted to the referee for this comment.)

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We say that $(R, \mathcal{T})$ is *minimal* if each equivalence class $[x]_R$, $x \in X$, is dense in $X$.

THEOREM 3.8. *Let $G$ be a countable group acting minimally and freely on the Cantor set $X$ and let $(R_G, \mathcal{T})$ be the associated étale equivalence relation (cf. Example 2.7(i)). Then $(R_G, \mathcal{T})$ is AF if and only if $G$ is locally finite.*

*Proof.* Assume $G$ is locally finite, and let $G_1 \subset G_2 \subset \cdots \subset G = \bigcup_n G_n$, where $G_n$ is a finite group for every $n$. It is easy to see that $(R_G, \mathcal{T}) = \varinjlim(R_{G_n}, \mathcal{T}_n)$, where $(R_{G_n}, \mathcal{T}_n)$ is obviously CEER. In fact, $R_{G_n}$ may be identified with $X \times G_n$ and $\mathcal{T}_n$ is the product topology. Hence $(R_G, \mathcal{T})$ is an $AF$-equivalence relation. Conversely, if $(R_G, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$ is an $AF$-equivalence relation, then—identifying $R_G$ with $X \times G$—let $F$ be a finite subset of $G$. To show that $G$ is locally finite it suffices to show that the subgroup generated by $F$ is finite. Since $X \times F$ is compact, it is contained in some $R_n$. Define $H = \{g \in G \mid X \times \{g\} \subset R_n\}$. Clearly $F \subset H$, and since $R_n$ is an equivalence relation it follows that $H$ is a subgroup of $G$. As $R_n$ is compact, $H$ is finite and so the subgroup generated by $F$ is finite. □

It is straightforward to verify that an $AF$-equivalence relation is an étale equivalence relation. Furthermore, we verify that the étale equivalence relation of Example 2.7(ii) is an $AF$-equivalence relation. The following theorem is the converse result, as alluded to in Example 2.7(ii).

THEOREM 3.9. *Let $(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$ be an AF-equivalence relation on $X$. There exists a Bratteli diagram $(V, E)$ such that $(R, \mathcal{T})$ is isomorphic to the AF-equivalence relation $AF(V, E)$ associated to $(V, E)$.*

*If $X$ is compact, the Bratteli diagram $(V, E)$ may be chosen to be standard. Furthermore, $(V, E)$ is simple if and only if $(R, \mathcal{T})$ is minimal.*

*Proof.* Choose an increasing sequence $\{K_m\}_{m=1}^{\infty}$ of compact, clopen subsets of $X$, such that $X = \bigcup_{m=1}^{\infty} K_m$ (if $X$ is itself compact, we let $K_1 = K_2 = \cdots = X$), and such that $r(R_m \setminus \Delta) \subset K_m$ for each $m$. We will choose an increasing sequence $\mathcal{P}_1 \prec \mathcal{P}_2 \prec \cdots$ of finite clopen partitions of $X$ whose union generates the topology of $X$, and which is related to the sequence $\{K_m\}$ in a way we now describe. In fact, if $\mathcal{P}_m = \{A_1, \ldots, A_{n_m}\}$, we require that $K_m = \bigcup_{i=1}^{n_m - 1} A_i$ and hence $A_{n_m} = X \setminus K_m$. (In particular, if $X$ is compact, $A_{n_m} = \emptyset$. Henceforth we will assume that $X$ is not compact—and so $A_{n_m} \neq \emptyset$ for every $m$—the compact case being of course a simplified version. We also assume that $K_m \subsetneqq K_{m+1}$ for every $m$, so that $A_{n_m} \setminus A_{n_{m+1}} \neq \emptyset$ for every $m$.) It is easily seen that $K_m$ and $X \setminus K_m$ are $R_m$-invariant sets, so $(R_m, \mathcal{T}_m)$ is the disjoint union of the restrictions of $R_m$ to $K_m$ and $X \setminus K_m$, respectively. (It should be understood that when we restrict we take the relative topology.) Now $R_m|_{X \setminus K_m}$ is equal to $\Delta|_{X \setminus K_m}$. By Lemma 3.4 we can decompose $K_m$ into a finite number of disjoint towers, with $R_m|_{K_m}$ being as described in Lemma 3.4 and in Remark 3.6. We will use the terminology suggested by Remark 3.6, including the terms 'tower', 'floor', 'height' and 'groupoid partition'. Keeping the notation above, as well as that used in Lemma 3.4 and Remark 3.6, we now describe how to construct the Bratteli diagram $(V, E)$. First, let $V_0 = \{v_0\}$ be a one-point set. We will do the next two steps in the construction, which should make it clear how we proceed.

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*Step 1.* Applying Lemma 3.4 to $(R_1, \mathcal{T}_1)$, we get a groupoid partition $\mathcal{O}_1'$ of $R_1$, so that $R_1|_{K_1}$ is represented (as explained in Remark 3.6) by $l_1$ (compact) towers of heights $h_1, \ldots, h_{l_1}$ and $R_1|_{X\setminus K_1} = \Delta|_{X\setminus K_1}$ is a single non-compact tower of height one, so that the associated clopen partition $\mathcal{O}_1$ of $X$ is finer than $\mathcal{P}_1$ and contains $A_{n_1} = X \setminus K_1$. We let $V_1 = \{v_1, \ldots, v_{l_1}, v_{l_1+1}\}$, where $v_i$ corresponds to the $i$th tower of height $h_i$ when $1 \leq i \leq l_1$, and $v_{l_1+1}$ corresponds to the non-compact tower $A_{n_1}$ of height one. The number of edges between $v_0$ and $v_i$ is the height $h_i$ of the tower $v_i$ for $1 \leq i \leq l_1$. There are no edges between $v_0$ and $v_{l_1+1}$ (so $v_{l_1+1}$ will be a source in $V$). This defines $E_1$.

*Step 2.* Let $\tilde{\mathcal{P}}_2 = \mathcal{P}_2 \vee \mathcal{O}_1$ be the join of $\mathcal{P}_2$ and $\mathcal{O}_1$ and so $\tilde{\mathcal{P}}_2$ is a finite clopen partition of $X$. Applying Corollary 3.5 to $(R_2, \mathcal{T}_2)$ we get a groupoid partition $\mathcal{O}_2'$ of $R_2$ that is finer than $\{\mathcal{O}_1', R_2 \setminus R_1\}$ and so that the associated clopen partition $\mathcal{O}_2$ of $X$ is finer than $\tilde{\mathcal{P}}_2$ and contains $A_{n_2} = X \setminus K_2$. Hence $R|_{K_2}$ will be represented by $l_2$ (compact) towers of heights $\tilde{h}_1, \ldots, \tilde{h}_{l_2}$ and $R|_{X\setminus K_2} = \Delta|_{X\setminus K_2}$ will be a single non-compact tower of height one. We let $V_2 = \{\tilde{v}_1, \ldots, \tilde{v}_{l_2}, \tilde{v}_{l_2+1}\}$, where $\tilde{v}_j$ corresponds to the $j$th tower of height $\tilde{h}_j$ when $1 \leq j \leq l_2$ and $\tilde{v}_{l_2+1}$ corresponds to the non-compact tower $A_{n_2}$ of height one. Let $\mathcal{O}_2'' = \{U \in \mathcal{O}_2' | U \subset R_1\}$. It is a simple observation that $\mathcal{O}_2''$ is a groupoid partition of $R_1$ that is finer than $\mathcal{O}_1'$ and with unit space equal to $\mathcal{O}_2$. The set $E_2$ of edges between $V_1$ and $V_2$ is labelled by $\mathcal{O}_2 \setminus \{A_{n_2}\}$ modulo $\mathcal{O}_2''$, i.e. $E_2$ consists of $\sim_{\mathcal{O}_2''}$ equivalence classes (denoted by $[\cdot]_{\mathcal{O}_2''}$) of $\mathcal{O}_2 \setminus \{A_{n_2}\}$. Specifically, if $A, B \in \mathcal{O}_2 \setminus \{A_{n_2}\}$, we have $A \sim_{\mathcal{O}_2''} B$ if there exists $U \in \mathcal{O}_2''$ so that $U^{-1}\cdot U = A$, $U \cdot U^{-1} = B$. As we explained in Remark 3.6, the vertex set $V_j$ ( $j = 1, 2$) may be identified with the $\sim_{\mathcal{O}_j'}$ equivalence classes $[\cdot]_{\mathcal{O}_j'}$ of $\mathcal{O}_j$. Doing this we may write down the source and range maps $i : E_2 \to V_1$, $f : E_2 \to V_2$, associated to the Bratteli diagram. In fact, if $[A]_{\mathcal{O}_2''} \in E_2$, where $A \in \mathcal{O}_2 \setminus \{A_{n_2}\}$, then $f([A]_{\mathcal{O}_2''}) = [A]_{\mathcal{O}_2'}$, $i([A]_{\mathcal{O}_2''}) = [B]_{\mathcal{O}_1'}$, where $B$ is the unique element of $\mathcal{O}_1$ such that $A \subset B$. The vertex $v_{l_2+1} \in V_2$ (corresponding to the tower $A_{n_2}$) will be a source in $V$.

Using the pictorial presentation of Figure 2, we can give a more intuitive explanation of our construction of the edge set $E_2$ between $V_1$ and $V_2$. In fact, for $1 \leq j \leq l_2$, let $x$ be any point in the tower $\tilde{v}_j \in V_2$. The $R_2$-equivalence class $[x]_{R_2}$ of $x$ consists of the $\tilde{h}_j$ vertically lying points of $\tilde{v}_j$ including $x$. Now $[x]_{R_2}$ is a disjoint union of distinct $R_1$-equivalence classes. If $[x]_{R_2}$ contains $h_{ij}$ distinct $R_1$-equivalence classes 'belonging' to the tower $v_i \in V_1$, we connect $v_i$ to $\tilde{v}_j$ by $h_{ij}$ edges. There are no edges between $\tilde{v}_{l_2+1}$ and any vertex in $V_1$.

Continuing in the same manner we construct the Bratteli diagram $(V, E)$. There is an obvious map $F$ from $X$ to the path space associated to $(V, E)$. In fact, if $x \in X$ let $v_i \in V_{n-1}$ be the tower at level $n-1$ that contains $x$. Likewise let $\tilde{v}_j \in V_n$ be the tower at level $n$ that contains $x$ and assume that $x$ lies in floor $A(\in \mathcal{O}_n)$ of $\tilde{v}_j$. The $n$th edge $e_n \in E_n$ of $F(x) = (e_1, e_2, \ldots)$ is then $[A]_{\mathcal{O}_n''}$. (We use similar notation at level $n$ as we used above at level two.) We verify that $F$ is an homeomorphism between $X$ and the path space associated to $(V, E)$. Furthermore, it is straightforward to show that the map $F$ establishes an isomorphism between $(R, \mathcal{T})$ and the $AF$-equivalence relation associated to $(V, E)$. In fact, $(x, y) \in R$ and $(x, y) \notin R_{n-1}$, $(x, y) \in R_n$ if and only if $F(x)$ and $F(y)$ become cofinal from level $n$ on. We omit the details.

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The two last assertions of the theorem are immediate consequences of our construction and the comments we made in Example 2.7(ii). □

*Remark 3.10.* Even though the Bratteli diagram model for $(R, \mathcal{T})$ is not unique, it is true that two such models give rise to isomorphic dimension groups and so the diagrams themselves are related by a telescoping procedure (cf. Example 2.7(ii) and Lemma 4.13).

We now consider the situation where $(V, E)$ is a standard Bratteli diagram and $(W, F)$ is a (standard) subdiagram, i.e. $W \subset V$, $F \subset E$ and $W_0 = V_0 = \{v_0\}$. It is easy to see that $X_{(W,F)}$ is a closed subset of $X_{(V,E)}$. It is also clear that $AF(W, F)$ is the intersection of $AF(V, E)$ with $X_{(W,F)} \times X_{(W,F)}$. Moreover, it is easy to check that the relative topology on $AF(W, F)$ coming from $AF(V, E)$ agrees with the usual topology on $AF(W, F)$. We have the following realization theorem for this situation.

THEOREM 3.11. *Let $(R, \mathcal{T})$ be an AF-equivalence relation on the compact (zero-dimensional) space $X$. Suppose that $Z$ is a closed subset of $X$ such that $R|_Z (= R \cap (Z \times Z))$, with the relative topology from $(R, \mathcal{T})$, is an étale equivalence relation on $Z$. (We say that $Z$ is $R$-étale.) Then there exists a Bratteli diagram $(V, E)$, a subdiagram $(W, F)$ and a homeomorphism $h : X_{(V,E)} \to X$ such that:*

(i) *$h$ implements an isomorphism between $AF(V, E)$ and $(R, \mathcal{T})$;*

(ii) *$h(X_{(W,F)}) = Z$ and the restriction of $h$ to $X_{(W,F)}$ implements on isomorphism between $AF(W, F)$ and $R|_Z$. In particular, $R|_Z$ is an AF-equivalence relation on $Z$.*

*Proof.* Let us take $(R', \mathcal{T}')$, a compact open subequivalence relation of $(R, \mathcal{T})$, where $\mathcal{T}'$ is the relative topology. As noted in Proposition 3.2, the topology $\mathcal{T}'$ on $R'$ coincides with the relative topology from $X \times X$. We note that $R' \cap (Z \times Z)$ is a clopen, compact subequivalence relation of $R \cap (Z \times Z)$ and therefore, in particular, is étale. For each pair $(x, y)$ in $R'$, we choose a clopen neighbourhood $U$ in $\mathcal{T}'$ (hence in $\mathcal{T}$) as follows. First, if $(x, y)$ is not in $Z \times Z$, we use the fact that $Z$, and hence $Z \times Z$, is closed to find a clopen neighbourhood $U$ which is disjoint from $Z \times Z$. If $(x, y)$ is in $Z \times Z$, then we use the fact that $R' \cap (Z \times Z)$ is étale to find a clopen subset $U \subset R'$ such that $r : U \cap (Z \times Z) \to r(U) \cap Z$ and $s : U \cap (Z \times Z) \to s(U) \cap Z$ are both homeomorphisms. To achieve this, let $V \in \mathcal{T}'$ be a clopen neighbourhood of $(x, y)$ such that $r : V \cap (Z \times Z) \to A \cap Z$ and $s : V \cap (Z \times Z) \to B \cap Z$ are homeomorphisms, where $A$ and $B$ are clopen subsets of $X$. Choose $U$ to be $V \cap r^{-1}(A) \cap s^{-1}(B)$. Moreover, we can choose $U$ sufficiently small so that $r, s$ are also local homeomorphisms from $U$ to $r(U)$ and $s(U)$, respectively. In this way, we cover $R'$ with clopen sets. We then extract a finite subcover and then choose a groupoid partition of $R'$ which is finer than this subcover, such that each member of the partition satisfies the properties just listed. (Compare the proof of Lemma 3.4 and Remark 3.6.) In the end, we obtain a groupoid partition $\mathcal{O}'$ of $R'$ such that

$$\mathcal{O}' \mid Z = \{U \cap (Z \times Z) \mid U \in \mathcal{O}', U \cap (Z \times Z) \neq \emptyset\}$$

is a groupoid partition of $R' \cap (Z \times Z)$. Moreover, it is clear from the construction that these may both be made finer than any pair of pre-assigned clopen partitions of $R'$ and $R' \cap (Z \times Z)$. We also note that it is easy to verify from the construction, any element

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$U$ of $\mathcal{O}'$ meets $Z \times Z$ if and only if $r(U)$ and $s(U)$ both meet $Z$. (Referring to Figure 2 as representing $\mathcal{O}'$, this means that if $Z$ intersects any two floors in the same tower, these intersections project onto each other.) Now we consider a sequence

$$R_1 \subset R_2 \subset R_3 \subset \cdots$$

of CEERs in $R$, with union $R$, each open in the next, so that $(R, \mathcal{T})$ is the inductive limit. For each $n$, we construct a groupoid partition $\mathcal{O}'_n$ of $R_n$ with the properties as above. We do this so that the restriction of $\mathcal{O}'_{n+1}$ to $R_n$ is finer than $\mathcal{O}'_n$ and so that the restriction of $\mathcal{O}'_{n+1} \mid Z$ to $R_n \cap (Z \times Z)$ is finer than $\mathcal{O}'_n \mid Z$. This sequence is also chosen so as to generate the respective topologies of $R$ and $R \cap (Z \times Z)$. In the proof of Theorem 3.9, it was shown how $\mathcal{O}'_n$ defines a Bratteli diagram $(V, E)$. We now describe the subdiagram $(W, F)$. Let $\Delta_X$ and $\Delta_Z$ denote the diagonals in $X \times X$ and $Z \times Z$, respectively (which we identify with $X$ and $Z$, respectively). The vertices of $V_n$ correspond to towers in $\mathcal{O}'_n$; that is, equivalence classes of sets in $\Delta_X \cap \mathcal{O}'_n$, modulo $\mathcal{O}'_n$. The vertices of $W_n$ are those classes having a representative which meets $\Delta_Z$. (Again referring to Figure 2, the vertices of $V_n$ correspond to the towers, while the vertices of $W_n$ correspond to those towers that meet $Z$.) The edges of $E_n$ are the equivalence classes of sets in $\Delta_X \cap \mathcal{O}'_n$, modulo $\mathcal{O}''_n = \{U \in \mathcal{O}'_n \mid U \subset R_{n-1}\}$. (For an interpretation of this in terms of Figure 2, we refer the reader to the remarks made in the proof of Theorem 3.9.) We let $F_n$ be those equivalence classes having a representative which meets $\Delta_Z$. It follows from the properties of the partitions $\mathcal{O}'_n$ described above that $(V, E)$ and $(W, F)$ satisfy the desired conclusion. We leave the details to the reader. □

The inductive limit $(R, \mathcal{T})$ of a sequence of étale equivalence relations $\{(R_n, \mathcal{T}_n)\}_{n=1}^{\infty}$ on $X$ ($(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$) is defined as in Definition 3.7, with the obvious modifications. It is an easy exercise to show that $(R, \mathcal{T})$ is an étale equivalence relation. The following proposition is a stabilization result with respect to $AF$-equivalence relations.

PROPOSITION 3.12.

(i) *Let $(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$ be an inductive limit of a sequence $\{(R_n, \mathcal{T}_n)\}$ of AF-equivalence relations on $X$. Then $(R, \mathcal{T})$ is an AF-equivalence relation on $X$.*

(ii) *Let $(R, \mathcal{T})$ be an AF-equivalence relation on $X$ and let $R' \subset R$ be a subequivalence relation which is open, i.e. $R' \in \mathcal{T}$. Then $(R', \mathcal{T}')$ is an AF-equivalence relation, where $\mathcal{T}'$ is the relative topology of $R$.*

*Proof.* (i) For each $n$, let $(R_n, \mathcal{T}_n) = \varinjlim(R_{n,k}, \mathcal{T}_{n,k})$, where $(R_{n,k}, \mathcal{T}_{n,k})$ is a CEER (Definition 3.1) on $X$ for every $k$. So we have

$$R_1 \subset R_2 \subset R_3 \subset \cdots \subset R = \bigcup_{n=1}^{\infty} R_n,$$

$$R_{n,1} \subset R_{n,2} \subset R_{n,3} \subset \cdots \subset R_n = \bigcup_{k=1}^{\infty} R_{n,k}, \quad n = 1, 2, \ldots,$$

where each set is open in the next one containing it with respect to the relevant topology. Define $R'_1 = R_{1,1}$. Now $R_{1,2} \subset R_1 \subset R_2 = \bigcup_{k=1}^{\infty} R_{2,k}$ and so we may choose $k_2 \geq 2$ so large that $R_{1,2} \subset R_{2,k_2}$. Define $R'_2 = R_{2,k_2}$. Continuing in this manner we get an ascending

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sequence $\{R'_n\}_{n=1}^{\infty}$ of equivalence relations on $X$ so that $R'_n$ contains all $R_{l,m}$, provided $l$ and $m$ are at most $n$. Clearly $R = \bigcup_{n=1}^{\infty} R'_n$ and we claim that $(R, \mathcal{T}) = \varinjlim(R'_n, \mathcal{T}'_n)$, where $(R'_n, \mathcal{T}'_n) = (R_{n,k_n}, \mathcal{T}_{n,k_n})$ is a CEER for each $n$. This will finish the proof of (i). (Note that $R'_n$ is open in $R'_{n+1}$, i.e. $R'_n \in \mathcal{T}'_{n+1}$ for every $n$.) Now, if $U \in \mathcal{T}$ then $U \cap R_n \in \mathcal{T}_n$ for every $n$. Hence $U \cap R'_n = U \cap R_{n,k_n} = (U \cap R_n) \cap R_{n,k_n} \in \mathcal{T}_{n,k_n} = \mathcal{T}'_n$ for every $n$ and so $U \in \tilde{\mathcal{T}}$, where by definition $(R, \tilde{\mathcal{T}})$ equals $\varinjlim(R'_n, \mathcal{T}'_n)$. Hence $\mathcal{T} \subset \tilde{\mathcal{T}}$. Conversely, assume $U \in \tilde{\mathcal{T}}$. Then

$$U \cap R'_n = U \cap R_{n,k_n} \in \mathcal{T}'_n = \mathcal{T}_{n,k_n} = \mathcal{T}_n \cap R_{n,k_n} \subset \mathcal{T}_n$$

for every $n$. To show that $U \in \mathcal{T}$ we must show that $U \cap R_m \in \mathcal{T}_m$ for any given $m$. This is again equivalent to showing that

$$U \cap R_{m,l} = (U \cap R_m) \cap R_{m,l} \in \mathcal{T}_{m,l}$$

for any $l$. Now choose $n \geq \max\{m, l\}$. Then $R_{m,l} \subset R'_n$ and, since clearly $R_{m,l} \subset R_m$, we get $U \cap R_{m,l} = ((U \cap R'_n) \cap R_m) \cap R_{m,l} \in \mathcal{T}_{m,l}$ from the fact that $U \cap R'_n \in \mathcal{T}_n$ and $R_{m,l}$ is open in $R_m$, which again is open in $R_n$. This proves that $U \in \mathcal{T}$ and so $\tilde{\mathcal{T}} \subset \mathcal{T}$, which finishes the proof of (i).

(ii) $R'$ being an open subequivalence relation of $R$ implies that $(R', \mathcal{T}')$ is an étale equivalence relation, where $\mathcal{T}'$ is the relative topology of $R$—a fact that is easily shown. Let $(R, \mathcal{T}) = \varinjlim(R_n, \mathcal{T}_n)$, where $(R_n, \mathcal{T}_n)$ is a CEER on $X$ for every $n$. It is easily verified that $(R', \mathcal{T}') = \varinjlim(R'_n, \mathcal{T}'_n)$, where $R'_n = R' \cap R_n$ is an open subequivalence relation of $R_n$ and $\mathcal{T}'_n$ is the relative topology of $R_n$. By (i) it is sufficient to show that $(R'_n, \mathcal{T}'_n)$ is an $AF$-equivalence relation for every $n$. So the proof boils down to showing that an open subequivalence relation of a CEER is an $AF$-equivalence relation. This again may be reduced further by Lemma 3.4 to showing that an open subequivalence relation of a CEER of type $m$, with $m \geq 2$, on a compact space is an $AF$-equivalence relation. (Indeed, it is straightforward to show that a finite disjoint union of $AF$-equivalence relations is again an $AF$-equivalence relation.) So we may assume that $(R, \mathcal{T})$ is equal to the product of the cotrivial and trivial equivalence relations on $X = Y \times \{1, \ldots, m\}$, where $Y$ is compact and $R' \subset R$ is an open subequivalence relation. The proof will be completed by showing that $(R', \mathcal{T}')$ is an $AF$-equivalence relation, where $\mathcal{T}'$ is the relative topology of $R$. By Proposition 3.2(i) we know that both $\mathcal{T}$ and $\mathcal{T}'$ are the relative topologies from $X \times X$. For $y \in Y$, let

$$R'(y) = \{(i, j) \mid ((y, i), (y, j)) \in R'\}$$

and observe that $R'(y)$ is an equivalence relation on $\{1, \ldots, m\}$. For $(i, j) \in R'(y)$, there exists a clopen neighbourhood $U_{i,j}$ of $y$ such that

$$\{((y', i), (y', j)) \mid y' \in U_{i,j}\} \subset R',$$

since $R'$ is open in $R$. Let $U_y = \bigcup_{(i,j) \in R'(y)} U_{i,j}$. Then $U_y$ is a clopen neighbourhood of $y$ such that $R'(y) \subset R'(y')$ for all $y' \in U_y$. Fix $\epsilon > 0$. For each $y \in Y$ select $U_y$ as above so that $U_y \subset B(y, \epsilon)$, where $B(y, \epsilon)$ is the open ball around $y$ of radius $\epsilon$. Select a finite subcover $U_{y_1}, \ldots, U_{y_k}$ of the clopen cover $\{U_y \mid y \in Y\}$ of $Y$.

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For $y \in Y$, let $R''(y) = \bigcap_{\{i \mid y \in U_i\}} R'(y_i)$. Let $\mathcal{P}$ be the clopen partition of $Y$ generated by $\{U_{y_1}, \ldots, U_{y_{k_n}}\}$. Then $R''|_E$ is constant for every $E \in \mathcal{P}$. In an obvious way $R''$ defines an equivalence relation on $X = Y \times \{1, \ldots, m\}$. Furthermore, we have shown that $R''$ is a subequivalence relation of $R'$, which is a CEER in the relative topology. Clearly $R''$ is an open subset of $R'$. Now we let $R'_1$ be some $R''$ corresponding to $\epsilon = 1$. Assume $R'_n$ has been defined to be some $R''$ corresponding to $\epsilon = 1/n$ and let $\{U_{y_1}, \ldots, U_{y_{k_n}}\}$ be the associated clopen cover of $Y$ as explained above. For $\epsilon = 1/(n+1)$, choose a finite clopen cover $\{U_{z_1}, \ldots, U_{z_{k_{n+1}}}\}$ of $Y$ as before so that every $U_{z_i}$ is contained in some $U_{y_j}$ and define $R'_{n+1}$ to be the associated $R''$. We observe that $R'_n \subset R'_{n+1}$, and that $R'_n$ is open in $R'_{n+1}$, where each equivalence relation is given the relative topology of $X \times X$. Now $\bigcup_{n=1}^{\infty} R'_n \subset R'$, since each $R'_n$ is contained in $R'$. On the other hand, let $y \in Y$ and let $U_y$ be, as before, a clopen neighbourhood of $y$ such that $y' \in U_y \Rightarrow R(y) \subset R(y')$. Choose $n$ so large that $B(y, 1/n) \subset U_y$. With $\epsilon = 1/n$ let $\{U_{y_1}, \ldots, U_{y_{k_n}}\}$ be the clopen cover of $Y$ associated to $R'_n$. In particular, $U_{y_i} \subset B(y_i, 1/n)$ for each $i$. If $y \in U_{y_i}$, then $y \in B(y_i, 1/n)$ and so $y_i \in B(y, 1/n) \subset U_y$. Hence $R'(y) \subset R'(y_i)$ and so we get $R'(y) \subset R'_n(y)$. Consequently, $R' = \bigcup_{n=1}^{\infty} R'_n$ and we easily verify that $(R', \mathcal{T}') = \varinjlim(R'_n, \mathcal{T}'_n)$, where $\mathcal{T}'_n$ is the relative topology of $X \times X$. This finishes the proof of (ii). □

## 4. *Affable equivalence relations*

*Definition 4.1.* (Affable equivalence relation) Let $R$ be a countable equivalence relation on the zero-dimensional (Hausdorff locally compact, second countable) space $X$. We say that $R$ is *affable* if $R$ may be given a topology $\mathcal{T}$ so that $(R, \mathcal{T})$ is an $AF$-equivalence relation.

*Remark 4.2.* To say that the countable equivalence relation $R$ on $X$ is affable is the same as saying that $R$ is orbit equivalent (cf. Definition 2.4) to some $(R', \mathcal{T}')$, where $(R', \mathcal{T}')$ is an $AF$-equivalence relation on $X'$; that is, there exists a homeomorphism $F : X \to X'$ such that $(x, y) \in R \iff \big(F(x), F(y)\big) \in R'$. In fact, using $F$ to pull back the topology $\mathcal{T}'$ on $R'$ to get the topology $\mathcal{T}$ on $R$, we get that $(R, \mathcal{T})$ is an $AF$-equivalence relation on $X$.

In Theorem 2.3 of [**6**], it is proved that for simple Bratteli diagrams $(V, E)$ and $(V', E')$, $AF(V, E)$ is orbit equivalent to $AF(V', E')$ if and only if

$$K_0(V, E)/\operatorname{Inf} K_0(V, E) \cong K_0(V', E')/\operatorname{Inf} K_0(V', E')$$

by a map preserving the canonical order units. On the other hand, Lemma 4.13 below states that $AF(V, E) \cong AF(V', E')$ if and only if $K_0(V, E) \cong K_0(V', E')$ by a map preserving the canonical order units. This implies that an affable equivalence relation usually carries many distinct structures of $AF$-equivalence relations.

The next result is an immediate consequence of Corollary 1.3. Because of its importance we state it is as a theorem.

THEOREM 4.3. *Let $(X, T)$ be a Cantor minimal system and let $(R, \mathcal{T})$ be the associated étale equivalence relation on $X$ (cf. Example 2.7(i)). Let $x$ be an arbitrary point of $X$. The subequivalence relation $R_{\{x\}}$ of $R$ whose equivalence classes are the full $T$-orbits,*

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*except that the $T$-orbit of $x$ split into two at $x$—the forward orbit $\{T^n x \mid n \geq 1\}$ and the backward orbit $\{T^n x \mid n \leq 0\}$—is open in $R$. Furthermore, $(R_{\{x\}}, \mathcal{T}_{\{x\}})$ is an AF-equivalence relation on $X$, where $\mathcal{T}_{\{x\}}$ is the relative topology. In particular, $R_{\{x\}}$ is affable.*

*Proof.* By Corollary 1.3 we may assume that $(X, T)$ is the Bratteli–Vershik system associated to the properly ordered Bratteli diagram $(V, E, \geq)$ and where $x$, respectively $Tx$, is the unique max path, respectively the unique min path. The set of paths that are cofinal with the unique max path equals the backward orbit of $x$, and the set of paths that are cofinal with the unique min path equals the forward orbit of $x$. As for the other $T$-orbits, they agree with the cofinal equivalence relation of $(V, E)$. Thus $R_{\{x\}}$ coincides with the cofinal equivalence relation associated to $(V, E)$. Let $(y, T^k y) \in R_{\{x\}}$, where $y = (e_1, e_2, \dots) \in X$ and $k$ is an integer. So $y$ and $T^k y$ are paths that agree from a certain level on, say $N$. Let $U$ be the open (and closed) neighbourhood of $y$ defined by

$$U = U_{(e_1, e_2, \dots, e_N)} = \{(f_1, f_2, \dots) \in X_B \mid (f_1, \dots, f_N) = (e_1, \dots, e_N)\}.$$

Then $W = \{(z, T^k z) \mid z \in U\}$ is an open neighbourhood of $(y, T^k y)$ in $(R, \mathcal{T})$. Furthermore, $W \subset R_{\{x\}}$ since $z$ and $T^k z$ agree from level $N$ on for every $z \in U$. Hence $R_{\{x\}}$ is open in $R$. The argument we have just given also shows that $\mathcal{T}_{\{x\}}$ coincides with the topology associated to $(V, E)$ as described in Example 2.7(ii) and so $(R_{\{x\}}, \mathcal{T}_{\{x\}}) = AF(V, E)$. Hence $(R_{\{x\}}, \mathcal{T}_{\{x\}})$ is an $AF$-equivalence relation on $X$. □

*Definition 4.4.* Let $(X, T)$ be a Cantor minimal system and let $Y$ be a non-empty closed subset of $X$. We say that $Y$ is *regular* (with respect to $(X, T)$) if the positive and negative return time maps $\lambda^+$ and $\lambda^-$ for $T$ on $Y$ are continuous, where $\lambda^+, \lambda^- : Y \to \mathbb{N} \cup \{+\infty\}$ are given by

$$\lambda^+(y) = \inf\{k \geq 1, +\infty \mid T^k y \in Y\},$$
$$\lambda^-(y) = \inf\{k \geq 1, +\infty \mid T^{-k} y \in Y\},$$

and $\mathbb{N} \cup \{+\infty\}$ is given the 'one-point compactification topology'.

*Remark 4.5.* Let $(X, T)$ be a Cantor minimal system. If $Y$ is a closed subset of $X$ that meets each $T$-orbit at most once, then $Y$ is regular. In fact, in this case $\lambda^+(y) = \lambda^-(y) = +\infty$ for each $y \in Y$.

The following theorem, besides being a considerable generalization of Theorem 4.3, turns out to be a useful tool in the study of affability of equivalence relations associated to certain group actions on the Cantor set. (See also the remarks prior to Theorem 4.8.)

THEOREM 4.6. *Let $(X, T)$ be a Cantor minimal system and let $(R, \mathcal{T})$ be the associated étale equivalence relation on $X$. Let $Y \subset X$ be a regular set that contains a point $y \in Y$ so that $\lambda^+(y) = \lambda^-(y) = +\infty$, i.e. $Y$ meets the $T$-orbit of $y$ at $y$ only. Let $R_Y$ be the subequivalence relation of $R$ defined by*

$$R_Y = \{(x, T^k x), (T^k x, x) \mid x \in X, k \geq 0, \#(\{0 \leq i < k \mid T^i x \in Y\}) \text{ is an even number}\}.$$

*In particular, if $Y$ meets each $T$-orbit at most once, $R_Y$ is obtained from $R$ by splitting the $T$-orbits meeting $Y$ into the forward and backward orbits at $Y$.*

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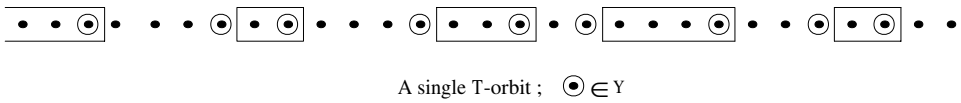



FIGURE 3. The equivalence classes of $R_Y$.

*Then $(R_Y, \mathcal{T}_Y)$ is an AF-equivalence relation on $X$, where $\mathcal{T}_Y$ is the relative topology from $R$. In particular, $R_Y$ is affable.*

*Proof.* Clearly $R_Y$ is a subequivalence relation of $R_{\{y\}}$. By Theorem 4.3 we have that $R_{\{y\}}$ is an open subrelation of $R$ and $(R_{\{y\}}, \mathcal{T}_{\{y\}})$ is an $AF$-equivalence relation, $\mathcal{T}_{\{y\}}$ being the relative topology from $R$. So by Proposition 3.12(ii) the proof will be completed if we can show that $R_Y$ is an open subset of $R$, i.e. $R_Y \in \mathcal{T}$. Let $x \in X$ and $k \geq 1$. It is obviously sufficient to show that $(x, T^k x) \in R_Y$ has an open neighbourhood $U \in \mathcal{T}$ so that $U \subset R_Y$. Let $T^{i_1}x, \ldots, T^{i_l}x$, where $0 \leq i_1 < \cdots < i_l \leq k-1$, be the points on the $T$-orbit of $x$ lying between $x$ and $T^{k-1}x$ that meet $Y$. By assumption $l$ is an even number. Obviously

$$\begin{aligned}
&\lambda^+(T^{i_1}x) = i_2 - i_1, \quad \lambda^-(T^{i_1}x) > i_1,\\
&\lambda^+(T^{i_j}x) = i_{j+1} - i_j, \quad \lambda^-(T^{i_j}x) = i_j - i_{j-1} \quad \text{if } 1 < j < l,\\
&\lambda^+(T^{i_l}x) > k - 1 - i_l, \quad \lambda^-(T^{i_l}x) = i_l - i_{l-1}.
\end{aligned}$$

By the continuity of $\lambda^+$ and $\lambda^-$ on $Y$, we may find an open neighbourhood $V$ of $x$ so that if $x' \in V$, the number $l'$ of points in $Y$ lying on the $T$-orbit of $x'$ between $x'$ and $T^{k-1}x'$ is even. In fact, we choose $V$ so small that for $1 \leq i \leq k-1$, $T^i V$ does not meet $Y$ if $i \notin \{i_1, \ldots, i_l\}$, and if $T^{i_j}x' \in Y, 1 \leq j \leq l$, then $\lambda^+(T^{i_j}x') = \lambda^+(T^{i_j}x)$ and $\lambda^-(T^{i_j}x') = \lambda^-(T^{i_j}x)$ if these values are finite (with obvious modification if some of these values are $+\infty$). By a simple argument it follows that if $T^{i_j}x' \in Y$ for some $i_j \in \{i_1, \ldots, i_l\}$, then $l' = l$. Thus either $l' = l$ or $l' = 0$. In either case $l'$ is even. This shows that $U = \{(x', T^k x') \mid x' \in V\}$ is contained in $R_Y$. By the definition of the topology $\mathcal{T}$ we have $U \in \mathcal{T}$. □

*Remark 4.7.* We illustrate the equivalence classes of $R_Y$ by a figure. Let us draw a $T$-orbit (ordered from left to right) as dots, and circle those dots that are in $Y$ (see Figure 3). The $T$-orbit splits into two $R_Y$-equivalence classes (assuming the orbit in question meets $Y$). One class is everything inside the boxes, while the other class is everything else. Note that if $Y$ meets the $T$-orbit at exactly one point, say $y$, we get the splitting at $y$ into the forward and backward $T$-orbits.

*Henceforth all Bratteli diagrams $(V, E)$ that we shall consider will be standard.* This entails in particular that the associated path space $X_{(V,E)}$ is compact.

One of our aims is to prove that the étale equivalence relation $(R, \mathcal{T})$ associated to a Cantor minimal system $(X, T)$ is affable. However, more than that, we want to prove that $(X, T)$—or $(R, \mathcal{T})$—is orbit equivalent (cf. Remark 4.2) to the subequivalence relation $R_Y$ of $R$, obtained by splitting the $T$-orbit into the forward and backward orbits at $Y$, where $Y$ is any (non-empty) closed subset $Y$ of $X$ that meets each $T$-orbit at most once. By Theorem 4.6 we know that $R_Y$ with the relative topology from $(R, \mathcal{T})$ is an

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$AF$-equivalence relation. We obtain our result without involving the full power of the main theorem (Theorem 2.2 in [**6**]) and hence avoid using homological algebra as well as $C^*$-algebra ingredients in our proof. More importantly, the key lemma (Lemma 4.15) that will give us our desired result as a corollary, is of independent interest. It turns out to be a powerful tool in handling more general group actions, something that will be treated in a forthcoming paper.

First, however, we want to prove a converse of the above—a result which is more easily accessible. This converse result is actually an immediate corollary of Lemma 6.1 in [**6**], but we give here a simplified and more direct proof avoiding any mentioning of $K$-theory and $C^*$-algebras.

THEOREM 4.8. *Let $(R, \mathcal{T})$ be a minimal $AF$-equivalence relation on the Cantor set $X$. Then $(R, \mathcal{T})$ is orbit equivalent to a Cantor minimal system $(Y, S)$, i.e. there exists a homeomorphism $F : X \to Y$ so that $F([x]_R) = \mathrm{orbit}_S(F(x))$ for every $x \in X$.*

*Proof.* By Theorem 3.9 we may assume at the outset that $(R, \mathcal{T})$ is the $AF$-equivalence relation $AF(V, E)$ associated to a simple Bratteli diagram $(V, E)$, where $X$ is the path space $X_{(V,E)}$ associated to $(V, E)$. Choose a proper ordering on $(V, E)$ and denote the associated Bratteli–Vershik system by $(X, T)$. Let $x_{\max}$, respectively $x_{\min}$, be the unique maximal, respectively minimal, path in $X$. Choose a point $x_0 \in X$ which is not cofinal with $x_{\max}$ or $x_{\min}$—in other words, $x_0 \notin \mathrm{orbit}_T(x_{\max})$. Let $Z = \{x_{\max}, x_0\}$. By Theorem 1.2 there exists an ordered Bratteli diagram $B_Z = (\tilde{V}, \tilde{E}, \geq)$, where $(\tilde{V}, \tilde{E})$ is simple, such that $(X, T)$ is conjugate to the Bratteli–Vershik system, denoted $(\tilde{X}, \tilde{T})$, associated to $B_Z$. Here $\tilde{X}$ is the path space $X_{(\tilde{V},\tilde{E})}$ associated to $(\tilde{V}, \tilde{E})$. Let $F : X \to \tilde{X}$ be the conjugating map and let $(\tilde{R}, \tilde{\mathcal{T}}) = AF(\tilde{V}, \tilde{E})$ be the $AF$-equivalence relation associated to $(\tilde{V}, \tilde{E})$. By Theorem 1.2 we get that $F$ implements an orbit equivalence between $AF(V, E)$ and the equivalence relation $\tilde{R} \subset \tilde{X} \times \tilde{X}$ generated by $AF(\tilde{V}, \tilde{E})(\subset \tilde{X} \times \tilde{X})$ and $(F(x_0), F(Tx_0)) \in \tilde{X} \times \tilde{X}$. Since $F(x_0)$, respectively $F(Tx_0)$, is a max path, respectively a min path, in $\tilde{X}$ with respect to the ordering $\geq$ on $(\tilde{V}, \tilde{E})$, we may give $(\tilde{V}, \tilde{E})$ a new ordering $\geq'$, which is proper, such that $F(x_0)$, respectively $F(Tx_0)$, is the unique max path, respectively unique min path. Let $(Y, S)$ denote the associated Bratteli–Vershik system, where $Y = \tilde{X}$. Then the map $F : X \to Y$ above implements an orbit equivalence between $(R, \mathcal{T}) = AF(V, E)$ and $(Y, S)$. □

Before we prove Lemma 4.15, alluded to above, we need some definitions and preliminary results.

*Definition 4.9.* Let $(R, \mathcal{T})$ be an étale relation on the space $X$. Let $\mu$ be a probability measure on $X$. Define the two $\sigma$-finite (regular) measures $\nu_r$ and $\nu_s$ on $(R, \mathcal{T})$ by

$$\nu_r(C) = \int_X \#(r^{-1}(x) \cap C)\, d\mu(x),\ \nu_s(C) = \int_X \#(s^{-1}(x) \cap C)\, d\mu(x)$$

where $C$ is a Borel subset of $R$, and $\#(S)$ denotes the cardinality of a set $S$. We say that $\mu$ is *$R$-invariant* if $\nu_r = \nu_s$.

By [**4**, §2] this is equivalent to saying that $\mu$ is $G$-invariant, where $G$ is any countable group such that $R = R_G$.

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*Remark 4.10.* Let $(R, \mathcal{T})$ be a minimal $AF$-equivalence relation on the Cantor set $X$. By Theorem 3.9 we may assume that $(R, \mathcal{T}) = AF(V, E)$, where $(V, E)$ is a simple Bratteli diagram with $X = X_{(V,E)}$. Then $\mu$ is an $R$-invariant probability measure iff $\mu$ is $T$-invariant, where $(X, T)$ is the Bratteli–Vershik system associated to any properly ordered diagram $(V, E, \geq)$, cf. [**7**, Theorem 5.5].

*Definition 4.11.* Let $(R, \mathcal{T})$ be an étale equivalence relation on the space $X$. We say that a (non-empty) closed subset $Z$ of $X$ is *thin* (with respect to $(R, \mathcal{T})$) if $\mu(Z) = 0$, for all $R$-invariant probability measures $\mu$ on $X$.

Let $(V, E)$ be a Bratteli diagram and $(W, F)$ be a subdiagram. If $X_{(W,F)}$ is a thin subset of $X_{(V,E)}$ (with respect to $AF(V, E)$), then we say that $(W, F)$ is a *thin* subdiagram of $(W, F)$.

LEMMA 4.12. *Let $(W, F)$ be a thin subdiagram of $(V, E)$, where $(V, E)$ is a simple Bratteli diagram and let $V_0 = W_0 = \{v_0\}$. Given $m \geq 1, K \geq 1$, there exists $n \geq m$ so that for $w \in W_m$, $w' \in W_n$,*

$$\begin{aligned}\#(\{\text{paths in } (W, F) \text{ from } w \text{ to } w'\}) &\leq K \cdot \#(\{\text{paths from } v_0 \text{ to } w' \text{ in } (W, F)\}) \\ &\leq \#(\{\text{paths in } (V, E) \text{ from } w \text{ to } w'\}).\end{aligned}$$

*Proof.* For $n \geq m$, let $U_n = \bigcup\{U_{(f_1,\ldots,f_n)} \mid (f_1, \ldots, f_n) \in P_n(W, F)\}$, where $P_n(W, F)$ denotes the paths in $(W, F)$ from the top vertex $v_0$ to a vertex $w \in W_n$. Here $U_{(f_1,\ldots,f_n)}$ is the clopen cylinder set in $X_{(V,E)}$ defined by $U_{(f_1,\ldots,f_n)} = \{(e_1, e_2, \ldots) \in X_{(V,E)} \mid (e_1, e_2, \ldots, e_n) = (f_1, f_2, \ldots, f_n)\}$. Clearly, $U_1 \supseteq U_2 \supseteq \cdots$ and $\bigcap_{n=1}^{\infty} U_n = X_{(W,F)}$. Let

$$0 < \delta < \frac{1}{K} \cdot \inf\{\mu(U_{(f_1,\ldots,f_m)}) \mid (f_1, \ldots, f_m) \in P_m(W, F); \mu \in M(V, E)\} \qquad (*)$$

where $M(V, E)$ denotes the $(V, E)$-invariant probability measures. Note that $\mu(A) > 0$ for any non-empty clopen set $A \subset X$ and any $\mu \in M(V, E)$, since $(V, E)$ is simple. In fact, $M(V, E)$ may be identified with the set of states on the simple dimension group $K_0(V, E)$ and $A$ (being a finite union of cylinder sets) may naturally be identified with a non-zero element $[A]$ in $K_0(V, E)^+$. Let $M_n = \{\mu \in M(V, E) \mid \mu(U_n) \geq \delta\}$. Then $M_n$ is $w^*$-compact, since the characteristic function $\chi_{U_n}$ is continuous. Since obviously $M_m \supseteq M_{m+1} \supseteq \cdots$, we get by the thinness of $X_{(W,F)}$ that $\bigcap_{n=m}^{\infty} M_n = \emptyset$. By compactness, there exists an $n_1 \geq m$ so that $M_{n_1} = \emptyset$ and so $\mu(U_{n_1}) < \delta$ for all $\mu \in M(V, E)$. By $(*)$, we then get $\mu(U_{(f_1,\ldots,f_m)}) > K\mu(U_{n_1})$ for all $\mu \in M(V, E)$ and all $(f_1, \ldots, f_m) \in P_m(W, F)$. Hence $[U_{(f_1,\ldots,f_m)}] > K[U_{n_1}]$ in $K_0(V, E)$ for all $(f_1, \ldots, f_m) \in P_m(W, F)$, cf. [**3**, Ch. 4]. This means that there exists $n \geq n_1$ so that $\chi_{U_{(f_1,\ldots,f_m)}}(v) > K\chi_{U_{n_1}}(v)$ for all $v \in V_n$ and all $(f_1, \ldots, f_m) \in P_m(W, F)$, where we have made the obvious identification of the characteristic function of a clopen set (being a finite union of cylinder sets) with a group element in $K_0(V, E)$. Now let $w \in W_m, w' \in W_n$

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and choose $(f_1, \ldots, f_m) \in P_m(W, F)$ so that $f(f_m) = w$. Then

$$\begin{aligned}\#(\{\text{paths from } w \text{ to } w' \text{ in } (V, E)\}) &= \chi_{U_{(f_1,\ldots,f_m)}}(w') > K \cdot \chi_{U_{n_1}}(w') \\ &= K \cdot \#(\{\text{paths } (e_1, \ldots, e_n) \text{ from } v_0 \text{ to } w' \text{ in } (V, E) \text{ such that } (e_1, \ldots, e_{n_1}) \\ &\quad \in P_{n_1}(W, F)\}) \\ &\geq K \cdot \#(\{\text{paths from } v_0 \text{ to } w' \text{ in } (W, F)\}) \\ &\geq K \cdot \#(\{\text{paths from } w \text{ to } w' \text{ in } (W, F)\}).\end{aligned}$$

This completes the proof. □

LEMMA 4.13. *Let $(V, E)$ and $(V', E')$ be two Bratteli diagrams. The following are equivalent:*

(i) *$AF(V, E) \cong AF(V', E')$;*

(ii) *$K_0(V, E) \cong K_0(V', E')$, i.e. $K_0(V, E)$ is order-isomorphic to $K_0(V', E')$ by a map preserving the canonical order units;*

(iii) *there exists a so-called 'aggregate' Bratteli diagram $(\tilde{V}, \tilde{E})$, so that telescoping $(\tilde{V}, \tilde{E})$ to odd levels $0 < 1 < 3 < 5 < \cdots$ yields a telescope of $(V, E)$, while telescoping $(\tilde{V}, \tilde{E})$ to even levels $0 < 2 < 4 < 6 < \cdots$ yields a telescope of $(V', E')$;*

(iv) *$(V, E) \sim (V', E')$, where $\sim$ denotes the equivalence relation on Bratteli diagrams generated by telescoping.*

*Proof.* The equivalence of (ii), (iii) and (iv) is well known, cf. [**6**, §3]. The implication (iii) $\Rightarrow$ (i) is immediate from the observation we made in Example 2.7(ii) concerning telescoping of Bratteli diagrams. In fact, $AF(\tilde{V}, \tilde{E})$ is isomorphic to both $AF(\tilde{V}_o, \tilde{E}_o)$ and $AF(\tilde{V}_e, \tilde{E}_e)$, where $(\tilde{V}_o, \tilde{E}_o)$, respectively $(\tilde{V}_e, \tilde{E}_e)$, is the telescope of $(\tilde{V}, \tilde{E})$ to odd levels, respectively even levels.

We prove (i) $\Rightarrow$ (iii). Let $AF(V, E) = \varinjlim(R_N, \mathcal{T}_N)$, $AF(V', E') = \varinjlim(R'_N, \mathcal{T}'_N)$, where $R_N, \mathcal{T}_N$, respectively $R'_N, \mathcal{T}'_N$, have the same meaning as in Example 2.7(ii). There is an obvious groupoid partition associated to $R_N$, respectively $R'_N$, which corresponds to the vertex set $V_N \in V$, respectively $V'_N \in V'$. Let $\alpha : X_{(V,E)} \to X_{(V',E')}$ implement the isomorphism between $AF(V, E)$ and $AF(V', E')$. Because of the compactness of $R_1$, $\alpha \times \alpha(R_1)$ is contained in $R_{n_1}$ for some $n_1 \geq 1$. We may choose $n_1$ so large that the groupoid partition associated to $R'_{n_1}$ is finer than that associated to $\alpha \times \alpha(R_1)$. By the same procedure as in the proof of Theorem 3.9 we associate edges between the vertices in $V_1$ and $V'_{n_1}$, keeping in mind that the groupoid partitions associated to $R_1$ and $\alpha \times \alpha(R_1)$, respectively, are isomorphic in an obvious way. Next we consider $\alpha^{-1} \times \alpha^{-1}(R'_{n_1})$, which by the compactness of $R'_{n_1}$ is contained in some $R_{n_2}$. We choose $n_2 > 1$ so large that the groupoid partition associated to $R_{n_2}$ is finer than that associated to $\alpha^{-1} \times \alpha^{-1}(R'_{n_1})$. Similarly as above, we associate edges between $V'_{n_1}$ and $V_{n_2}$. Continuing in this way we construct a Bratteli diagram $(\tilde{V}, \tilde{E})$. It is now a simple matter to show that $(\tilde{V}, \tilde{E})$ is an aggregate diagram with respect to $(V, E)$ and $(V', E')$. We omit the details. □

*Remark 4.14.* The equivalence (i) $\Leftrightarrow$ (ii) in Lemma 4.13 is Corollary 3.6 of [**11**]. Let $(X, T)$ and $(Y, S)$ be two Cantor minimal systems with associated étale equivalence

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relations $R$ and $R'$, respectively (cf. Example 2.7(i)). Let $x \in X$, $y \in Y$, and let $R_{\{x\}}$, respectively $R'_{\{y\}}$, denote the $AF$-subequivalence relation of $R$, respectively $R'$ (cf. Theorem 4.3). Then $R_{\{x\}} \cong R'_{\{y\}}$ iff $(X, T)$ and $(Y, S)$ are strong orbit equivalent. This is an immediate consequence of the lemma and Corollary 1.3, in combination with Theorem 2.1 of [**6**].

LEMMA 4.15. (Key lemma) *Let $(R, \mathcal{T})$ be isomorphic to $(R', \mathcal{T}')$, where $(R, \mathcal{T})$ and $(R', \mathcal{T}')$ are minimal AF-equivalence relations on the Cantor sets $X$ and $X'$, respectively. Let $Z$ and $Z'$ be closed, thin subsets of $X$ and $X'$ (with respect to $(R, \mathcal{T})$ and $(R', \mathcal{T}')$), respectively. Assume that:*

(i) *the (not necessarily minimal) restrictions $R|_Z = R \cap (Z \times Z)$ (respectively $R'|_{Z'} = R' \cap (Z' \times Z')$) with the relative topologies are étale equivalence relations on $Z$ (respectively $Z'$), i.e. $Z$ and $Z'$ are $R$-étale;*

(ii) *there exists a homeomorphism $\alpha : Z \to Z'$ which implements an isomorphism between $R|_Z$ and $R'|_{Z'}$.*

*There exists an extension $\tilde{\alpha} : X \to X'$ of $\alpha$ such that $\tilde{\alpha}$ implements an isomorphism between $(R, \mathcal{T})$ and $(R', \mathcal{T}')$.*

*Proof.* By Theorem 3.11, we may assume that $(R, \mathcal{T}) = AF(V, E), (R', \mathcal{T}') = AF(V', E')$ and $R|_Z = AF(W, F)$, $R'|_{Z'} = AF(W', F')$, where $(W, F)$ and $(W', F')$ are thin subdiagrams of the (simple) Bratteli diagrams $(V, E)$ and $(V', E')$, respectively. So $X = X_{(V,E)}$, $X' = X_{(V',E')}$, $Z = X_{(W,F)}$, $Z' = X_{(W',F')}$. The idea of the proof is to construct a Bratteli diagram $(\tilde{V}, \tilde{E})$, together with a subdiagram $(\tilde{W}, \tilde{F})$, so that $(\tilde{V}, \tilde{E})$ is an aggregate diagram with respect to $(V, E)$ and $(V', E')$, while $(\tilde{W}, \tilde{F})$ is an aggregate diagram with respect to $(W, F)$ and $(W', F')$. Furthermore, we do this in such way that we can 'read off' the map $\alpha : X_{(W,F)} \to X_{(W',F')}$ from $(\tilde{W}, \tilde{F})$ as an 'intertwining map' (to be explained below).

We then use $(\tilde{V}, \tilde{E})$ to extend $\alpha$ to $\tilde{\alpha} : X_{(V,E)} \to X_{(V',E')}$, $\alpha$ again being an intertwining map. We begin the proof by first noting that by telescoping $(V, E)$, say, we automatically get a corresponding telescoping of $(W, F)$, which again will be a thin subdiagram. Now there is a natural homeomorphism between the path spaces associated to a Bratteli diagram and a telescope of it (cf. Example 2.7(ii)). Thus, by Lemma 4.13(i), we may assume at the start that there is a Bratteli diagram $(\overline{V}, \overline{E})$, so that telescoping $(\overline{V}, \overline{E})$ to odd levels $0 < 1 < 3 < \cdots$, we get $(V, E)$, while telescoping $(\overline{V}, \overline{E})$ to even levels $0 < 2 < 4 < \cdots$, we get $(V', E')$. Using the notation introduced in Example 2.7(ii), let $V = V_0 \cup V_1 \cup V_3 \cup \cdots$, $E = E_1 \cup E_2 \cup \cdots$, $W = W_0 \cup W_1 \cup W_2 \cup \cdots$, $F = F_1 \cup F_2 \cup \cdots$ and similarly for $V'$, $E'$, $W'$, $F'$. Also, let $AF(W, F) = \varinjlim(R_N, \mathcal{T}_N)$, $AF(W', F') = \varinjlim(R'_N, \mathcal{T}'_N)$. Since $R_1$ is compact, we get by assumption (ii) that there exists $n'_1 \geq 1$ so that $\alpha \times \alpha(R_1) \subset R'_{n'_1}$. Furthermore, we may assume $n'_1$ is chosen so large that the groupoid partition associated to $R'_{n'_1}$ is finer than that associated to $\alpha \times \alpha(R_1)$. As in the proof of (i) $\Rightarrow$ (iii) in Lemma 4.13 we associate edges between vertices in $W_1$ and $W'_{n'_1}$. Denote these edges by $L$. Now $V_1$ and $V'_{n'_1}$ correspond to level $n_1 = 1$ and level $2n'_1$, respectively, of $(\overline{V}, \overline{E})$ and so by telescoping between these levels we get edges connecting vertices in $V_1 = V_{n_1}$ with vertices in $V'_{n'_1}$. Denote these edges by $M$. Since $(W', F')$ is

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thin in $(V', E')$ by condition (ii), we may apply Lemma 4.12 and choose $n_1'$ so large that for any $v \in W_1 \subset V_1$ and $v' \in W'_{n_1'} \subset V'_{n_1'}$, the number of edges in $M$ between $v$ and $v'$ is larger than the number of edges in $L$ between $v$ and $v'$.

Next we consider $\alpha^{-1} \times \alpha^{-1}(R'_{n_1'})$. Arguing in the same way as above, we find $n_2 > n_1'$ with edge set $L$, respectively $M$, between vertices in $W'_{n_1'}$ and $W_{n_2}$, respectively $V'_{n_1'}$ and $V_{n_2}$, with the same properties as above. Continuing in this way we construct a Bratteli diagram $(\tilde{V}, \tilde{E})$, which we show has the desired properties. In fact, by its very construction there is a Bratteli subdiagram $(\tilde{W}, \tilde{F})$ of $(\tilde{V}, \tilde{E})$, so that $(\tilde{W}, \tilde{F})$ is an aggregate diagram with respect to $(W, F)$ and $(W', F')$, while $(\tilde{V}, \tilde{E})$ itself is an aggregate diagram with respect to $(V, E)$ and $(V', E')$. In fact, by our previous deliberations we may assume that telescoping $(\tilde{V}, \tilde{E})$, respectively $(\tilde{W}, \tilde{F})$, to odd levels $0 < 1 < 3 < \cdots$, we get $(V, E)$, respectively $(W, F)$—not just a telescope of these. Likewise, telescoping $(\tilde{V}, \tilde{E})$, respectively $(\tilde{W}, \tilde{F})$, to even levels $0 < 2 < 4 < \cdots$, we get $(V', E')$, respectively $(W', F')$. Let $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2 \cup \cdots$ and $\tilde{F} = \tilde{F}_1 \cup \tilde{F}_2 \cup \cdots$. We define the composition of edges between levels $k-1$ and $k+1$, where $k \geq 1$, by

$$\tilde{E}_k \circ \tilde{E}_{k+1} = \{(\tilde{e}_k, \tilde{e}_{k+1}) | \tilde{e}_k \in \tilde{E}_k, \tilde{e}_{k+1} \in \tilde{E}_{k+1}, f(\tilde{e}_k) = i(\tilde{e}_{k+1})\}$$
$$\tilde{F}_k \circ \tilde{F}_{k+1} = \{(\tilde{f}_k, \tilde{f}_{k+1}) | \tilde{f}_k \in \tilde{F}_k, \tilde{f}_{k+1} \in \tilde{F}_{k+1}, f(\tilde{f}_k) = i(\tilde{f}_{k+1})\}.$$

We will establish bijections

$$(a) \begin{cases} \tilde{E}_{k-1} \circ \tilde{E}_k \longleftrightarrow E'_{k/2} \\ \tilde{E}_k \circ \tilde{E}_{k+1} \longleftrightarrow E_{k/2+1} \end{cases} \qquad (b) \begin{cases} \tilde{F}_{k-1} \circ \tilde{F}_k \longleftrightarrow F'_{k/2} \\ \tilde{F}_k \circ \tilde{F}_{k+1} \longleftrightarrow F_{k/2+1} \end{cases}$$

for every even number $k = 2, 4, \ldots$, which will respect the range and source maps—recalling that $\tilde{V}_m = V_{(m+1)/2}$, $\tilde{W}_m = W_{(m+1)/2}$ for $m$ odd and $\tilde{V}_m = V'_{m/2}$, $\tilde{W}_m = W'_{m/2}$ for $m$ even. (In addition we have $\tilde{E}_1 = E_1$, $\tilde{F}_1 = F$ and $\tilde{V}_0 = V_0 = V'_0 = \tilde{W}_0 = W_0 = W'_0$, all being equal to the top vertex of $(\tilde{V}, \tilde{E})$.) The bijections will be chosen successively and in such a way that we will be able to read off the given map $\alpha : X_{(W,F)} \to X_{(W',F')}$ by this intertwining process. Furthermore, the bijections in $(a)$ extend the ones in $(b)$, keeping in mind that the various edge sets occurring in $(b)$ are contained in the corresponding ones occurring in $(a)$.

We first consider the bijections $(b)$, starting with $\tilde{F}_1 = F_1$. By the embedding scheme that we outlined above, the inclusion $\alpha \times \alpha(R_1) \subset R_1'$ determines uniquely the edges $\tilde{F}_2$ between $\tilde{W}_1 = W_1$ and $\tilde{W}_2 = W_1'$. This in turn sets up a bijection $\tilde{F}_1 \circ \tilde{F}_2 \longleftrightarrow F_1'$ in an obvious way, respecting the range and source maps. Similarly, the inclusion $\alpha^{-1} \times \alpha^{-1}(R_1') \subset R_2$ uniquely determines the edges $\tilde{F}_3$ between $\tilde{W}_2 = W_1'$ and $\tilde{W}_3 = W_2$, and this in turn sets up a bijection $\tilde{F}_2 \circ \tilde{F}_3 \longleftrightarrow F_2$ in an obvious way, respecting the range and source maps. Continuing in this way we get all the bijections in $(b)$. Now these bijections induce a map between $X_{(W,F)}$ and $X_{(W',F')}$, which will be equal to $\alpha$. In fact, if $x = (f_1, f_2, \ldots) \in X_{(W,F)}$, where $f_i \in F_i$, then the successive bijections $\tilde{F}_k \circ \tilde{F}_{k+1} \longleftrightarrow F_{k/2+1}$ of $(b)$, starting with $\tilde{F}_1 = F_1$, will determine a unique path in $X_{(\tilde{W},\tilde{F})}$. Then, using the bijections $\tilde{F}_{k-1} \circ \tilde{F}_k \longleftrightarrow F'_{k/2}$ of $(b)$, we conclude that $x$ determines a unique path $y = (\tilde{f}_1, \tilde{f}_2, \ldots) \in X_{(W',F')}$, where $f_i' \in F_i'$. We easily show

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that $y = \alpha(x)$—we omit the details. We say that $\alpha : X_{(W,F)} \to X_{(W',F')}$ is defined as an *intertwining map* (via the aggregate diagram $(\tilde{W}, \tilde{F})$). (Compare with [**6**, Theorem 2.1, proof of (ii) $\Rightarrow$ (i)].)

To conclude the proof of the lemma we extend the bijection in $(b)$ to $(a)$. We do this successively, starting with $\tilde{E}_1 = E_1$. We may choose the various bijections in $(a)$, which extend those in $(b)$, in an arbitrary way, the only proviso being that the source and range maps are respected. The associated intertwining map $\tilde{\alpha} : X_{(V,E)} \to X_{(V',E')}$ will clearly be a homeomorphism that extends $\alpha$. By its very construction, $\tilde{\alpha}$ clearly preserves cofinality. Furthermore, the $n$ first edges $\{f'_1, f'_2, \ldots, f'_n\}$ of $y = \tilde{\alpha}(x) = (f'_1, f'_2, \ldots) \in X_{(V',E')}$, $f'_i \in E'_i$, are determined by the $n$ first edges of $x = (e_1, e_2, \ldots) \in X_{(V,E)}$, $e_i \in E_i$. (A similar statement is true for $\tilde{\alpha}^{-1}$.) This implies that $\tilde{\alpha}$ implements an isomorphism between $AF(V, E)$ and $AF(V', E')$ (cf. the description we gave of convergence in Example 2.7(ii)). □

We are now in a position to prove the following non-trivial result, which is a converse to Theorem 4.8.

THEOREM 4.16. *Let $(X, T)$ be a Cantor minimal system and let $R$ denote the equivalence relation associated to $(X, T)$, i.e. the $R$-equivalence classes are the $T$-orbits. Then $R$ is affable.*

*In fact, more is true: $R$ is orbit equivalent to $R_Y$, where $Y$ is any non-empty closed subset of $X$ that meets each $T$-orbit at most once (cf. Theorem 4.6 for the definition of $R_Y$ and the proof that $R_Y$ is affable).*

*Proof.* The main ingredient of the proof is Lemma 4.15, in combination with Theorem 1.2. However, we need a preliminary result in order to set the stage, so to say, before we can apply Lemma 4.15. So let $Y$ be any non-empty closed subset of $X$ that meets each $T$-orbit at most once (such a set is thin since $\{T^k(Y)\}_{k=-\infty}^{\infty}$ are disjoint, closed sets having the same measure with respect to a $T$-invariant probability measure). We need the following technical result, which we state as a sublemma, and whose proof we postpone to the end in order not to interrupt the flow of the main argument.

SUBLEMMA. *There exists a point $y \in X \setminus Y$ and a sequence of pairwise disjoint, non-empty clopen sets $\{U_n\}_{n=1}^{\infty}$, each of which are disjoint from $Y$, together with a sequence of non-empty closed sets $\{Y_n\}_{n=1}^{\infty}$, where $Y_n \subset U_n$ for each $n$, so that:*

(i) *$U_n \to y$ (i.e. for each neighbourhood $U$ of $y$, there exists $N$ so that $n \geq N$ implies $U_n \subset U$);*

(ii) *the set $Z = Y \cup \{y\} \cup \bigcup_{n=1}^{\infty} Y_n$ is a closed set such that $Z$ meets each $T$-orbit at most once;*

(iii) *there exists a sequence $\{h_n\}_{n=0}^{\infty}$ of homeomorphisms $h_0 : Y \to Y_1, h_1 : Y_1 \to Y_2, h_2 : Y_2 \to Y_3, \ldots$ .*

Applying the sublemma we may define a homeomorphism $h : Z \to Z'$, where $Z' = \{y\} \cup \bigcup_{n=1}^{\infty} Y_n$. In fact, define $h$ by $h|_{Y_n} = h_n$, $n = 0, 1, 2, \ldots$, and set $h(y) = y$. Using Theorem 1.2 we construct a Bratteli–Vershik model for $(X, T)$—which we still denote by $(X, T)$—based on the closed set $Z$ and let $B = (V, E, \geq)$ denote the associated ordered Bratteli diagram. Hence $X = X_{(V,E)}$ and $T$ is the Vershik map. Furthermore,

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since $Z$ are the maximal paths and $T(Z)$ are the minimal paths in $X$, $Z \cup T(Z)$ will be the path space associated to a thin subdiagram $(W, F)$ of $(V, E)$, i.e. $Z \cup T(Z) = X_{(W,F)}$. In fact, $(W, F)$ is a *tree* and it is obviously thin because of condition (ii) of the sublemma. By Theorem 1.2, $R_Z$ is equal to the cofinal relation associated to $(V, E)$. Furthermore, it is clear that

$$R = R_Z \vee \{(z, Tz) | z \in Z\}, \quad R_Y = R_Z \vee \{(z, Tz) | z \in Z'\}, \tag{$**$}$$

where we here let $A \vee B$ denote the equivalence relation on $X$ generated by the two subsets $A, B$ of $X \times X$.

Set $(V', E') = (V, E)$ and let $(W', E')$ be the thin subdiagram of $(V', E')$ associated to $Z' \cup T(Z')$, i.e. $Z' \cup T(Z') = X_{(W',F')}$. (Note that $(W', F')$ is a subdiagram of $(W, F)$.) There is a homeomorphism $\alpha : X_{(W,F)} \to X_{(W',F')}$, namely $\alpha|_Z = h$ and $\alpha(Tz) = T(h(z))$, $z \in Z$. Because of condition (ii) of the sublemma, $\alpha$ is well defined and is a homeomorphism. Also, $\alpha$ clearly implements an isomorphism between $AF(W, F)$ and $AF(W', F')$, which is an immediate consequence of the fact that both the Bratteli diagrams $(W, F)$ and $(W', F')$ are trees. Hence, all the conditions of Lemma 4.15 are satisfied. Let $\tilde{\alpha} : X_{(V,E)} \to X_{(V',E')}$ be the extension of $\alpha$, with the properties stated in Lemma 4.15. Clearly $\tilde{\alpha} \times \tilde{\alpha}(R_Z) = R_Z$, since $\tilde{\alpha}$ preserves the cofinal relation associated to $(V, E) = (V', E')$. Also, $\tilde{\alpha} \times \tilde{\alpha}(\{(z, Tz) | z \in Z\}) = \alpha \times \alpha(\{(z, Tz) | z \in Z\}) = \{(z, Tz) | z \in Z'\}$. Hence, by $(**)$, $\tilde{\alpha} \times \tilde{\alpha}(R) = R_Y$.

*Proof of the sublemma.* We first show that if $V$ is a non-empty clopen subset of $X$ that is disjoint from $Y$, then we may find a closed subset $Y'$ of $V$, such that $Y \cup Y'$ meets each $T$-orbit at most once and, furthermore, there exists a homeomorphism $h : Y \to Y'$. To obtain this we pick a finite partition $\{A_1, \ldots, A_{n_1}\}$ of $Y$ consisting of non-empty clopen sets ($Y$ is given the relative topology from $X$) such that the diameters of each $A_i$ is less than $\frac{1}{2}$. Also, pick $B_1, \ldots B_{n_1}$ to be non-empty clopen and pairwise disjoint subsets of $V$, each of diameter less than $\frac{1}{2}$, such that:

(i) $T^i(B_k) \cap T^j(B_l) = \emptyset$ for $-2 \leq i, j \leq 2$, $1 \leq k, l \leq n_1$;

(ii) $B_k \cap T^i(Y) = \emptyset$ for $-2 \leq i \leq 2$, $1 \leq k \leq n_1$.

This can clearly be achieved—as for (ii) we note that $\bigcup_{i=-2}^{2} T^i(Y)$ is a closed subset of $X$ with empty interior. Next we partition each of the $A_i$ into non-empty clopen sets of diameter less than $\frac{1}{3}$, with corresponding picking of clopen subsets of $B_i$. For example, say the partition of $A_1$ is $\{A_1^1, \ldots, A_1^{m_1}\}$. We pick $m_1$ non-empty and pairwise disjoint clopen subsets $B_1^1, \ldots, B_1^{m_1}$ of $B_1$, each with diameter less than $\frac{1}{3}$, such that they satisfy properties (i) and (ii), where $i$ and $j$ now range between $-3$ and 3, and $k$ and $l$ range from 1 to $m_1$. Continuing like this we get nested sequences of $A$ converging to every point in $Y$, together with nested sequences of $B$ converging to points in $V$, and we denote this set of points by $Y'$. There is a map $h : Y \to Y'$ induced by the obvious one-to-one correspondence between nested $A$- and $B$-sequences, and it is a routine matter to verify that $Y'$ and $h$ satisfy the desired properties. To finish the proof of the sublemma, choose $y \in X \setminus Y$ so that $Y_0' = Y \cup \{y\}$ meets each $T$-orbit at most once and let $\{U_n\}_{n=1}^{\infty}$ be a sequence of pairwise disjoint, non-empty clopen sets (each disjoint from $Y$) converging to $y$. Applying what we have just proved, choose successively non-empty closed sets $Y_1', Y_2', Y_3', \ldots$ such that $Y_1' \subset U_1, Y_2' \subset U_2, Y_3' \subset U_3, \ldots$ and homeomorphisms $h_0' : Y_0' \to Y_1', h_1' : Y_0' \cup Y_1' \to$

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$Y_2', h_2' : Y_0' \cup Y_1' \cup Y_2' \to Y_3', \ldots$ so that for each $k$, $Y_0' \cup Y_1' \cup Y_2' \cup \ldots \cup Y_k'$ meets each $T$-orbit at most once. Define successively $Y_0 = Y, Y_1 = h_1'(Y_0), Y_2 = h_1'(Y_1), Y_3 = h_2'(Y_2), \ldots$ and $h_k = h_k'|_{Y_k}$ for $k \geq 0$. Then $Z = Y \cup \{y\} \cup \bigcup_{n=1}^{\infty} Y_n$ is a closed set that meets each $T$-orbit at most once. This finishes the proof of the sublemma and consequently the proof of the theorem. □

The following corollary of Theorem 4.16, can succinctly be stated to say that a finite extension of a minimal $AF$-equivalence relation is affable.

COROLLARY 4.17. *Let $(R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set $X$. Let $(x_1, y_1), \ldots, (x_n, y_n)$ be $n$ pairs of points in $X \times X$. Let $R'$ be the equivalence relation on $X$ generated by $R$ and $(x_1, y_1), \ldots, (x_n, y_n)$. Then $R'$ is affable.*

*Proof.* Clearly it is enough to prove the statement for the special case $n = 1$, since the general case follows by induction. By Theorem 3.9 we may assume that $(R, \mathcal{T}) = AF(V, E)$, where $(V, E)$ is a simple, standard Bratteli diagram and $X = X_{(V,E)}$. If $x_1$ and $y_1$ are cofinal paths, then $R' = R$ and there is nothing to prove. So assume $x_1$ and $y_1$ are not cofinal. According to Proposition 1.1, there exists a Cantor minimal system $(X, T)$ such that $T$ preserves cofinality, except that $Tx_1 = y_1$. By Theorem 4.16 the equivalence relation $R'$ associated to $(X, T)$ is affable. Now $R'$ is the equivalence relation generated by $R$ and $(x_1, y_1)$. □

The following theorem is a vast generalization of Corollary 4.17 and will be a powerful tool in relating the orbit structure of minimal group actions (as homeomorphisms on the Cantor set) to $AF$-equivalence relations. Recall some terminology. If $(R, \mathcal{T})$ is an étale equivalence relation on $X$ and $Y$ is a closed subset of $X$, we say that $Y$ is $R$-étale if $R|_Y (= R \cap (Y \times Y))$, with the relative topology, is an étale equivalence relation on $Y$.

THEOREM 4.18. *Let $(R, \mathcal{T})$ be a minimal AF-equivalence relation on the Cantor set $X$ and let $Y_0, Y_1$ be two closed $R$-étale and thin subsets of $X$. Assume $R \cap (Y_0 \times Y_1) = \emptyset$ (and so, in particular, $Y_0 \cap Y_1 = \emptyset$), and let $h : Y_0 \to Y_1$ be a homeomorphism such that $h \times h : R|_{Y_0} \to R|_{Y_1}$ is an isomorphism. Then the equivalence relation $\widehat{R} = R \vee \{(y, h(y)) | y \in Y_0\}$ on $X$, generated by $R$ and $\{(y, h(y)) | y \in Y_0\}$, is affable. In fact, $\widehat{R}$ is orbit equivalent to $R$.*

*Proof.* By our assumption, $R|_{Y_0 \cup Y_1}$ is isomorphic to the disjoint union $R|_{Y_0} \sqcup R|_{Y_1}$ of $R|_{Y_0}$ and $R|_{Y_1}$ (with the relative topologies), and so $R|_{Y_0 \cup Y_1}$ is an $AF$-equivalence relation. (Recall that $R|_{Y_0}$ and $R|_{Y_1}$ are $AF$-equivalence relations by Theorem 3.11 (ii).) Since $R|_{Y_0} \cong R|_{Y_1}$, we have by Theorem 3.9 that $R|_{Y_0} \cong AF(W, F) \cong R|_{Y_1}$ for some Bratteli diagram $(W, F)$. Hence we get that $R|_{Y_0 \cup Y_1} \cong AF(\overline{W}, \overline{F})$, where $(\overline{W}, \overline{F})$ consists of two disjoint copies of $(W, F)$, obtained by adding a new level at the top consisting of two edges, as shown in Figure 4(a). By Theorem 3.11 we may assume that $(R, \mathcal{T}) = AF(V, E)$, $R|_{Y_0 \cup Y_1} = AF(\widetilde{W}, \widetilde{F})$, $X = X_{(V,E)}$ and $Y_0 \cup Y_1 = X_{(\widetilde{W}, \widetilde{F})}$, where $(V, E)$ is a simple Bratteli diagram and $(\widetilde{W}, \widetilde{F})$ is a thin subdiagram. Since $AF(\widetilde{W}, \widetilde{F}) \cong AF(\overline{W}, \overline{F})$, we may assume as a consequence of Lemma 4.13 ((i) $\Leftrightarrow$ (iv))—applying a combination of telescoping and its converse, microscoping, of $(V, E)$ (and hence of $(\widetilde{W}, \widetilde{F})$) (cf. [**6**, §3])—that $(\widetilde{W}, \widetilde{F})$ consists of two disjoint replicas (in particular, the vertex sets are disjoint) of

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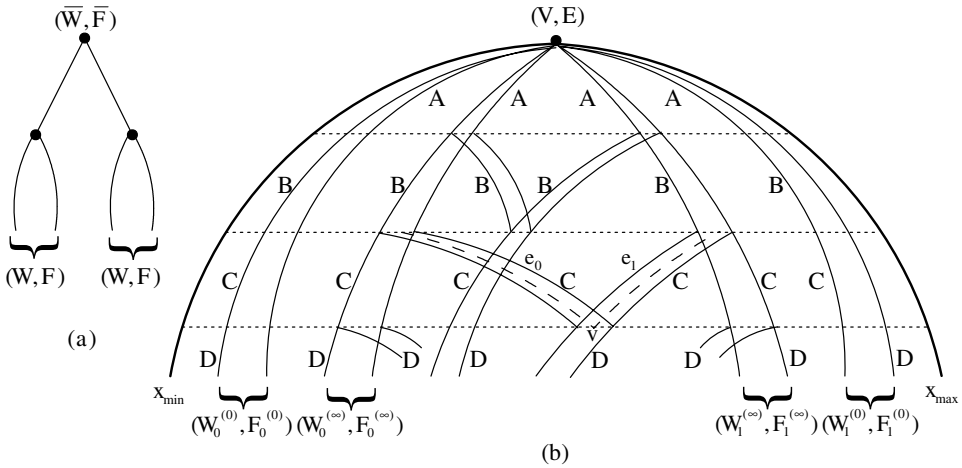



FIGURE 4. (a) $(\overline{W}, \overline{F})$ obtained by adding a new level at the top consisting of two edges. (b) The transformed diagram.

the same Bratteli diagram, the latter being equivalent to $(W, F)$. We denote the two thin subdiagrams by $(W_0^{(0)}, F_0^{(0)})$ and $(W_1^{(0)}, W_1^{(0)})$, respectively. We identify $Y_0$ and $Y_1$ with the path spaces $X_{(W_0^{(0)}, F_0^{(0)})}$ and $X_{(W_1^{(0)}, F_1^{(0)})}$, respectively. The map $h : Y_0 \to Y_1$ becomes the obvious map between $X_{(W_0^{(0)}, F_0^{(0)})}$ and $X_{(W_1^{(0)}, F_1^{(0)})}$. We now transform the Bratteli diagram $(V, E)$ (and hence the subdiagram $(\widetilde{W}, \widetilde{F})$) by a succession of telescopings and microscopings, obtaining an equivalent Bratteli diagram that lends itself to a construction involving the use of Theorem 4.6, whereby we set the stage for the application of the key lemma (Lemma 4.15). In Figure 4(b) we give a presentation of the transformed diagram, which we again denote by $(V, E)$ and which can be described as follows.

There are four disjoint thin subdiagrams of $(V, E)$, each of these are replicas of the same diagram, the latter being equivalent to $(W, F)$, and for convenience we retain the notation $(W, F)$ for this diagram. Two of the thin subdiagrams are the transformed subdiagrams of $(W_i^{(0)}, F_i^{(0)})$, $i = 0, 1$, above, and we retain the notation for these. The two other replicas of $(W, F)$ are denoted by $(W_0^{(\infty)}, F_0^{(\infty)})$ and $(W_1^{(\infty)}, F_1^{(\infty)})$, respectively.

At every level $n$ there 'emanates' from each of the subdiagrams $(W_i^{(\infty)}, F_i^{(\infty)})$, $i = 0, 1$, a copy of the $n$th tail of $(W, F)$. These coalesce at level $n + 1$ and continue together from then on. The various tail copies of $(W, F)$ have disjoint sets of vertices at each level and are also disjoint from the four (disjoint) sets of vertices belonging to the four subdiagrams $(W_i^{(0)}, F_i^{(0)})$, $(W_i^{(\infty)}, F_i^{(\infty)})$, $i = 1, 2$. We have indicated in Figure 4(b) by $A$, $B$, $C$, $D$, etc., the various edge sets that are equal, respectively, at level 1, level 2, level 3, level 4, etc. We refer to the collection of 'emanating' tail copies of $(W, F)$ as the 'tail parts of $(W, F)$'. We denote by $(W_0^{(k)}, F_0^{(k)})$, $k = 1, 2, 3, \ldots$, the subdiagram that consists of the initial part of $(W_0^{(\infty)}, F_0^{(\infty)})$ to the $k$th level and then the tail part of $(W, F)$ emanating at this level from $(W_0^{(\infty)}, F_0^{(\infty)})$. Similarly we define $(W_1^{(k)}, F_1^{(k)})$, $k = 1, 2, 3, \ldots$ with respect to $(W_1^{(\infty)}, F_1^{(\infty)})$. All the subdiagrams in question will be thin in $(V, E)$. Furthermore,

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the transformed diagram contains two infinite paths, denoted in Figure 4(b) by $x_{\max}$ and $x_{\min}$, that at each level go through distinct vertices, which are disjoint from the vertices associated to the subdiagrams described above. Also, between any pair of vertices at consecutive levels of $(V, E)$ there are at least two edges in $E$.

A transformed diagram with these properties can be obtained by utilizing Lemma 4.12 and the fact that $(\widetilde{W}, \widetilde{F})$ is a thin subdiagram of $(V, E)$. We omit the details, which are rather lengthy, but we briefly say that the procedure starts by telescoping the original diagram $(V, E)$, using Lemma 4.12, in such a way that between the telescoped levels there are sufficiently many edges, making 'room' for microscoping (followed by appropriate telescoping), in order to fit in $(W_i^{(\infty)}, F_i^{(\infty)})$, $i = 1, 2$, and the various tail copies of $(W, F)$ as thin subdiagrams.

We now introduce an auxiliary closed subset $Z$ of $X = X_{(V,E)}$ and give $(V, E)$ a proper ordering so that $Z$ becomes a regular set (cf. Definition 4.4) with respect to the lexicographic ('Vershik') map. We then apply Theorem 4.6 to 'split' the various coalescing tails of $(W, F)$, to obtain for each $k = 1, 2, \ldots$, two copies $(W_0^{(k)}, F_0^{(k)})$ and $(W_1^{(k)}, F_1^{(k)})$ of $(W, F)$. Then we will be in a position to apply the key lemma, Lemma 4.15, to finish the proof.

First we give $(V, E)$ a proper ordering so that the unique min path, respectively the unique max path, is the path that we described above and denoted by $x_{\min}$, respectively $x_{\max}$, in Figure 4(b). We denote the associated Vershik map by $T$. Furthermore, the ordering is chosen so that for the edges ranging at any given vertex $v \in V$, those edges sourcing at the vertex that the unique min path goes through are ordered consecutively, starting with the minimal edge. Similarly, those edges sourcing at the vertex that the unique max path goes through are ordered consecutively, ending with the maximal edge. Let $v \in V_n$ be a vertex at level $n, n = 2, 3, \ldots$, where edges from the two coalescing tail copies of $(W, F)$—emanating at level $n-1$ from $(W_0^{(\infty)}, F_0^{(\infty)})$ and $(W_1^{(\infty)}, F_1^{(\infty)})$, respectively—meet. There is a natural bijection between the two edge sets in $f^{-1}(v)$ belonging to the two tail copies of $(W, F)$. We order these edges so that if $e_0$ and $e_1$ correspond by this bijection, $e_0$ sourcing in $(W_0^{(\infty)})_{n-1}$ and $e_1$ sourcing in $(W_1^{(\infty)})_{n-1}$, then $e_1$ follows immediately after $e_0$ in the ordering. (In Figure 4 we illustrate this at level $n = 3$.) As for all the other edges in $E$, these may be ordered arbitrarily if consistent with the requirements we have imposed.

Let $(L, G)$ be the subdiagram of $(V, E)$ consisting of:

(i) the 'tail parts of $(W, F)$' (see above);
(ii) the edges that make up the unique max path;
(iii) the max and next-to-max ('max$-1$') edges that range at vertices belonging to the coalescing tails of the various emanating $(W, F)$;
(iv) the max edges ranging at $W_0^{(\infty)} \cup W_1^{(\infty)}$, i.e. ranging at the set of vertices at the very top of the emanating tail copies of the $(W, F)$.

In Figure 5(a) we exhibit the subdiagram $(L, G)$. Note that from each vertex $v$ belonging to the 'tail parts of $(W, F)$' there is a max (or max$-1$) edge ranging at $v$. In Figure 5(a) we have not drawn all these edges in order to make the figure clearer.

We claim that $Z = X_{(L,G)}$ is a regular subset of $X = X_{(V,E)}$ with respect to the Vershik map $T$. This is an immediate consequence of the fact that $Z$ is the path space

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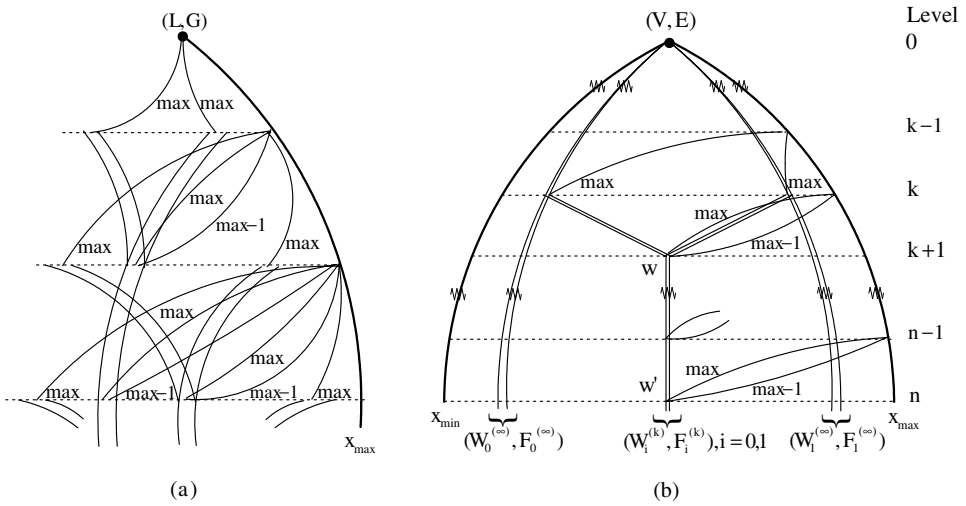



FIGURE 5. (a) The subdiagram $(L, G)$. (b) The subdiagram $(W_i^{(k)}, F_i^{(k)})$ in $(V, E)$.

associated to a subdiagram of $(V, E)$. In fact, the return map on $Z$ is the same as the Vershik map on $Z = X_{(L,G)}$, determined by the ordering on $(L, G)$ (induced in an obvious way from the ordering on $(V, E)$). From this it follows easily that the return time maps $\lambda^+, \lambda^- : Z \to \mathbb{N} \cup \{+\infty\}$ are continuous. We omit the details. Also, $\lambda^+(x_{\max}) = \lambda^-(x_{\max}) = +\infty$. Hence $Z$ satisfies the necessary conditions so that we can apply Theorem 4.6 to $Z$ and $(X, T)$.

By Theorem 4.6 the subequivalence relation $R_Z$ of $R$ is $AF$, and so $R_Z \cong AF(\widetilde{V}, \widetilde{E})$ for some Bratteli diagram $(\widetilde{V}, \widetilde{E})$. We now argue that $(\widetilde{V}, \widetilde{E})$ is a simple diagram (which is equivalent to showing that $R_Z$ is a minimal equivalence relation, cf. Theorem 3.9). Furthermore, we argue that $(\widetilde{V}, \widetilde{E})$ may be given a ('suggestive') presentation as shown in Figure 6, where the subdiagrams $(W_i^{(k)}, F_i^{(k)})$ are thin in $(\widetilde{V}, \widetilde{E})$ and $AF(W_i^{(k)}, F_i^{(k)}) \cong AF(W, F)$ for all $k = 0, 1, 2, \ldots, \infty$, and $i = 0, 1$. This will set the stage for applying Lemma 4.15.

Recall that an $R$-equivalence class $[x]_R$ is either equal to the $R_Z$-equivalence class $[x]_{R_Z}$, or $[x]_R$ is partitioned into two $R_Z$-equivalence classes. In fact, it is easily seen that the only $R$-equivalence classes that may split into two $R_Z$-equivalence classes are $[x]_R$, where $x \in X_{(L,G)}$, $x \neq x_{\max}$. In Claim 2 below we will show that this splitting does occur.

(In the following we refer the reader to Example 2.7(ii) for notation and terminology.)

CLAIM 1. *$R_Z$ is minimal, i.e. each $R_Z$-equivalence class is dense in $X = X_{(V,E)}$.*

*Proof.* By the remark above we must show that if $x \in X_{(L,G)}$ and $U$ is a cylinder set, say $U = U_{(f_1,\ldots,f_k)} = \{(e_1, e_2, \ldots) | e_i = f_i, i = 1, \ldots, k\}$, then $[x]_{R_Z}$ meets $U$. Let $x = (g_1, g_2, \ldots)$. Let $e$ be an edge in $E$ connecting $f(f_k)$ at level $k$ to the vertex at level $k + 1$ that $x_{\max}$ goes through. Let $e_{\max-1}$ and $e_{\max}$ denote the max$-1$ and max edges, respectively, that source at $f(e)$ (at level $k + 1$) and range at $i(g_{k+3})$ (at level $k + 2$). Let $x_1 = (f_1, \ldots, f_k, e, e_{\max-1}, g_{k+3}, g_{k+4}, \ldots)$, $x_2 =$

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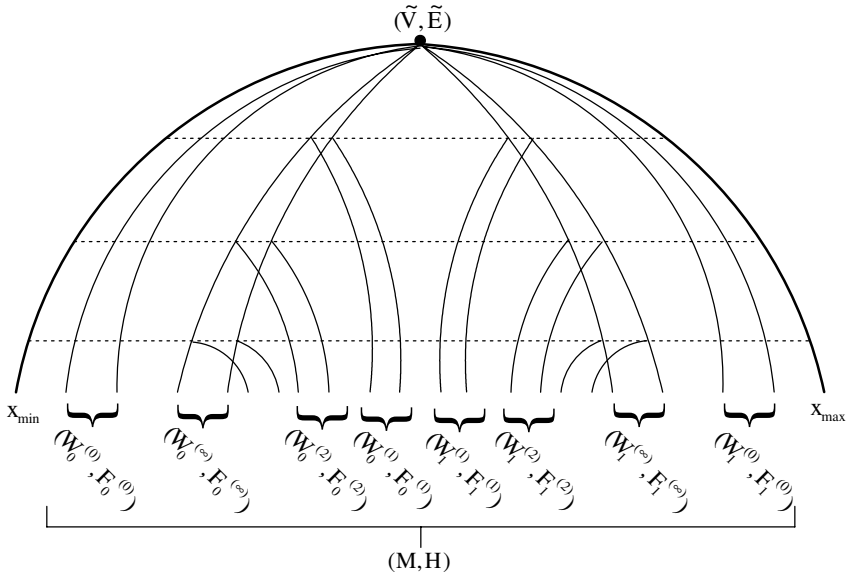



FIGURE 6. Subdiagrams of $(\widetilde{V}, \widetilde{E})$.

$(f_1, \ldots, f_k, e, e_{\max}, g_{k+3}, g_{k+4}, \ldots)$. Clearly, $x_1$ and $x_2$ lie in $U$. Also, $x_1$ and $x_2$ are cofinal with $x$, hence in the same $R$-equivalence class. It is easy to see that $(x_1, x_2) \notin R_Z$. In fact, if $T^l x_1 = x_2, l \geq 1$, there is exactly one point among $x_1, Tx_1, T^2x_1, \ldots, T^{l-1}x_1$ lying in $Z$, namely the path following the unique max path $x_{\max}$ from level 0 to level $k+1$, and from there on being cofinal with $x_1$. Since either $(x, x_1) \in R_Z$ or $(x, x_2) \in R_Z$, we conclude that $[x]_{R_Z}$ meets $U$. □

Let $k \in \{1, 2, 3, \ldots\}$. Recall that $(W_0^{(k)}, F_0^{(k)})$ and $(W_1^{(k)}, F_1^{(k)})$ denote the two subdiagrams of $(V, E)$ associated to the two tail copies of $(W, F)$ emanating at level $k$ from $(W_0^{(\infty)}, F_0^{(\infty)})$ and $(W_1^{(\infty)}, F_1^{(\infty)})$, respectively. (By construction, $(W_0^{(k)}, F_0^{(k)})$ and $(W_1^{(k)}, F_1^{(k)})$ are replicas of $(W, F)$, coalescing at level $k+1$; see Figure 5(b).)

CLAIM 2. *Let $x_0$ and $x_1$ be two cofinal paths in $X_{(W_i^{(k)}, F_i^{(k)})}$, $i = 0$ or $i = 1$. Then $(x_0, x_1) \in R_Z$. Let $y_0 \in X_{(W_0^{(k)}, F_0^{(k)})}$ and $y_1 \in X_{(W_1^{(k)}, F_1^{(k)})}$ be cofinal paths. Then $(y_0, y_1) \notin R_Z$.*

*Proof.* If $x_0$ and $x_1$ are cofinal from level $k$ on, it is easy to see that $(x_0, x_1) \in R_Z$. In fact, if $x_0 < x_1$ in the lexicographic ordering, then $T^l x_0 = x_1$ for some $l \geq 1$. There will be no points in $Z$ among $x_0, Tx_0, \ldots, T^{l-1}x_0$ and so $(x_0, x_1) \in R_Z$. Let $x_0$ and $x_1$ be cofinal from level $k+1$ on and, say, $x_0 < x_1$ in the lexicographic ordering. Then $x_1 = T^l x_0$ for some $l \geq 1$. We must show that the number of points that lie in $Z$ among $x_0, Tx_0, \ldots, T^{l-1}x_0$ is even. This, however, is a consequence of the way we have introduced our ordering on $(V, E)$ and the fact that the 'max' and the 'max $-1$' edges are part of the edge set that make up the closed set $Z$ (see Figure 5(b)). This is easy to show, but we omit the details. Similarly, we show that $(x_0, x_1) \in R_Z$ if $x_0$ and $x_1$ are cofinal from any level $n \geq k+1$. For $y_0$ and $y_1$ we argue as follows. Let $y_0'$ be the obvious 'mirror'

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path of $y_0$ in $(W_1^{(k)}, F_1^{(k)})$ (using the fact that $(W_0^{(k)}, F_0^{(k)})$ and $(W_1^{(k)}, F_1^{(k)})$ are replicas of each other). So $y_0$ and $y_0'$ are cofinal from level $k+1$ on. It is a consequence of the ordering we have introduced that $y_0 < y_0'$ and that if $T^l y_0 = y_0'$ (for some $l \geq 1$), then the number of points in $y_0, Ty_0, \ldots, T^{l-1}y_0$ meeting $Z$ is odd, a fact that is easily shown. Hence $(y_0, y_0') \notin R_Z$. Now $(y_0', y_1) \in R_Z$ by the first part and so $(y_0, y_1) \notin R_Z$.

By Claim 1, the Bratteli diagram for $R_Z$ must be simple. The path spaces $X_{(W_i^{(k)}, F_i^{(k)})}$ associated to the subdiagrams $(W_i^{(k)}, F_i^{(k)})$, $k = 0, 1, 2, \ldots, \infty$; $i = 0, 1$, of $(V, E)$, are pairwise disjoint. Let $Y$ be the union of these. Then $Y$ is a closed subset of $X$, and by Claim 2 it is clear that $R_Z|_Y$ is an $AF$-equivalence relation with the relative topology. By Theorem 3.11 there exists a (simple) Bratteli diagram $(\widetilde{V}, \widetilde{E})$, a subdiagram $(M, H)$ and a homeomorphism $h : X_{(\widetilde{V}, \widetilde{E})} \to X = X_{(V,E)}$, such that $h$ implements an isomorphism between $AF(\widetilde{V}, \widetilde{E})$ and $R_Z$ (given the relative topology from $(R, \mathcal{T})$). Furthermore, $h(X_{(M,H)}) = Y$ and the restriction of $h$ to $X_{(M,H)}$ implements an isomorphism between $AF(M, H)$ and $R_Z|_Y$. Now we claim that $R_Z|_Y$ may be represented by the particular subdiagram of $(\widetilde{V}, \widetilde{E})$ occurring in Figure 6. In fact, by invoking Lemma 4.13, we may assume that $(M, H)$ is of this form, where $AF(W_i^{(k)}, F_i^{(k)}) \cong AF(W, F)$ for all $k = 0, 1, 2, \ldots, \infty$; $i = 0, 1$. (We retain the notation $(W_i^{(k)}, F_i^{(k)})$ for the various subdiagrams occurring in Figure 6—these may be telescopes of those occurring in Figure 4(b), however.) The subdiagrams $(W_i^{(k)}, F_i^{(k)})$ are all replicas of each other. For each $k = 0, 1, 2, \ldots$, let $h_k : X_{(W_0^{(k)}, F_0^{(k)})} \to X_{(W_1^{(k)}, F_1^{(k)})}$ be the obvious homeomorphism, using the fact that $(W_0^{(k)}, F_0^{(k)})$ and $(W_1^{(k)}, F_1^{(k)})$ are replicas of each other. (We let $h_k = h$, where we have identified $Y_i$ with $X_{(W_i^{(0)}, F_i^{(0)})}$, $i = 0, 1$.) Then we have

$$\begin{aligned} R &= R_Z \vee \{(x, h_k(x)) | x \in X_{(W_0^{(k)}, F_0^{(k)})}, k = 1, 2, \ldots\} \\ \widehat{R} &= R_Z \vee \{(x, h_k(x)) | x \in X_{(W_0^{(k)}, F_0^{(k)})}, k = 0, 1, 2, \ldots\}. \end{aligned} \qquad (***)$$

Let $(M', H')$ denote the subdiagram of $(\widetilde{V}, \widetilde{E})$ consisting of $\{(W_i^{(k)}, F_i^{(k)}) : k = 1, 2, \ldots, \infty; i = 0, 1\}$. Assume for the time being that we can prove that $(M, H)$ and hence *a fortiori* $(M', H')$ is a thin subdiagram of $(\widetilde{V}, \widetilde{E})$. Let $\alpha_0^{(k)} : X_{(W_0^{(k)}, F_0^{(k)})} \to X_{(W_0^{(k+1)}, F_0^{(k+1)})}$, respectively $\alpha_1^{(k)} : X_{(W_1^{(k)}, F_1^{(k)})} \to X_{(W_1^{(k+1)}, F_1^{(k+1)})}$, $k = 0, 1, 2, \ldots$, be the obvious homeomorphisms (using that all the diagrams in question are replicas of each other), which implements isomorphisms between $AF(W_0^{(k)}, F_0^{(k)})$ and $AF(W_0^{(k+1)}, F_0^{(k+1)})$, respectively $AF(W_1^{(k)}, F_1^{(k)})$ and $AF(W_1^{(k+1)}, F_1^{(k+1)})$. Let $\alpha_0^{(\infty)} : X_{(W_0^{(\infty)}, F_0^{(\infty)})} \to X_{(W_0^{(\infty)}, F_0^{(\infty)})}$ and $\alpha_1^{(\infty)} : X_{(W_1^{(\infty)}, F_1^{(\infty)})} \to X_{(W_1^{(\infty)}, F_1^{(\infty)})}$ be the identity maps. We let $\alpha : X_{(M,H)} \to X_{(M',H')}$ be defined by $\alpha|_{(W_0^{(k)}, F_0^{(k)})} = \alpha_0^{(k)}$, $\alpha|_{(W_1^{(k)}, F_1^{(k)})} = \alpha_1^{(k)}$, for $k = 0, 1, 2, \ldots, \infty$. Then $\alpha$ is a homeomorphism implementing an isomorphism between $AF(M, H)$ and $AF(M', H')$. By the key lemma (Lemma 4.15), $\alpha$ can be extended to $\widetilde{\alpha} : X_{(\widetilde{V}, \widetilde{E})} \to X_{(\widetilde{V}, \widetilde{E})}$ such that $\widetilde{\alpha}$ implements an automorphism of $AF(\widetilde{V}, \widetilde{E})$. Since $\widetilde{\alpha}$ preserves cofinality, we get that $\widetilde{\alpha} \times \widetilde{\alpha}(R_Z) = R_Z$. By its very construction, it is clear that $\widetilde{\alpha} \times \widetilde{\alpha}(\{(x, h_k(x)) | x \in X_{(W_0^{(k)}, F_0^{(k)})}\}) = \{(y, h_{k+1}(y)) | y \in X_{(W_0^{(k+1)}, F_0^{(k+1)})}\}$ for every $k = 0, 1, 2, \ldots$. By $(***)$ this implies that $\widetilde{\alpha} \times \widetilde{\alpha}(\widehat{R}) = R$, which finishes the proof. □

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It remains to prove the following claim.

CLAIM 3. *The subdiagram* $(M, H)$ *is thin in* $(\widetilde{V}, \widetilde{E})$.

*Proof.* Let $\mu$ be a $R_Z$-invariant probability measure. It is enough to prove that $\mu(X_{(W_i^{(k)}, F_i^{(k)})}) = 0$ for $k = 0, 1, 2, \ldots, \infty; i = 0, 1$. Here we prove this for $k \in \{1, 2, \ldots\}, i = 0$ or 1. (For $k = 0$ or $k = \infty$ the proof is similar, but less complicated.) So fix $k$ and $i$. In what follows we use the same notation $(W_i^{(k)}, F_i^{(k)})$ for the subdiagrams occurring in either $(V, E)$ or $(\widetilde{V}, \widetilde{E})$. This causes no problem, since the associated path spaces $X_{(W_i^{(k)}, F_i^{(k)})}$ may be identified in a natural way. Let $\varepsilon > 0$. Choose $K > 0$ such that $1/K < \varepsilon$. Since $(W_i^{(k)}, F_i^{(k)})$ is thin in $(V, E)$, we can use Lemma 4.12 to find $n \geq k + 1$ such that the inequalities in Lemma 4.12 are satisfied. We connect any path in $(V, E)$ between $w \in (W_i^{(k)})_{k+1}$ and $w' \in (W_i^{(k+1)})_n$ to the top vertex by following $x_{\max}$ from level 0 to level $k$, and then the max edge, respectively the max$-1$ edge, to $w$ (see Figure 5(b)). The number of paths of each of these two types is larger than $K$ times the number of paths from the top vertex to $w'$ lying in $(W_i^{(k)}, F_i^{(k)})$. Let us denote two paths $p$ and $q$ from the top vertex to $w'$ for *dual*, if $p$ and $q$ are of the types just described, $p$ and $q$ differing only at the max and max$-1$ edges between levels $k$ and $k + 1$ (see Figure 5(b)). Now let $x$ and $y$ be two paths in $X_{(V,E)}$, which are cofinal from level $n$ on, such that the initial $n$th parts are dual. Then it is easy to show that $(x, y) \notin R_Z$. Hence it follows that if $z$ is a path in $X_{(W_i^{(k)}, F_i^{(k)})}$, cofinal with $x$ and $y$ from level $n$ on, then either $(x, z) \in R_Z$ or $(y, z) \in R_Z$. We draw the following conclusion. Let $r$ be a path in $(W_i^{(k)}, F_i^{(k)})$ from the top vertex to $w'$ and let $p$ and $q$ be dual paths. Then either $\mu(U_p) = \mu(U_r)$, or $\mu(U_q) = \mu(U_r)$, where $U_p$, $U_q$ and $U_r$ denote the cylinder sets in $X_{(V,E)}$ associated to $p$, $q$ and $r$. Summing up all this we get that the $\mu$-measure of $A_n = \bigcup\{U_r | r$ path in $(W_i^{(k)}, F_i^{(k)})$ from the top vertex to $w', w' \in (W_i^{(k)})_n\}$ is $\leq 1/K < \varepsilon$. Since $X_{(W_i^{(k)}, F_i^{(k)})}$ is the intersection $\bigcap_{n=k+1}^{\infty} A_n$, we get finally that $\mu(X_{(W_i^{(k)}, F_i^{(k)})}) = 0$. □

This finishes the proof of the theorem. □

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