Incidence scrolls. By Mr W. G. WELCHMAN, Sidney Sussex College.

[*Received* 10 February, *read* 6 March 1933.]

1. The work of this paper was undertaken with a view to finding out what ruled surfaces can be determined by incidences, i.e. generated by the lines which meet a certain *set of spaces* which I shall call a *base*. Such ruled surfaces I shall call *incidence scrolls*. In [3] the lines which meet three lines generate a quadric surface. In [4] it is easy to show that a base consisting of a line and three planes gives the general rational quartic scroll, while the lines which meet five planes in [4] give the general elliptic quintic scroll. One might be tempted to think that at least all the rational normal scrolls could be obtained as incidence scrolls by taking for base a suitable number of spaces containing directrix curves, but unfortunately there is a residual surface except in the case of the rational scrolls of general type and of those with a directrix line.

In the present paper several formulae are obtained which show that except in these two cases the incidence scrolls soon cease to be simple. The formulae only deal with some of the more simple cases, the order of the scroll given by a general base being given by Giambelli's formula*. The results in [3], [4], [5], ..., [10] are tabulated.

2. At the outset we may notice that a non-degenerate incidence scroll cannot have a double point, for, if two lines through a point meet all the spaces of the base, so will all the lines of the plane pencil determined by them, and the incidence scroll is degenerate, containing the plane pencil. Thus the generators of the scroll meet each base space in points of a curve through each point of which passes one generator. These curves are directrix curves of the scroll, and the scroll may be regarded as generated by a (1, 1) correspondence between any two of the curves. In particular if there is a plane in the base the directrix curve in it can have no double points, so that we know the genus when we know the order of the directrix curve.

Also, to save references in the text, it is as well to recall the general types of ruled surfaces. A rational ruled surface of order n is normal in $[n+1]$. The general type has one directrix curve of order $\frac{1}{2}(n-1)$ or ∞^1 of order $\frac{1}{2}n$. There are also particular surfaces with a directrix curve of lower order. An elliptic ruled surface of order n is normal in $[n-1]$, and the general type has two

* See Segre, *Encykl. Math. Wiss.*, III C 7, 818.

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

directrix curves of order $\frac{1}{2}n$ or an infinity of order $\frac{1}{2}(n+1)$. There are particular surfaces with a directrix curve of lower order and when n is even the two directrix curves of order $\frac{1}{2}n$ may coincide or there may be an infinite number of directrix curves of order $\frac{1}{2}n$. A ruled surface of order n and genus p is in general normal in $[n-2p+1]$, and the general type has 2^p directrix curves of order $\frac{1}{2}(n+p-1)$ or ∞^1 of order $\frac{1}{2}(n+p)$. There are also particular surfaces with directrix curves of lower order, and, for $p>1$, "special" surfaces which are normal in a space of more than $n-2p+1$ dimensions*.

Finally, we notice that in $[n]$ it is no use taking an $[r]$ and an $[s]$ among the base spaces of a scroll if $r+s<n-1$, for all the generators of the scroll would then lie in the $[r+s+1]$ which contains these two spaces.

3. In $[2n+1]$ consider the scroll R generated by the lines which meet *three* $[n]$*'s and one* $[n+1]$. This scroll will have directrix curves in the $[n]$'s, of order x, say, and a directrix curve in the $[n+1]$ of order y. Then y is the number of generators of the scroll which meet an $[n]$ lying in the base $[n+1]$, i.e. the number of lines in $[2n+1]$ which meet four $[n]$'s. Call this number z_n.

Take a prime S_{2n} through the base $[n+1]$. The generators of the scroll which lie in this prime must meet three $[n-1]$'s and an $[n+1]$ therein. But lines which meet two $[n-1]$'s must lie in the $[2n-1]$ which contains the two $[n-1]$'s, so that the generators of R which lie in S_{2n} must lie in a $[2n-3]$ and meet four $[n-2]$'s therein. The number of these generators is therefore z_{n-2}. But through the point of intersection of the base $[n+1]$ and a base $[n]$ there passes one line which meets the other two base $[n]$'s, as we see by projection; this line is a generator of the scroll, but does not lie in S_{2n}. Thus the order x of the directrix curve in a base $[n]$ is $z_{n-2}+1$.

The scroll is determined by a (1, 1) correspondence between any two of its directrix curves, so that, if X is the order of the scroll, we have

$$X = 2x = 2(z_{n-2}+1),$$

and

$$X = x+y-1 = z_n + z_{n-2},$$

whence

$$z_n = z_{n-2}+2.$$

But z_1 is the number of lines meeting four lines in [3], and z_2 is the number of lines meeting four planes in [5], so that

$$z_1 = 2, \quad z_2 = 3, \quad \text{and} \quad z_n = n+1,$$

and the order of the scroll is $2n$.

* For a fuller statement and for references, see Segre, *Encykl. Math. Wiss.*, III C 7, 911, 912.

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

Through the base $[n+1]$ and $n-1$ generators there passes a pencil of primes each of which contains one other generator, so that the scroll is a *rational normal ruled surface of general type.* It is clear, moreover, that any such surface may be obtained in this manner, by taking as base three $[n]$'s containing directrix C_0^n of the surface, and one $[n+1]$ containing a directrix C_0^{n+1}. Thus

(1) *In* $[2n+1]$ *a base consisting of three* $[n]$*'s and one* $[n+1]$ *gives an* R_0^{2n} *of general type.*

4. We now make this surface degenerate. Suppose that two of the base $[n]$'s lie in a prime S_{2n} and meet in a point P. Then R_0^{2n} degenerates into a plane pencil with vertex P and a surface in S_{2n} generated by the lines which meet one $[n-1]$ and three $[n]$'s. The plane pencil will have one generator in S_{2n}, and this will also belong to the surface in S_{2n}, which is therefore rational. We have therefore the following result:

(2) *In* $[2n]$ *a base consisting of one* $[n-1]$ *and three* $[n]$*'s gives an* R_0^{2n-1}. The surface is of general type, for the directrix curve in the $[n-1]$ is a C_0^{n-1} and those in the $[n]$'s are of order n. Further, it is evident that any R_0^{2n-1} in $[2n]$ which is of general type may be obtained in this manner.

5. The degeneration in the last paragraph is most useful. If we have in $[m]$ an R_P^N given by the lines which meet the spaces of a certain base, and if the base contains an $[r]$ and an $[s]$ such that $r+s=m-1$, then in the degeneration in which the $[r]$ and the $[s]$ lie in a prime the surface splits up into a plane pencil and a scroll in $[m-1]$ with one generator in common. Thus we obtain an incidence scroll in $[m-1]$, of order $N-1$ and genus P, whose base consists of $[r]$, $[s]$ and the prime sections of the other base spaces in $[m]$. I shall refer to this process as "*combining* $[r]$ *and* $[s]$".

Again, if the base of R_P^N in $[m]$ consists of an $[r]$ and an $[s]$ such that $r+s=m$ and other spaces $[t]$, $[u]$, ..., we deduce that in $[m+1]$ the surface given by a base which consists of

$$\text{an } [r], \quad \text{an } [s], \quad \text{a } [t+1], \quad \text{an } [u+1], \ldots$$

is of order $N+1$ and genus P. I shall refer to this process as "*separating* $[r]$ *and* $[s]$".

6. Now take *two* $[n]$*'s, two* $[n+1]$*'s and one* $[2n-1]$ as a base in $[2n+1]$, giving a scroll R. Suppose that one of the base $[n+1]$'s and the base $[2n-1]$ lie in a prime S_{2n} and meet in an $[n]S_n$. The scroll R breaks up into

(a) a scroll formed by lines which meet three $[n]$'s and one $[n+1]$, which we know to be an R_0^{2n}, and

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

(*b*) a scroll formed by lines in S_{2n} which meet two $[n-1]$'s, one $[n+1]$ and one $[n]$ therein. These lines must lie in the $[2n-1]$ which contains the two $[n-1]$'s and meet three $[n-1]$'s and one $[n]$ therein, giving an R_0^{2n-2}.

The scroll R_0^{2n} has a directrix C_0^n in S_n and has therefore n generators in S_{2n}, which also belong to R_0^{2n-2}, and we have the following theorem:

(3) *In* $[2n+1]$ *a base consisting of two* $[n]$*'s, two* $[n+1]$*'s, and one* $[2n-1]$ *gives an incidence scroll of order* $4n-2$ *and genus* $n-1$, and the directrix curve in each base $[n]$ is a C_{n-1}^{2n-1}.

In [3] we obtain a quadric. In [5] we obtain an R_1^6 which is of general type, having two directrix cubics. In [7] the surface is an R_2^{10} which is not of general type, having two directrix quintics whereas the general R_2^{10} has a single infinity of C_2^6 as minimum directrices.

7. Combining the two base $[n]$'s of (3) as explained in § 5 we have the following theorem:

(4) *In* $[2n]$ *a base consisting of four* $[n]$*'s and one* $[2n-2]$ *gives an incidence scroll of order* $4n-3$ *and genus* $n-1$, whose directrix curves in the four base $[n]$'s are C_{n-1}^{2n-1}.

In [4] we obtain the well-known R_1^5 given by lines meeting five planes. In [6] we obtain an R_2^9 which is of general type having four directrix quintics lying in solids. In [8] we obtain an R_3^{13} with four directrix septimics lying in [4]'s, which is not a surface of general type.

Again, by separating an $[n]$ and an $[n+1]$ in (3) we see that in $[2n+2]$ the incidence scroll given by an $[n]$, two $[n+1]$'s, an $[n+2]$ and a $[2n]$ is of order $4n-1$ and genus $n-1$. Thus:

(5) *In* $[2n]$ *the incidence scroll given by an* $[n-1]$, *two* $[n]$*'s, an* $[n+1]$ *and a* $[2n-2]$ *is of order* $4n-5$ *and genus* $n-2$, the directrix curve in the $[n-1]$ being of order $2n-3$.

In [6] this gives an R_1^7 generated by lines which meet a plane, two [3]'s and two [4]'s; the surface is not of general type, having a directrix cubic. Also in [8] we obtain an R_2^{11} given by one [3], two [4]'s, one [5] and one [6].

From (5), by separating the two $[n]$'s, we obtain the following theorem:

(6) *In* $[2n+1]$ *the incidence scroll given by* 3 $[n]$*'s, one* $[n+2]$ *and one* $[2n-1]$ *is of order* $4n-4$ *and genus* $n-2$, the directrix curve in each base $[n]$ being of order $2n-2$.

For $n=2$ we merely get an R_0^4 in [5], but the case $n=3$ is more interesting. We obtain in [7] an R_1^8 formed by the lines which meet three solids and two [5]'s, the directrix curves in the base solids being elliptic quartics. Any three generators will lie in a [5],

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

which meets the base spaces in three planes and two solids, and therefore contains a fourth generator (since the lines of [5] which meet three planes and one solid generate an R_0^4). Thus the surface R_1^8 has a single infinity of directrix quartics and no directrix quintics. The surface, therefore, is not of general type.

Again, in [9] we have an R_2^{12} generated by the lines which meet three [4]'s, a [6] and a [7]. The spaces of seven dimensions which contain four generators also contain two others, so that the surface has a single infinity of directrix C_2^6 lying in [4]'s.

Finally, from (5), by separating the $[n-1]$ and the $[n+1]$ we obtain another theorem:

(7) *In* $[2n+1]$ *the incidence scroll given by an* $[n-1]$, *three* $[n+1]$*'s and a* $[2n-1]$ *is of order* $4n-4$ *and genus* $n-2$, the order of the directrix curve in the $[n-1]$ being $2n-3$. In [5] this gives the rational quartic scroll with a directrix line, and in [7] the lines which meet a plane, three [4]'s and a [5] generate an R_1^8 with a directrix cubic, which is not a surface of general type.

8. In $[2n+1]$ we may take as base one $[n]$, three $[n+1]$'s and one $[2n-2]$. If the $[n]$ and the $[2n-2]$ lie in S_{2n} and meet in S_{n-2}, the scroll degenerates into (*a*) an R_{n-1}^{4n-3} in S_{2n}, generated by lines meeting four $[n]$'s and one $[2n-2]$*, and (*b*) a scroll formed by the lines which meet one $[n-2]$ and three $[n+1]$'s. But an $[n-2]$ and an $[n+1]$ lie in a $[2n]$, so the scroll (*b*) lies in $[2n-2]$ and is generated by lines which meet one $[n-2]$ and three $[n-1]$'s therein. Thus the scroll (*b*) is an R_0^{2n-3} with a directrix C_0^{n-2} in S_{n-2}, and has therefore $n-1$ generators in S_{2n} which also belong to R_{n-1}^{4n-3}, the scroll (*a*). We have then the following theorem:

(8) *In* $[2n+1]$ *the incidence scroll determined by one* $[n]$, *three* $[n+1]$*'s and one* $[2n-2]$ *is of order* $6n-6$ *and genus* $2n-3$. Each base $[n]$ contains a directrix curve of order $3n-3$. In [5] this gives the R_1^6 already obtained from theorem (3), and in [7] we obtain an R_3^{12} which is not of general type.

By a similar degeneration we can also prove the following theorem:

(9) *In* $[2n]$ *the incidence scroll determined by three* $[n]$*'s, one* $[n+1]$ *and one* $[2n-3]$ *is of order* $6n-9$ *and genus* $2n-4$, with a directrix curve in each $[n]$ of order $3n-4$. In [6] this gives the R_2^9 already obtained from theorem (4), and in [8] we obtain an R_4^{15} with directrix octavics which is not of general type.

9. By applying the method of § 5 to theorem (8) we obtain in succession the following results:

(10) *In* $[2n]$ *the scroll given by* $[n-1]$, $[n]$, *two* $[n+1]$*'s,* $[2n-3]$ *is of order* $6n-11$ *and genus* $2n-5$.

* By theorem (4).

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

(11) *In* $[2n+1]$ *the scroll given by* $[n-1]$, *two* $[n+1]$*'s*, $[n+2]$, $[2n-2]$ *is of order* $6n-10$ *and genus* $2n-5$.

(12) *In* $[2n]$ *the scroll given by* $[n-2]$, *three* $[n+1]$*'s*, $[2n-3]$ *is of order* $6n-15$ *and genus* $2n-7$.

Similarly from (9) we obtain three more theorems:

(13) *In* $[2n+1]$ *the scroll given by two* $[n]$*'s*, $[n+1]$, $[n+2]$, $[2n-2]$ *is of order* $6n-8$ *and genus* $2n-4$.

(14) *In* $[2n]$ *the scroll given by* $[n-1]$, *two* $[n]$*'s*, $[n+2]$, $[2n-3]$ *is of order* $6n-13$ *and genus* $2n-6$.

(15) *In* $[2n+1]$ *the scroll given by three* $[n]$*'s*, $[n+3]$, $[2n-2]$ *is of order* $6n-12$ *and genus* $2n-6$.

For all the scrolls in this paragraph the order of the directrix curve in a base $[r]$ is $r+p$, where p is the genus of the scroll.

Similarly we can prove that in $[2n]$ the incidence scroll given by an $[n-1]$, three $[n+1]$'s and one $[2n-4]$ is of order $8n-17$ and genus $3n-9$. From this and similar results we can deduce a series of scrolls, of order involving $8n$, but these scrolls do not occur in space of less than ten dimensions. However, we can obtain other simple surfaces by another series of theorems.

10. We first prove the following theorem:

(16) *In* $[n]$ *the scroll given by a line and* $n-1$ $[n-2]$*'s is the rational normal ruled surface of order* $n-1$ *with a directrix line.*

For supposing the theorem true in $[n]$, we can add an $[n-1]$ to the base without affecting the scroll, and separating the line and the $[n-1]$, as in § 5, we obtain the theorem in $[n+1]$. But the theorem is true in [3], so it is true in general.

11. Take, in $[m]$, a plane and m $[m-2]$'s, and let the lines which meet them generate a scroll of order x_m and genus p_m. If the plane and an $[m-2]$ lie in a prime S_{m-1} and meet in a line, the scroll degenerates into an $R^{x_{m-1}}_{p_{m-1}}$ in S_{m-1} and an R_0^{m-1} with $m-2$ generators in S_{m-1}.

Thus $$x_m = x_{m-1} + m - 1,$$

and $$p_m = p_{m-1} + m - 3.$$

But in [4] we know that an elliptic quintic scroll is generated by the lines which meet five planes; thus

$$x_m = \binom{m}{2} - 1, \quad p_m = \binom{m-2}{2},$$

and the order of the directrix curve in the base plane is seen to be $m-1$ (a curve without double points, as we should expect).

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

Separating the base plane and a base $[m-2]$, we see that, in $[m+1]$, the scroll given by a plane, an $[m-2]$ and $m-1$ $[m-1]$'s is of order $\binom{m}{2}$ and genus $\binom{m-2}{2}$. Thus, in $[m]$, the scroll given by a plane, an $[m-3]$, and $m-2$ $[m-2]$'s is of order $\binom{m-1}{2}$ and genus $\binom{m-3}{2}$. As before we may separate the plane and an $[m-2]$, and we see that, in $[m]$, the scroll given by a plane, two $[m-3]$'s, and $m-4$ $[m-2]$'s is of order $\binom{m-2}{2}+1$ and genus $\binom{m-4}{2}$. Continuing this process we obtain the following theorem:

(17) *In* $[m]$ *the scroll given by a plane,* i $[m-3]$*'s, and* $m-2i$ $[m-2]$*'s is of order* $\binom{m-i}{2}+i-1$ *and genus* $\binom{m-i-2}{2}$, the directrix curve in the plane being of order $m-i-1$.

This theorem is powerful, as it gives all the incidence scrolls whose bases contain a plane. For instance, the rational ones are given by taking $m=i+3$, and there are three of them, R_0^3 in [4], R_0^4 in [5], and R_0^5 in [6]. The elliptic ones are given by taking $m=i+4$, with $i\leqslant 4$ to make $m-2i\geqslant 0$ and these are R_1^5 in [4], R_1^6 in [5], ..., and R_1^9 in [8], only the first two being of general type.

The scrolls of genus three with a base plane are R_3^9 in [5], R_3^{10} in [6], ..., and R_3^{14} in [10]. These are of interest in connexion with a paper on special ruled surfaces which is to be published shortly.

12. It is possible to establish a theorem similar to (17) for solids instead of planes. For, in [5], the scroll given by seven solids is an R_8^{14}, and from this it may be proved inductively that in $[n]$ the scroll given by a solid and $n+1$ $[n-2]$'s is of order $\binom{n+1}{3}-n-1$ and genus $\binom{n}{3}+\binom{n-1}{3}-2n+4$, with a directrix curve in the solid of order $\binom{n}{2}-1$. The [3] and an $[n-2]$ may be "separated" in a way analogous to § 5, but with a rational ruled surface with a directrix line breaking off instead of a plane pencil. This introduces base $[n-3]$'s in $[n]$, and the solid and an $[n-3]$ may be separated to introduce $[n-4]$'s into the base. By a combination of these two processes all scrolls with a base solid may be obtained. However, the formulae are complicated, and most of the simpler surfaces have already been obtained.

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

Containing Space	Base Spaces [1] [2] [3] [4] [5] [6] [7] [8]	Scroll Order Genus	Theorems
[3]	3	2 0	(1), (3), (16)
[4]	1 3	3 0	(2), (5), (9), (16), (17)
	5	5 1	(4), (17)
[5]	1 4	4 0	(7), (16)
	3 1	4 0	(1), (6), (13), (17)
	2 3	6 1	(3), (8), (17)
	1 5	9 3	(17)
	7	14 8	(17)
[6]	1 5	5 0	(16)
	1 3	5 0	(2), (14), (17)
	1 2 2	7 1	(5), (10), (17)
	1 1 4	10 3	(17)
	1 6	14 6	(17)
	4 1	9 2	(4), (9)
	3 3	13 5	
	2 5	19 11	
	1 7	28 22	
	9	42 43	
[7]	1 6	6 0	(16)
	1 3 1	8 1	(7), (11), (17)
	1 2 3	11 3	(17)
	1 1 5	15 6	(17)
	1 7	20 10	(17)
	3 1	6 0	(1), (15)
	3 2	8 1	(6)
	2 2 1	10 2	(3), (13)
	2 1 3	14 5	
	2 5	20 11	
	1 4	12 3	(8)
	1 3 2	17 7	
[8]	1 7	7 0	(16)
	1 4	9 1	(12), (17)
	1 3 2	12 3	(17)
	1 2 4	16 6	(17)
	1 3	7 0	(2)
	1 2 1 1	11 2	(5), (14)
	1 2 3	15 5	
	1 1 3	13 3	(10)
	1 1 2 2	18 7	
	4 1	13 3	(4)
	3 2	15 4	(9)

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

Containing Space	Base Spaces [1] [2] [3] [4] [5] [6] [7] [8]	Scroll Order Genus	Theorems
[9]	1 8	8 0	(16)
	1 4 1	13 3	(17)
	1 3 2	17 6	(17)
	1 3 1	12 2	(7)
	1 2 2	14 3	(11)
	3 1	8 0	(1)
	3 1 1	12 2	(6), (15)
	2 2 1	14 3	(3)
	2 1 2	16 4	(13)
	1 3 1	18 5	(8)
[10]	1 9	9 0	(16)
	1 5	14 3	(17)
	1 4 2	18 6	(17)
	1 3 1	15 3	(12)
	1 3	9 0	(2)
	1 2 1 1	15 3	(5)
	1 2 2	17 4	(14)
	1 1 2 1	19 5	(10)
	4 1	17 4	(4)
	3 1 1	21 6	(9)

13. The above table shows the results of the theorems proved in this paper. The list of incidence scrolls is complete up to [6], but in the rest of the table I have only included the simpler scrolls. The surfaces given by the first 16 theorems are not "special", but soon become of very "particular" type, while the surfaces given by (17) soon become "special". In fact the only surfaces of general type which have appeared as incidence scrolls, apart from the rational ones, are the elliptic quintic in [4], the elliptic sextic in [5], and the novenic of genus two in [6].

The orders of the directrix curves of any surface in the table may easily be found by referring to the theorem involved. A few scrolls are mentioned in the table which do not depend on the theorems, but they tend to show that the simplest scrolls are given by the theorems.

14. We might have expected to obtain other surfaces by incidences. For instance, in [7] the rational sextic ruled surface with a directrix conic has ∞^2 directrix quartics and ∞^4 directrix quintics. By taking as base the plane of the directrix conic, three [4]'s containing directrix quartics, and one [5] containing a directrix quintic, we clearly obtain R_0^6, but we also obtain a quadric, the two scrolls forming a degenerate elliptic octavic surface.

Again, the general septimic surface in [6] has ∞^1 directrix quartics lying in solids and ∞^3 directrix quintics lying in [4]'s.

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press

Taking as base four of these solids and one [4] we must obtain an R_1^7, but there will again be a residual quadric, since four solids and a [4] in general position give an R_2^9. We can specialise the position of the base spaces by making the [4] contain the point of intersection of two of the solids and the point of intersection of the other two, so that two plane pencils split off.

Again, the general R_2^7 in [4] has four plane directrix quartics, but it cannot be obtained by incidences since there is no other plane which is met by all the generators. In fact scrolls which may be obtained directly by incidences are rare.

15. It should be noticed that in $[n]$ the fundamental incidence scroll is that generated by the lines which meet $2n-3$ $[n-2]$'s, all the others being portions of degenerate forms of this. For any $[n-2-i]$ in the base may be replaced by $i+1$ $[n-2]$'s through it, and if this is done for every space of the base we have $2n-3$ $[n-2]$'s, and the scroll determined by the base is part of the degenerate scroll of order $\frac{1}{n-1}\binom{2n-2}{n}$ determined by the $[n-2]$'s. The residual scrolls should be studied, but they are not as interesting as those given by a set of spaces in general position.

However, it is perhaps interesting to remark that in [4], if we take five planes, $\sigma_1, \ldots, \sigma_5$, of which three, $\sigma_1, \ldots, \sigma_3$, meet in a point O, the elliptic quintic scroll splits up into a plane pencil and a rational quartic scroll with a double point at O. For the lines through O which meet σ_4 and σ_5 form a pencil in a plane π which is part of the scroll so that there must be a residual R^4. The lines which meet σ_1, σ_2 and σ_3 lie in the planes ϖ of one system of a quadric point cone with vertex O, and each plane ϖ meets σ_4 and σ_5 in two points whose join is a generator of R^4. But the quadric cone meets the plane π in two lines through O, which are generators of R^4, so that the quartic scroll has a double point at O. Also, having two generators in common with the plane pencil, R^4 is rational, as is also obvious from the fact that R^4 is given by a (1, 1) correspondence between the two conics in which the quadric cone is met by σ_4 and σ_5.

Further, the general rational quartic scroll in [4] can be obtained in this manner, for it possesses a single infinity of directrix cubics which lie in planes through the unique double point, and we may take three of these planes and two conic planes as a base.

https://doi.org/10.1017/S0305004100011038 Published online by Cambridge University Press