NOTE ON A METHOD FOR COMPUTING INFINITE INTEGRALS OF OSCILLATORY FUNCTIONS

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1. *Formulation of the method.* In many branches of applied mathematics one is often faced with the numerical evaluation of integrals of the form $\int_a^\infty f(x)\,dx$, where a is a constant and where $f(x)$ oscillates about zero in such a way that the integral over each half-cycle is smaller in absolute magnitude than (and opposite in sign to) that over the preceding half-cycle. It is sometimes possible to evaluate such an integral by transforming it in the complex plane into the integral of a monotone decreasing function of constant sign (cf. also Luke (2)). The method indicated here shows an alternative procedure when such a transformation is undesirable or impossible.

The method is based on Euler's transformation of slowly convergent alternating series (Bromwich (1), p. 62). According to this transformation, if we have a series

$$V_0 - V_1 + V_2 - V_3 + V_4 - \ldots,$$

where $$V_n > 0, \quad V_{n+1} < V_n, \quad \text{for all } n,$$

and if we write $$\Delta V_n = V_{n+1} - V_n, \quad \Delta^{r+1} V_n = \Delta^r V_{n+1} - \Delta^r V_n,$$

then $$\sum_0^\infty (-1)^n V_n = \tfrac{1}{2}V_0 - \tfrac{1}{4}\Delta V_0 + \tfrac{1}{8}\Delta^2 V_0 - \ldots,$$

where the series on the right can be shown to be convergent whenever the original series is.

We can write down a simple expression for the remainder R_p after p terms if $V_n = \phi(n)$ where $\phi(x)$ is a function of x such that $\phi^{(p)}(x)$ has a fixed sign for all positive values of x, and steadily decreases in numerical value as x increases. It can then be shown (Bromwich (1)) that

$$|R_p| < 2^{-p} |\Delta^p V_0|.$$

Suppose that our function $f(x)$ has zeros (greater than a) at

$$x = x_i \quad (i = 1, 2, 3, \ldots),$$

with $a \leqslant x_1 < x_2 < x_3 < \ldots$, and suppose that we wish to evaluate $\int_a^\infty f(x)\,dx$. We split up our range of integration (Fig. 1) into

$$\int_a^\infty f(x)\,dx = \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \int_{x_2}^{x_3} f(x)\,dx + \ldots.$$

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We may suppose without loss of generality that $f(a) \geqslant 0$; if $f(a) = 0$ we take $x_1 = a$. Then the integrals over the separate ranges are alternately positive and negative and (after the first) decrease in absolute value:

$$\int_a^\infty f(x)\,dx = \int_a^{x_1} f(x)\,dx - \int_{x_1}^{x_2} \{-f(x)\}\,dx + \int_{x_2}^{x_3} f(x)\,dx - \ldots$$

$$= \int_a^{x_1} f(x)\,dx - (V_0 - V_1 + V_2 - V_3 + V_4 - \ldots),$$

where

$$V_0 - V_1 + V_2 - V_3 + V_4 - \ldots$$

is a slowly convergent alternating series. Then, according to the transformation,

$$\int_a^\infty f(x)\,dx = \int_a^{x_1} f(x)\,dx - (\tfrac{1}{2}V_0 - \tfrac{1}{4}\Delta V_0 + \tfrac{1}{8}\Delta^2 V_0 - \ldots).$$

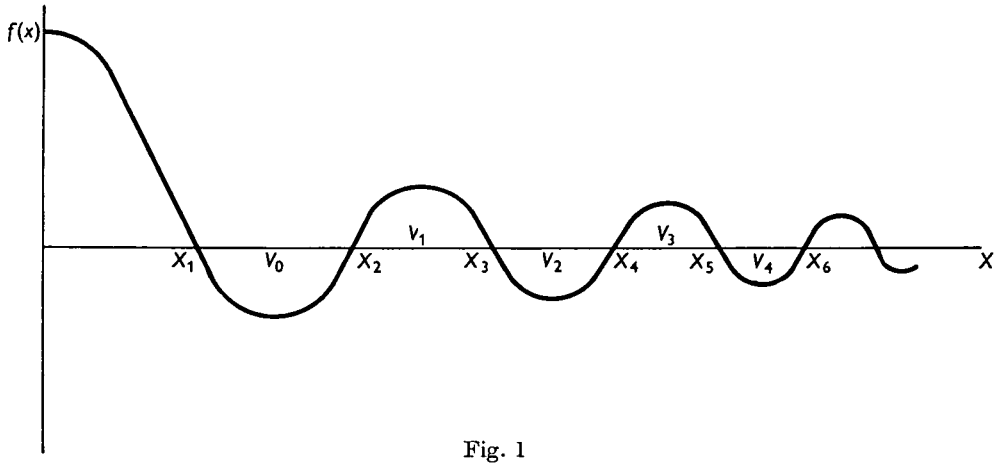


Fig. 1

2. *Illustration of the method.* (i) Consider the well-known integral

$$\int_0^\infty J_0(x)\,dx = 1,$$

where $J_0(x)$ is the Bessel function of order zero of the first kind. Table 1 is obtained from Watson ((4), p. 752):

Table 1

n	$\int_0^{x_n} J_0(x)\,dx$	$\pm\int_{x_n}^{x_{n+1}} J_0(x)\,dx$	Δ^2	Δ^4	Δ^6
1	1·470442				
		0·801596			
2	0·668846				
		0·599322			
3	1·268168				
		0·499049			
4	0·769119		−62514		
		0·436535		18802	
5	1·205654		−43712		−7865
		0·392823		10946	
6	0·812831		−32766		−3916
		0·360057		7030	
7	1·172888		−25736		−2200
		0·334321		4830	
8	0·838567		−20906		−1343
		0·313415		3487	
9	1·151982		−17419		− 872
		0·295996		2615	
10	0·855986		−14804		− 596
		0·281192		2019	
11	1·137178		−12785		
		0·268407			
12	0·868771				

Δ^4	Δ^6	
3940		
	−2224	
1716		1365
	− 859	
857		473
	− 386	
471		191
	− 195	
276		

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Choosing the second advancing row of differences in the table, we get

$$\int_0^\infty J_0(x)\,dx = 0{\cdot}769119 + \tfrac{1}{2}(0{\cdot}436535) + \tfrac{1}{4}(0{\cdot}043712) + \tfrac{1}{8}(0{\cdot}010946) + \tfrac{1}{16}(0{\cdot}003916) + \tfrac{1}{32}(0{\cdot}001716) + \tfrac{1}{64}(0{\cdot}000859) + \tfrac{1}{128}(0{\cdot}000473) + \tfrac{1}{256}(0{\cdot}000282) = 0{\cdot}999999.$$

(ii) Consider the integral

$$\int_0^\infty \frac{J_0(2x)\,x(x^2+\frac{1}{3})^{\frac{1}{2}}\,[2x^2\exp\{-\frac{1}{5}(x^2+1)^{\frac{1}{2}}\} - (2x^2+1)\exp\{-\frac{1}{5}(x^2+\frac{1}{3})^{\frac{1}{2}}\}]}{(2x^2+1)^2 - 4x^2(x^2+\frac{1}{3})^{\frac{1}{2}}\,(x^2+1)^{\frac{1}{2}}}\,dx,$$

which was recently given to the author for evaluation by Prof. C. L. Pekeris.

Numerical integration was performed over the ranges $(0, x_1)$, (x_1, x_2), (x_2, x_3), ..., (x_7, x_8), where x_i $(i = 1, \ldots, 8)$ are the first eight zeros of $J_0(2x)$. The results obtained are given in Table 2 and differenced.

Table 2

n	$\pm\int_{x_n}^{x_{n+1}} f(x)\,dx$	Δ^2	Δ^4	
0	0·145234*			
1	0·206401			
		−42062		
2	0·150723		12689	
		−29373		−4253
3	0·108661		8436	
		−20937		−2685
4	0·079288		5751	
		−15186		−1702
5	0·058351		4049	
		−11137		
6	0·043165			
7	0·032028			

	Δ^6
1568	
983	−585

* This figure is the value of $-\int_0^{x_1} f(x)\,dx$.

Choosing the second row of advancing differences we have

$$\int_0^\infty f(x)\,dx = -0{\cdot}145234 + 0{\cdot}206401 - [\tfrac{1}{2}(0{\cdot}150723) + \tfrac{1}{4}(0{\cdot}042062) + \tfrac{1}{8}(0{\cdot}012689) + \tfrac{1}{16}(0{\cdot}004253) + \tfrac{1}{32}(0{\cdot}001568) + \tfrac{1}{64}(0{\cdot}000585)] = -0{\cdot}026620.$$

Here the last figure has little meaning in view of the magnitude of the last difference taken into account, and so we write the result as

$$\int_0^\infty f(x)\,dx = -0{\cdot}02662.$$

The integral above has also been computed by numerical integration after transformation in the complex plane. The value obtained was −0·026614.

3. *Alternative method.* A variation on the present method has been suggested to the author by J. C. P. Miller in a private communication. According to this, instead of integrating from one zero of the function to the next, one integrates between numbers $x_{n+\frac{1}{2}}$ which are obtained by interpolating between the zeros x_n of the function to half-integral values of n. That is to say, we integrate between (precise) points which are

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near to the turning points of $f(x)$, but not at these turning points, and then obtain the integral in the form of an alternating series with smaller terms. Less integration is then needed to get the same accuracy after the use of Euler's transformation. In the case where the zeros of $f(x)$ arise from a Bessel function factor $J_r(x)$, we may use the zeros of $Y_r(x)$, the Bessel function of the second kind of order r.

4. *Illustration of alternative method.* The method is illustrated by considering again the above examples.

(i) The integral $\int_0^\infty J_0(x)\,dx$. The following values of $\int_0^{y_i} J_0(x)\,dx$ $(i = 1, 2, \ldots, 9)$, where y_i is the ith zero of $Y_0(x)$, were obtained partly by interpolating in a table (3) of $\int_0^x J_0(t)\,dt$ and partly by use of the series

$$\int_0^x J_0(t)\,dt = 2[J_1(x) + J_3(x) + J_5(x) + \ldots].$$

Table 3

i	$\int_0^{y_1} J_0(x)\,dx$	$\pm\int_{y_i}^{y_{i+1}} J_0(x)\,dx$	Δ^2	Δ^4	Δ^6	
1	1·041597					
		0·061149				
2	0·980448		−29883			
		0·031266	−11586	18297		
3	1·011714			5723	−12574	
		0·019680	−5863	2425	−3298	9276
4	0·992034			1218	−1207	2091	−7185
		0·013817	−3438	696	−522	685	−1406	5779
5	1·005851			417	−279	243	−442	964	−4817
		0·010379	−2220					
6	0·995472		−1524					
		0·008159						
7	1·003631		−1107					
		0·006635						
8	0·996996							
		0·005528						
9	1·002524							

From Table 3 we get for $\int_0^\infty J_0(x)\,dx$, using the third row of advancing differences,

$$\begin{aligned}\int_0^\infty J_0(x)\,dx &= 1{\cdot}011714 - [\tfrac{1}{2}(0{\cdot}019680) + \tfrac{1}{4}(0{\cdot}005863) + \tfrac{1}{8}(0{\cdot}002425)\\ &\quad + \tfrac{1}{16}(0{\cdot}001207) + \tfrac{1}{32}(0{\cdot}000685) + \tfrac{1}{64}(0{\cdot}000442)]\\ &= 1{\cdot}000001.\end{aligned}$$

This example illustrates a definite advantage in using the method suggested by Miller, since when we used the zeros of $J_0(x)$, we had to integrate from $x = 0$ to

$$x = x_{12} = 36{\cdot}9170984,$$

while using the zeros of $Y_0(x)$ we can obtain the same accuracy by integrating from $x = 0$ to $x = x_{9\frac{1}{2}} = y_9 = 25{\cdot}9229577$.

Now consider again the second example treated above. The following values of

$$\int_{y_{i-1}}^{y_i} \frac{J_0(2x)\,x(x^2+\frac{1}{3})^{\frac{1}{2}}[2x^2\exp\{-\frac{1}{5}(x^2+1)^{\frac{1}{2}}\} - (2x^2+1)\exp\{-\frac{1}{5}(x^2+\frac{1}{3})^{\frac{1}{2}}\}]}{(2x^2+1)^2 - 4x^2(x^2+\frac{1}{3})^{\frac{1}{2}}(x^2+1)^{\frac{1}{2}}}\,dx$$

$(i = 1, 2, \ldots, 9)$,

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where y_i are the zeros of $Y_0(2x)$ and $y_0 = 0$, were obtained by numerical integration. The values are differenced as shown below in Table 4. From this table we have

$$\int_0^\infty f(x)\,dx = -0{\cdot}045402 + 0{\cdot}008325 + 0{\cdot}018338 - [\tfrac{1}{2}(0{\cdot}013311) + \tfrac{1}{4}(0{\cdot}004027) + \tfrac{1}{8}(0{\cdot}001375) + \tfrac{1}{16}(0{\cdot}000546) + \tfrac{1}{32}(0{\cdot}000262)]$$

$$= -0{\cdot}026615.$$

Table 4

n	$\pm\int_{y_{n-1}}^{y_n} f(x)\,dx$	Δ^2		Δ^4
1	0·045402			
2	0·008325			
3	0·018338	−5027		
4	0·013311	−4027	1000	
5	0·009284	−2652	1375	−546
6	0·006632	−1823	829	−284
7	0·004809	−1278	545	262
8	0·003531			

(Here $y_0 = 0$.)

Here again we have a slight improvement in accuracy. (We could also have obtained the same accuracy as before with less numerical integration.)

It thus appears that the suggestion of Miller represents a definite improvement on the proposed method in the case where the zeros x_n come from a Bessel function factor (or of course a sine or cosine factor) where the values of $x_{n+\frac{1}{2}}$ can be obtained from tables, but it is the opinion of the author that the method is difficult to use in the case where the $x_{n+\frac{1}{2}}$ must be obtained by actual interpolation. The author has in fact tried to interpolate between the zeros of $J_0(x)$, and it was not found possible to interpolate accurately between the lower zeros without taking into account a very large number of zeros of $J_0(x)$.

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REFERENCES

(1) Bromwich, T. J. I'A. *An introduction to the theory of infinite series* (Macmillan, 1942).
(2) Luke, Y. L. *Proc. Camb. Phil. Soc.* 50 (1954), 269.
(3) *Table of functions and of zeros of functions* (U.S. Dep. of Commerce—Nat. Bureau of Standards Appl. Math. Series 37 (1954)).
(4) Watson, G. N. *A treatise on the theory of Bessel functions* (Cambridge, 1948).

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