Note on the modified Heine-Borel Theorem.

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§1. The Heine-Borel Theorem for one-dimensional intervals may be enunciated as follows:

(A) If a set D of intervals, all in the closed interval (a, b), be such that every point of (a, b) is an interior point of at least one interval of the set D (the end points a, b being regarded as interior to an interval when either of them is an end point of such interval), then a finite set E of intervals all belonging to D exists such that every point of (a, b) is interior to at least one interval of E.

It is sometimes convenient to use modified forms of this theorem. One such form is given by Whittaker and Watson (*Modern Analysis* (1927) 53). It reads:

(B) Given (i) a straight line CD and (ii) a law by which, corresponding to each point P of CD we can determine a closed interval $I(P)$ of CD, P being an interior point of $I(P)$, except when P is at C or D, when it is an end point; then the line CD can be divided into a finite number of closed intervals $J_1, J_2, \ldots, J_k$, such that each interval J_r contains at least one point (not an end point) P_r, such that no point of J_r lies outside $I(P_r)$.

Whittaker and Watson attempt to prove (B) by the following argument.

A closed interval contained in CD, which possesses the property which (B) asserts CD to possess, is called a suitable interval. On the assumption that CD is not a suitable interval, they infer by the method of repeated bisections that there exists a sequence s_n of intervals contained in CD such that

(i) the length of s_n is $2^{-n}\,CD$,

(ii) no point of s_{n+1} is outside s_n,

(iii) the interval s_n is not a suitable interval.

The sequence s_n tends to a point Q. From and after a certain value of n, s_n is interior to $I(Q)$. They then assert that this implies that s_n is a suitable interval.

This, however, is not so. For it may happen that from and after a certain n, all the intervals s_n have an end point in common, namely Q. In this case, Q is not an interior point of s_n, and hence by the definition of a suitable interval, it does not follow that s_n is such an interval.

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Proposition (B) as it stands requires correction. For it is easy to construct examples in which the point P_1 must be the point C, and the point P_k must be the point D. In this case they are not interior points of J_1, J_k respectively. We expect, however, that if (B) be amended so as to allow for this possibility, the proposition will become true. We therefore consider

(C) Proposition (B) is true provided that P_1 be allowed to be the left-hand end point of J_1 and P_k the right-hand end point of J_k.

§ 2. Concerning the proof of (C) we remark that the method of Whittaker and Watson breaks down. We can, however, proceed as follows:

The theorem is true for all intervals $C\xi$ contained in $I(C)$. If the theorem is false, the points x with the property that the theorem is true for Cx have an upper bound G, and $G \leqslant D$. Suppose first that $G < D$.

If the theorem is true for CG, let $J_1, J_2, \ldots J_k$ be the division of CG into closed intervals. Then J_k cannot be $I(G)$ since G is an end point of J_k and an interior point of $I(G)$. If $P_k \equiv G$, then

$$J_1,\ J_2,\ \ldots,\ J_{k-1},\ (J_k + GG'),$$

where G' is the right-hand end point of $I(G)$, is a decomposition of CG' which satisfies the conditions of the theorem. Since $G' > G$ we have a contradiction with the definition of G.

Suppose, then, that P_k is an interior point J_k. Let the left-hand end point of J_k be denoted by α_k. Since $I(P_k)$ contains J_k, we can find a point β such that $P_k < \beta < G$, and such that β is an interior point of $I(P_k)$. Then the intervals

$$J_1,\ J_2,\ \ldots,\ J_{k-1},\ (\alpha_k, \beta), (\beta G')$$

will be a decomposition of CG' which satisfies the conditions of the theorem; again a contradiction.

We now suppose that the theorem is false for CG. We can find a point $L < G$ such that L is interior to $I(G)$ and such that the theorem is true for CL. Let $J_1, J_2, \ldots, J_k$ be the division of CL into closed intervals. If P_k is interior to J_k, the intervals

$$J_1,\ J_2,\ \ldots,\ J_k,\ LG'$$

will be a suitable decomposition of CG'. If $P_k \equiv L$, let the end points of J_k be λ, L. Then we can find a point λ' satisfying $L < \lambda' < G$, when λ' is interior to $I(L)$ and

$$J_1,\ J_2,\ \ldots,\ J_{k-1},\ (\lambda\lambda'), (\lambda' G')$$

is a suitable decomposition of CG'; again a contradiction.

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We finally consider the case $G = D$. Here, if the theorem is true for CG, there is nothing to prove. If not, we can find the point L as before. If P_k is interior to J_k,

$$J_1,\ J_2,\ \ldots,\ J_k LG$$

is a suitable decomposition of CD. If $P_k \equiv L$,

$$J_1,\ J_2,\ \ldots,\ J_{k-1},\ (\lambda\lambda'),\ (\lambda' G)$$

is a suitable decomposition; a contradiction in both cases. This completes the proof.

The above proof is adapted from an argument given by Carathéodory for proving (A). The corresponding proposition for space of higher dimensions may be proved by induction*.

* Cf. Carathéodory, *Vorlesungen über Reelle Funktionen* (1927), 42–45.

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