Math. Proc. Camb. Phil. Soc. (1978), **83**, 37
Printed in Great Britain

Dirichlet's theorem on diophantine approximation

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(*Received* 24 *August* 1976)

1. *Introduction.* Let N take the values 1, 2, ... A theorem of Dirichlet asserts that for any (x, y) in the Euclidean plane R_2 the inequality

$$|q_1 x + q_2 y - p| < N^{-2}$$

is soluble in integers q_1, q_2, p with $0 < \max(|q_1|, |q_2|) \leqslant N$.

Suppose now $\mu < 1$. For almost all (x, y) in R_2 there are infinitely many N for which the inequalities

$$|q_1 x + q_2 y - p| < \mu N^{-2}, \quad 0 < \max(|q_1|, |q_2|) \leqslant N$$

are insoluble in integers q_1, q_2, p. This follows from a deep theorem of H. Davenport and W. M. Schmidt (4).

Delicate problems arise if we assume a functional relationship between x and y. One result of this kind has been proved, by Davenport and Schmidt (3).

THEOREM A. *Suppose* $0 < \mu < \frac{1}{4}$. *For almost all* x *there are infinitely many* N *for which the inequalities*

$$|q_1 x + q_2 x^2 - p| < \mu N^{-2}, \quad 0 < \max(|q_1|, |q_2|) \leqslant N$$

are insoluble in integers q_1, q_2, p.

In the present paper I replace the curve $y = x^2$ by a smooth curve, whose curvature is at most λ (say) and is almost everywhere non-zero. The gain in generality is balanced by a reduction of the constant $\frac{1}{4}$ to a small value depending on λ.

Throughout the paper let $x(s)$, $y(s)$ $(a \leqslant s \leqslant b)$ denote a curve Γ in R_2, the parameter s being arc length. We always suppose $x'''(s)$, $y'''(s)$ exist and are continuous. We write $k(s)$ for the curvature,

$$k(s) = x'(s)\, y''(s) - y'(s)\, x''(s).$$

THEOREM 1. *Suppose* $k(s)$ *is almost everywhere non-zero and* $|k(s)| \leqslant \lambda$ $(a \leqslant s \leqslant b)$. *Define* $\chi = \min(10^{-3}, 10^{-8}\lambda^{-1/3})$. *For almost all* s *there are infinitely many* N *for which the inequalities*

$$|q_1 x(s) + q_2 y(s) - p| < \chi N^{-2} \quad (0 < q \leqslant N) \tag{1}$$

are insoluble in integers q_1, q_2, p. *Here and subsequently* q *denotes max* $(|q_1|, |q_2|)$.

THEOREM 2. *Suppose* $k(s)$ *is non-zero for almost all* s. *Let* $\sum_{n=1}^{\infty} \psi(n)$ *be a convergent series of positive monotonic decreasing terms. Then for almost all* s *the inequality*

$$|q_1 x(s) + q_2 y(s) - p| < \psi^2(q) \tag{2}$$

has at most finitely many solutions in integers q_1, q_2, p.

Theorem 2 is a slightly sharpened form of a theorem of Schmidt (7). For the curve

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$y = x^2$ a still sharper result can be proved ((8), chapter I, p. 33). The sharpening is important for the proof of Theorem 3 below.

For complex numbers θ, ϕ write $w(\theta, \phi)$ for the supremum of the positive numbers w for which

$$|q_1\theta + q_2\phi - p| < q^{-w} \quad \text{for infinitely many integer triples } q_1, q_2, p.$$

By Dirichlet's theorem $w(x, y) \geqslant 2$ for all (x, y) in R_2, and the hypotheses of Theorem 2 clearly imply $w(x(s), y(s)) = 2$ for almost all s.

THEOREM 3. *Suppose $k(s)$ is non-zero except for a set of s of Hausdorff dimension 0. Write*

$$X(\Gamma, \mu) = \{s \in [a, b] : w(x(s), y(s)) \geqslant \mu\},$$

$$X'(\Gamma, \mu) = \{s \in [a, b] : w(x(s), y(s)) = \mu\}.$$

Then for all $\mu > 2$ the sets $X(\Gamma, \mu)$ and $X'(\Gamma, \mu)$ both have Hausdorff dimension $3/(\mu+1)$.

Theorem 3 was proved for the particular curve (x, x^2) by A. Baker and W. M. Schmidt (1). For corresponding results for $(x, x^2, \ldots, x^k)$ (where $k \geqslant 3$) see (2).

The following theorem of V. G. Sprindžuk ((8), chapter I, lemma 14) concerns planar Lebesgue measure.

THEOREM B. *Let m, n be integers, $1 \leqslant m < n$. Let $\sum_{q=1}^{\infty} \psi(q)$ be a convergent series of positive terms. For almost all complex θ, the inequality*

$$|q_1\theta^n + q_2\theta^m - p| < \psi^{\frac{1}{2}}(k), \quad k = \max(|q_1|, |q_2|, p)$$

has at most finitely many solutions in integers q_1, q_2, p.

In analogy with Theorem 3 we have

THEOREM 4. *Define $Y(\mu) = \{\theta : \theta$ non-real, $w(\theta^n, \theta^m) \geqslant \mu\}$ and $Y'(\mu) = \{\theta : \theta$ non-real, $w(\theta^m, \theta^n) = \mu\}$. Then for $1 \leqslant m < n$, $\mu > \frac{1}{2}$ we have*

$$\dim Y(\mu) = \dim Y'(\mu) = 3/(\mu+1).$$

Since the proof of Theorem 1 is long and involved, we give here an outline of the principal steps, at the same time explaining the connections with previous work (3, 6, 7). $\| \ \|$ denotes distance from the nearest integer.

(*a*) Let $U = (a_0 - \delta, a_0 + \delta)$ be an interval in (a, b) and write for a lattice point $\mathbf{q} = (q_1, q_2)$,

$$E(\mathbf{q}) = \{s \in U : \|Y(\mathbf{q}, s)\| < \chi N^{-2}\},$$

where $Y(\mathbf{q}, s) = q_1 x(s) + q_2 y(s)$. Apart from the value of the constant, Theorem 1 follows if we can give an estimate for the Lebesgue measure of

$$\bigcup_{0 < q \leqslant N} E(\mathbf{q})$$

of the type

$$\Big|\bigcup_{0 < q \leqslant N} E(\mathbf{q})\Big| \leqslant F_\lambda(\chi)\,\delta,$$

where $F_\lambda(\chi) \to 0$ as $\chi \to 0+$. This reduction uses the 'density principle' just as in (3).

(*b*) The sum

$$\sum_{0 < q \leqslant N} |E(\mathbf{q})|$$

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is split into two parts, according as

$$|Y'(\mathbf{q}, a_0)| < C\tilde{q}\delta \tag{i}$$

or not, where $\tilde{q} = (q_1^2 + q_2^2)^{\frac{1}{2}}$ and C is a suitable numerical constant, $C > 1$. In the 'large derivative' case there is a fairly obvious bound for $|E(\mathbf{q})|$. Here again I follow (3).

(*c*) In the case (i), $(d/ds)\, Y(\mathbf{q}, s)$ can be expressed in the form

$$\tilde{q} \sin(\alpha(s) - \beta),$$

where $\alpha(s)$ is the angle between the curve and the x-axis, and $\beta = \tan^{-1}(-q_1/q_2)$. Now $\beta = \alpha(s_0)$ for some s_0 and $Y'(s_0) = 0$. If δ is small, then

$$Y''(s) = \tilde{q} \cos(\alpha(s) - \beta) . k(s)$$

is of the same order as q throughout U. This enables us to bound suitably

$$\sum_{0<q\leqslant N} \sum_{p \neq p_0} |\{s: |Y(\mathbf{q}, s) - p| < \chi N^{-2}\}|,$$

where p_0 is the nearest integer to $Y(s_0)$. The ideas here come mostly from (7), but (3) is used slightly (for the last time).

(*d*) To estimate

$$\sum_{0<q\leqslant N} |\{s: |Y(\mathbf{q}, s) - p_0| < \chi N^{-2}\}|$$

we combine the ideas of (*c*) with a bound for the number of $\mathbf{q}$, $q \leqslant N$, with

$$\| Y(\mathbf{q}, s_0) \| \ll q^{-\nu},$$

where $\nu \in [1-\epsilon, \frac{3}{2}+\epsilon]$. Here one can reduce the problem to counting solutions of

$$\|g(x, y)\| \ll x^{-\nu} \quad (1 \leqslant x \leqslant k), \tag{ii}$$

where g is a certain homogeneous function and y varies over a sector $t_1 x \leqslant y \leqslant t_2 x$, narrow in the sense that

$$t_2 - t_1 < C(\lambda)\, \delta. \tag{iii}$$

This reduction closely follows (7).

(*e*) Finally, to count solutions of (ii), we employ the method of (6), which bounds the number of lattice points in double sequences using combinatorial and geometrical notions. As the condition (iii) does not occur in (6) a good deal of new calculation is needed to apply Schmidt's method. This is carried out in section 2.

These ideas are used again to prove Theorem 2, except that (6) is now directly applicable. Similarly for the proof of $\dim X(\Gamma, \mu) \leqslant 3/(\mu+1)$. The reverse inequality depends on the ideas of (1), especially on the method of A. Baker and Schmidt for constructing well-spaced algebraic numbers. We adapt their argument to get well-spaced points on the curve Γ which also lie on lines

$$q_1 x + q_2 y - p = 0.$$

Finally Theorem 4 is easier, as a more direct method of Sprindžuk (8) yields $\dim Y(\mu) \leqslant 3/(\mu+1)$ quite quickly.

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2. *Lattice points near a cone.* This section contains some general estimates concerning lattice points in R_2 that will be needed in section 4. We write $\mathbf{u}$, $\mathbf{v}$, for lattice points in R_2, that is, points with integer coordinates. We write $\tilde{u} = (u_1^2+u_2^2)^{\frac{1}{2}}$ where $\mathbf{u} = (u_1, u_2)$.

Let σ, τ be real numbers, $\sigma < \tau$. We write M_0 for the set of non-zero lattice points in R_2, $M_1 = M_1(\sigma, \tau)$ for the set of $\mathbf{u} = (u_1, u_2)$ in M_0 for which

$$u_1 > 0 \quad \text{and} \quad \sigma u_1 \leqslant u_2 \leqslant \tau u_1.$$

Let M_0 be totally ordered by a relation $\prec$.

Let $\mathscr{U}$ be a finite sequence $\mathbf{u}_1, \ldots, \mathbf{u}_n$ in M_0. We say a pair $\mathbf{v}$, $\mathbf{v}'$ *occurs in* $\mathscr{U}$ if for some i $(1 \leqslant i < n)$, we have $\mathbf{v} = \mathbf{u}_i$, $\mathbf{v}' = u_{i+1}$.

We write

$$n(\mathscr{U}) = n, \quad l(\mathscr{U}) = \sum_{i=1}^{n} \tilde{\mathbf{u}}_i.$$

$\mathscr{U}$ is said to be *regular* if $\mathbf{u}_i \prec \mathbf{u}_j$ for $1 \leqslant i < j \leqslant n$. We allow the possibility that $n = 0$ and $\mathscr{U}$ is empty.

Let $D \geqslant 1$. Let $\mathscr{U}_1, \ldots, \mathscr{U}_A$ be finite sequences of lattice points. We call $\mathscr{U}_1, \ldots, \mathscr{U}_A$ a *D-regular double sequence* if

(a) each $\mathscr{U}_i$ is regular,

(b) no pair $\mathbf{v}, \mathbf{v}'$ with $\tilde{\mathbf{v}} \geqslant D$, $\tilde{\mathbf{v}}' \geqslant D$ occurs in two sequences $\mathscr{U}_i$. By a theorem of Schmidt (6), the bound

$$\sum_{i=1}^{A} l(\mathscr{U}_i) \leqslant C \tag{3}$$

implies

$$\sum_{i=1}^{A} n(\mathscr{U}_i) \leqslant 150 C^{\frac{2}{3}} A^{\frac{1}{3}} + 40AD^2, \tag{4}$$

whenever $\mathscr{U}_1, \ldots, \mathscr{U}_A$ is a D-regular double sequence in M_0.

Let M be a subset of M_0 with at least k elements. We write

$$l_k(M) = \min(\tilde{\mathbf{v}}_1 + \ldots + \tilde{\mathbf{v}}_k)$$

where the minimum is taken over all k-tuples of different points $\mathbf{v}_i \in M$. The proof of 4) depe nds (in the first instance) on the inequality

$$l_k(M_0) \geqslant k^{\frac{3}{2}}/8. \tag{5}$$

If all the points of $\mathscr{U}_1, \ldots, \mathscr{U}_A$ lie in the (narrow) sector M_1, (4) can be strengthened.

LEMMA 1. *Let $\mathscr{U}_1, \ldots, \mathscr{U}_A$ be a D-regular double sequence in M_1 satisfying* (3). *Then*

$$\sum_{i=1}^{A} n(\mathscr{U}_i) \leqslant 150(\tau-\sigma)^{\frac{1}{3}} C^{\frac{2}{3}} A^{\frac{1}{3}} + 40AD^2$$

provided $A > A_1$. Here $A_1 = A_1(\sigma, \tau)$.

Proof. We first show that

$$l_k(M_1) > (\tau-\sigma)^{-\frac{1}{2}} k^{\frac{3}{2}}/8 \tag{6}$$

provided $k > k_1 = k_1(\sigma, \tau)$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be distinct points of M_1, ordered so that $\tilde{\mathbf{v}}_1 \leqslant \tilde{\mathbf{v}}_2 \leqslant \ldots \leqslant \tilde{\mathbf{v}}_k$. The unit squares $\mathbf{v}_i + [0,1)^2$ whose bottom left-hand vertices are $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are disjoint and lie in the plane region

$$1 \leqslant x \leqslant \tilde{\mathbf{v}}_r + 1, \quad \sigma x - |\sigma| \leqslant y \leqslant \tau x + 1 + |\tau|$$

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whose area is

$$(\tau-\sigma)\left(\frac{\tilde{\mathbf{v}}_r^2}{2}+\tilde{\mathbf{v}}_r\right)+(|\tau|+|\sigma|+1)\,\tilde{\mathbf{v}}_r<(\tau-\sigma)\,\tilde{\mathbf{v}}_r^2$$

if $r > r_1(\sigma,\tau)$. From this we can deduce that

$$r<(\tau-\sigma)\,\tilde{\mathbf{v}}_r^2\ (r>r_1(\sigma,\tau))$$

so that

$$\tilde{\mathbf{v}}_1+\ldots+\tilde{\mathbf{v}}_k>(\tau-\sigma)^{-\frac{1}{2}}\sum_{r=r_1+1}^{k} r^{\frac{1}{2}}$$

$$>\tfrac{1}{2}(\tau-\sigma)^{-\frac{1}{2}}k^{\frac{3}{2}}$$

if $k > k_1(\sigma,\tau)$; this proves (6).

Now the proof of Lemma 1 may be completed by repeating almost verbatim the proof of Satz 5 of (6), but using (6) in place of (5).

The principal application of (4) in (6) is

THEOREM C. *Let $0<\lambda_1<\lambda_2$. For natural numbers q let $\mathscr{B}_q$ denote the region $1\leqslant x\leqslant q$, $\lambda_1 x\leqslant y\leqslant\lambda_2 x$.*

Let $g(x,y)$ be defined for $x>0$, $\lambda_1 x\leqslant y\leqslant\lambda_2 x$ and homogeneous of degree 1,

$$g(tx,ty)=tg(x,y)\quad \textit{for}\quad t>0.$$

Further let $g_y(x,y)=l(y/x)$, where the function $l(t)$ is twice continuously differentiable in $\lambda_1\leqslant t\leqslant\lambda_2$ and $l'(t)$ is positive $(\lambda_1\leqslant t\leqslant\lambda_2)$.

Denote by $N_\mu(q)$ the number of integral $(x,y)\in\mathscr{B}_q$ with

$$\|g(x,y)\|<x^{-\mu}. \tag{7}$$

Then for $1<\mu\leqslant\frac{3}{2}$,

$$N_\mu(q)\ll q^{\frac{3}{2}+(\frac{3}{2}-\mu)/7}.$$

We remark here that the constant implied by '$P\ll Q$' is always independent of the quantities N, δ, x, H, q_1, q_2, p, k, q, h appearing in the expressions P, Q.

The purpose of Lemma 1 is to enable us to prove a modified version of Theorem C.

PROPOSITION 1. *Let g be as in Theorem C and suppose $\lambda_1\leqslant\nu_1<\nu_2\leqslant\lambda_2$. Denote by $M_\mu(q)$ the number of integral (x,y) satisfying (7) in the subregion $\nu_1 x\leqslant y\leqslant\nu_2 x$ of $\mathscr{B}_q$. Write*

$$\kappa=\sup_{[\lambda_1,\lambda_2]}|l(t)|,\quad \kappa_1=\sup_{[\lambda_1,\lambda_2]}|l'(t)|.$$

Then for $\mu>\frac{3}{2}$,

$$M_\mu(q)<151(2+\kappa)^{\frac{2}{3}}\kappa_1^{\frac{1}{3}}(\nu_2-\nu_1)\,q^{\frac{3}{2}}$$

for all sufficiently large q.

We require several more lemmas for the proof of Proposition 1. In the next lemma let the real functions $\phi_1(x)$, $\phi_2(x)$ be defined and differentiable in $1\leqslant x\leqslant A$, and suppose $0<\phi_2(x)-\phi_1(x)\leqslant B$. Here A is an integer, $A\geqslant 2$. Let $\mathscr{B}$ denote the region $1\leqslant x\leqslant A$, $\phi_1(x)\leqslant y\leqslant\phi_2(x)$ of R_2. We now totally order the points of R_2 in such a way that for $\mathbf{v}=(v_1,v_2)$, $\mathbf{w}=(w_1,w_2)$, and $v_1>0$, $w_1>0$,

$$v_2/v_1<w_2/w_1\quad\text{implies}\quad \mathbf{v}\prec\mathbf{w}.$$

LEMMA 2. *Suppose $f(x,y)$ is a real valued function on $\mathscr{B}$ and the partial derivatives*

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$f_y, f_{yy}, f_{yyy}, f_{yx}, f_{yyx}$ exist and are continuous on $\mathscr{B}$. Suppose f_y grows monotonically along each curve $y = \phi_1(x)$, $y = \phi_2(x)$. Suppose that throughout $\mathscr{B}$,

$$|f_y| \leqslant \kappa, \quad f_{yy} > 0,$$

and

$$\mathscr{D}(x,y) = \begin{vmatrix} f_{yy} & f_{yx} \\ f_{yyy} & f_{yyx} \end{vmatrix} \neq 0.$$

Denote by $E(i)$ the minimum of $|\mathscr{D}f_{yy}^{-1}|$ in the subregion $i \leqslant x \leqslant i+1$ of $\mathscr{B}$. Let $\delta(1), \ldots, \delta(A)$ satisfy $0 < \delta(i) < \frac{1}{2}$ $(i = 1, \ldots, A)$. Let D be real, $D \geqslant 1$. Suppose that

(i) $D^3 > 4 + 2\kappa$,

(ii) $f_{yy}(i,y) > 4\delta(i) \quad (1 \leqslant i \leqslant A, \phi_1(i) \leqslant y \leqslant \phi_2(i))$,

(iii) $(8+4\kappa) f_{yy}(i,y) > D|f_{yyy}(i,y)| \quad (1 \leqslant i \leqslant A, \phi_1(i) \leqslant y \leqslant \phi_2(i))$,

(iv) $D^2 E(i) > 64(2+\kappa)^2 \max(\delta(i), \delta(i+1)) \quad (1 \leqslant i \leqslant A-1)$.

Let i denote any of $1, 2, \ldots, A$. If there are fewer than 2 integer solutions of

$$|z - f(i,y)| < \delta(i), \quad \phi_1(i) + D^2 \leqslant y \leqslant \phi_2(i) - D^2,$$

let $\mathscr{U}_i$ be empty. If there are at least two solutions, say $(y_1, z_1), \ldots, (y_k, z_k)$, where $y_1 < \ldots < y_k$, let $\mathscr{U}_i$ be the sequence $\mathbf{u}_1, \ldots, \mathbf{u}_{k-1}$ where $\mathbf{u}_j = (y_{j+1} - y_j, z_{j+1} - z_j)$ $(1 \leqslant j \leqslant k-1)$. Then

(a) $l(\mathscr{U}_i) \leqslant (2+\kappa) B$ $(i = 1, \ldots, A)$,

(b) $\mathscr{U}_1, \ldots, \mathscr{U}_A$ is a D-regular double sequence.

Proof. Essentially this is the content of Hilfssätze 10, 11 and 12 of (6). The only nontrivial difference is that $\mathscr{D}(x,y)$ is assumed positive in $\mathscr{B}$ in proving (b). If $\mathscr{D}(x,y) < 0$ in $\mathscr{B}$, very slight changes are needed in the argument of (6).

LEMMA 3. *Suppose all the hypotheses of Lemma 2 are satisfied. Suppose further that*

$$\sigma \leqslant f_y(i,y) \leqslant \tau \quad (1 \leqslant i \leqslant A, \phi_1(i) \leqslant y \leqslant \phi_2(i)).$$

Then provided $A > A_1$, the number of solutions in integers of

$$|z - f(x,y)| < \delta(x)$$

with $(x,y) \in \mathscr{B}$ is at most

$$150(2+\kappa)^{\frac{2}{3}} (\tau - \sigma)^{\frac{1}{3}} A^{\frac{1}{3}} B^{\frac{2}{3}} + 45AD^2.$$

Proof. Suppose $(y_{j+1} - y_j, z_{j+1} - z_j)$ is a member of $\mathscr{U}_i$, as described in Lemma 2. It is shown in the proof of Hilfssatz 11 of (5) that

$$f_y(i, y_j)(y_{j+1} - y_j) < z_{j+1} - z_j < f_y(i, y_{j+1})(y_{j+1} - y_j).$$

Thus

$$\sigma(y_{j+1} - y_j) < z_{j+1} - z_j < \tau(y_{j+1} - y_j)$$

and $\mathscr{U}_1, \ldots, \mathscr{U}_A$ is a D-regular double sequence in M_1.

The number of integer solutions of $|f(i,y) - z| < \delta(i)$ for fixed i is at most

$$n(\mathscr{U}_i) + 2D^2 + 3 \leqslant n(\mathscr{U}_i) + 5D^2.$$

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By Lemma 2(*a*), we may apply Lemma 1 with $C = (2+\kappa)AB$. The number of solutions of $|f(x,y)-z| < \delta(x)$ with $(x,y)\in\mathscr{B}$ is thus at most

$$\sum_{i=1}^{A} n(\mathscr{U}_i) + 5AD^2 \leqslant 150(\tau-\sigma)^{\frac{1}{3}} C^{\frac{2}{3}} A^{\frac{1}{3}} + 40AD^2 + 5AD^2$$
$$= 150(\tau-\sigma)^{\frac{1}{3}} (2+\kappa)^{\frac{2}{3}} A^{\frac{1}{3}} B^{\frac{2}{3}} + 45AD^2.$$

Proof of Proposition 1. Suppose $\mu < 2$,

$$l'(t) \geqslant \kappa_2 > 0, \quad |l''(t)| \leqslant \kappa_3 \quad (\lambda_1 \leqslant t \leqslant \lambda_2).$$

Write $k = k(q) = [q^{\frac{1}{2}}]$; let γ be any number satisfying $1-\frac{1}{2}\mu < \gamma < \frac{1}{4}$, and write $D = D(q) = q^{\gamma}$. The number of solutions of (7) with $1 \leqslant x \leqslant k$ is $\ll q$. It thus suffices to estimate the number of solutions with $k+1 \leqslant x \leqslant q$. Write $f(x,y) = g(k+x,y)$, $\delta(x) = (k+x)^{-\mu}$. $\mathscr{B}$ denotes the region

$$1 \leqslant x \leqslant q-k,$$

$$\phi_1(x) \equiv \nu_1(k+x) \leqslant y \leqslant \nu_2(k+x) \equiv \phi_2(x).$$

Simple computations show that

$$f_y = l(y/(k+x)),$$

$$f_{yy} = (k+x)^{-1} l'(y/(k+x)),$$

$$f_{yyy} = (k+x)^{-2} l''(y/(k+x)),$$

$$\begin{vmatrix} f_{yy} & f_{yx} \\ f_{yyy} & f_{yyx} \end{vmatrix} f_{yy}^{-1} = -(k+x)^{-2} l'(y/(k+x)).$$

The hypotheses of Lemma 2, other than (i)–(iv), are easily verified, where $A = q-k$, $B = (\nu_2-\nu_1)q$. Moreover, (i) is obviously satisfied for large q, and for $i \geqslant 1$,

$$f_{yy}(i,y) \geqslant (k+i)^{-1}\kappa_2 > 4(k+i)^{-\mu}$$

if q (and thus k) is sufficiently large; this deals with (ii). Also, as $\gamma < \frac{1}{2}$,

$$(8+4\kappa) f_{yy}(i,y) \geqslant (k+i)^{-1}\kappa_2 > D(q)\kappa_3(k+i)^{-2} \geqslant D|f_{yyy}(i,y)|$$

$(1 \leqslant i \leqslant A, \phi_1(i) \leqslant y \leqslant \phi_2(i))$ if q is sufficiently large; this proves (iii). As for (iv), since $2\gamma > 2-\mu$,

$$D^2 E(i) \geqslant q^{2\gamma}\kappa_2(k+i+1)^{-2} > 64(2+\kappa)^2 (k+i)^{-\mu}$$
$$= 64(2+\kappa)^2 \max(\delta(i), \delta(i+1))$$

$(1 \leqslant i \leqslant A-1)$ if q is sufficiently large. All hypotheses of Lemma 2 are satisfied. Since

$$l(\nu_1) \leqslant f_y(i,y) \leqslant l(\nu_2) \quad (1 \leqslant i \leqslant A, \phi_1(i) \leqslant y \leqslant \phi_2(i))$$

an application of Lemma 3 yields the bound

$$150(2+\kappa)^{\frac{2}{3}} (l(\nu_2)-l(\nu_1))^{\frac{1}{3}} q^{\frac{2}{3}} (\nu_2-\nu_1)^{\frac{2}{3}} + 45qD^2(q)$$

for the number of solutions of

$$\|g(k+x,y)\| < (k+x)^{-\mu} \quad (1 \leqslant x \leqslant q-k, \nu_1(k+x) \leqslant y \leqslant \nu_2(k+x)),$$

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if q is sufficiently large. Since $\gamma < \frac{1}{4}$, it is now easy to complete the proof of Proposition 1.

It will be useful in section 4 to have a variant of Theorem C in which the lower bound on μ is reduced. For this we need several lemmas. Write $e(y) = e^{2\pi i y}$.

LEMMA 4. *Let $x_1, \dots, x_N$ be real numbers. Suppose there are exactly H solutions of $\|x_j\| < \gamma < \frac{1}{2}$ $(j \leqslant N)$. Then for $M = 1, 2, \dots$,*

$$H - 2\gamma N \ll NM^{-1} + \sum_{h=1}^{M} h^{-1} \left| \sum_{j=1}^{N} e(hx_j) \right|.$$

The implied constant is absolute.

LEMMA 5. *Suppose the real function f satisfies $f''(y) \geqslant \rho > 0$ $(A \leqslant y \leqslant B)$. Then*

$$\sum_{A \leqslant x \leqslant B} e(f(n)) \ll (f'(B) - f'(A) + 1)(\rho^{-\frac{1}{2}} + 1).$$

The implied constant is absolute.

Lemma 4 is theorem III of (5), and Lemma 5 is lemma 4·6 of (9), chapter 5.

LEMMA 6. *With the hypotheses of Theorem C, the function $N_\mu(q)$ defined in Theorem C satisfies*

$$N_\mu(q) \ll q^{\max(\frac{5}{3}, 2-\mu)} \quad (0 < \mu \leqslant 1). \tag{8}$$

Proof. For $x \geqslant 1$ write $N(x)$ for the number of solutions of (7) with $\lambda_1 x \leqslant y \leqslant \lambda_2 x$. By Lemma 4,

$$N(x) \ll 2x^{-\mu}([\lambda_2 x] - [\lambda_1 x]) + xM^{-1} + \sum_{h=1}^{M} h^{-1} \Big| \sum_{\lambda_1 x \leqslant y \leqslant \lambda_2 x} e(hg(x, y)) \Big|.$$

We apply Lemma 5 to $f(y) = hg(x, y)$. We have $f'(y) = hl(y/x) \ll h$,

$$f''(y) = hx^{-1}l'(y/x) \gg hx^{-1},$$

so

$$N(x) \ll x^{1-\mu} + xM^{-1} + \sum_{h=1}^{M} (h^{-\frac{1}{2}} x^{\frac{1}{2}} + 1)$$

$$\ll x^{1-\mu} + xM^{-1} + x^{\frac{1}{2}} M^{\frac{1}{2}} + M.$$

Choosing $M = [x^{\frac{1}{2}}]$ and summing over x, we obtain (8).

3. *Lemmas on plane curves.* In this section we assume that $x(s)$, $y(s)$ satisfies the following conditions for $a^* \leqslant s \leqslant b^*$:

$$x'^2(s) + y'^2(s) = 1, \tag{9}$$

$$0 < x_1 \leqslant x'(s), \quad 0 < y_1 \leqslant y'(s), \tag{10}$$

$$0 < c \leqslant |k(s)| \leqslant d, \tag{11}$$

$$b^* - a^* \leqslant 1/10(1 + d). \tag{12}$$

We write $U = [a_0 - \delta, a_0 + \delta]$ for an interval lying in (a^*, b^*).

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We introduce some further functions of s. We write $t = t(s) = y'(s)/x'(s)$. Thus $y_1 \leqslant t \leqslant x_1^{-1}$. One finds that $dt/ds = (x'y'' - y'x'')/x'^2 = k(s)/x'^2$, thus $c \leqslant |dt/ds| \leqslant d/x_1^2$. Write $[t_1, t_2] = t([a^*, b^*])$, then $t_1 \leqslant t(s) \leqslant t_2$. The inverse function $s = s(t)$ is defined in $t_1 \leqslant t \leqslant t_2$ and twice continuously differentiable there, as are the functions

$$X(t) = x(s(t)), \quad Y(t) = y(s(t)) \quad (t_1 \leqslant t \leqslant t_2).$$

Note that $X'(t) = t^{-1}Y'(t)$.

Finally let $\alpha(s)$ be defined by $\tan \alpha = t(s)$, $0 < \alpha < \frac{1}{2}\pi$. $\alpha(s)$ is strictly monotonic in $a^* \leqslant s \leqslant b^*$ and because of (9) we have $x'(s) = \cos\alpha(s)$, $y'(s) = \sin\alpha(s)$. We put $[\alpha_1, \alpha_2] = \alpha([a_0 - \delta, a_0 + \delta])$, $[\alpha_1^*, \alpha_2^*] = \alpha([a^*, b^*])$, so that

$$\alpha_1^* < \alpha_1 < \alpha_2 < \alpha_2^*.$$

Let $\lambda(n)$ be numbers in $(0, \frac{1}{8})$ $(n = 1, 2, \ldots)$.

If (q_1, q_2) is a non-zero lattice point we consistently write $q_1 = \tilde{q}\sin\beta$, $q_2 = -\tilde{q}\cos\beta$ where $\tilde{q} > 0$, $0 \leqslant \beta < 2\pi$. We write S for the set of (q_1, q_2) for which $0 \leqslant \beta < \pi$, $\tilde{S}$ for the subset of S for which $\beta \notin [\alpha_1^*, \alpha_2^*]$, and S' for the remaining part of S. If p is an integer we write for $(q_1, q_2) \in S$

$$E(q_1, q_2, p) = \{s \in U : |q_1 x(s) + q_2 y(s) - p| < \lambda(q)\}.$$

The following general result is Hilfssatz 3 of (7).

LEMMA 7. *Let f be differentiable in an interval I. To each v in an interval J suppose there is exactly one $u = u(v)$ in I with $f(u) = v$, and $f'(u(v)) \neq 0$. The u for which $f(u) \in J$ lie in a subinterval K of I, of length*

$$|K| = \int_J |f'(u(v))|^{-1} dv.$$

In general, we write $|E|$ for the Lebesgue measure of a real set E. Our first application of Lemma 7 is

LEMMA 8. *Let $(q_1, q_2) \in \tilde{S}$. Then $E(q_1, q_2, p)$ is an interval and*

$$|E(q_1, q_2, p)| \leqslant c_1 \lambda(q) q^{-1},$$

where $c_1 = c_1(\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*)$.

Proof. For $s \in U$,

$$\begin{aligned} |q_1 x'(s) + q_2 y'(s)| &= \tilde{q}|\cos\alpha(s)\sin\beta - \sin\alpha(s)\cos\beta| \\ &= \tilde{q}|\sin(\alpha(s) - \beta)| \\ &\geqslant \tilde{q}\sin(\min(\alpha_1 - \alpha_1^*, \alpha_2^* - \alpha_2)) = \tilde{q}c_2. \end{aligned}$$

By Lemma 7, then, $E(q_1, q_2, p)$ is an interval and

$$|E(q_1, q_2, p)| \leqslant 2\lambda(q)\tilde{q}^{-1}c_2^{-1} \leqslant 2c_2^{-1}\lambda(q)q^{-1}$$

as required.

Note that we allow the empty set as an interval. For the remainder of this section suppose $(q_1, q_2) \in S'$. There is exactly one s_0 in $a^* \leqslant s \leqslant b^*$ such that $\alpha(s_0) = \beta$. We write $Y(s) = q_1 x(s) + q_2 y(s)$, $Y_0 = Y(s_0)$.

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LEMMA 9. *We have* $Y'(s_0) = 0$, *and*

$$|Y'(s)| \geqslant (3\tfrac{1}{2}c\tilde{q}|Y(s)-Y_0|)^{\frac{1}{2}} \quad (a^* \leqslant s \leqslant b^*).$$

Proof. $Y'(s) = \tilde{q}(\sin\beta\cos\alpha(s) - \cos\beta\sin\alpha(s))$

$$= \tilde{q}\sin(\beta-\alpha(s)).$$

Thus $Y'(s_0) = 0$. Now

$$Y''(s) = -\tilde{q}\cos(\beta-\alpha(s))\,k(s).$$

Hence
$$|Y'(s)| \leqslant \sup|Y''(s)|(b^*-a^*) \leqslant d\tilde{q}|b^*-a^*| \leqslant \tilde{q}/2 \quad (a^* \leqslant s \leqslant b^*)$$

in view of (12). Thus

$$|\sin(\beta-\alpha(s))| \leqslant \tfrac{1}{2}, \quad |\cos(\beta-\alpha(s))| \geqslant \sqrt{3}/2, \quad |Y''(s)| \geqslant \sqrt{3}\,\tilde{q}c/2 \quad (a^* \leqslant s \leqslant b^*).$$

It follows that in $(s_0, b^*]$ Y' takes only one sign and $Y = Y(s)$ has a differentiable inverse $s = s(Y)$. Moreover, writing $s = s(Y)$,

$$d/dY(Y'(s))^2 = 2Y'(s)\,Y''(s)\,ds/dY = 2Y''(s),$$

$$|d/dY(Y'(s))^2| \geqslant 3\tfrac{1}{2}\tilde{q}c,$$

and so $|Y'(s)|^2 \geqslant 3\tfrac{1}{2}\tilde{q}c|Y(s)-Y_0|$ $(s_0 < s \leqslant b^*)$. Similar arguments apply for $s < s_0$.

LEMMA 10. *Let* p_0 *be the integer in*

$$Y_0 - \tfrac{1}{2} \leqslant p_0 < Y_0 + \tfrac{1}{2}.$$

Then for $p \neq p_0$, $E(q_1, q_2, p)$ *is the union of two intervals and*

$$|E(q_1, q_2, p)| \leqslant 8c^{-\frac{1}{2}}\lambda(q)\,q^{-\frac{1}{2}}|p-p_0|^{-\frac{1}{2}}. \tag{13}$$

Proof. For integer p write

$$E_+(q_1, q_2, p) = \{s \in E(q_1, q_2, p) : s \geqslant s_0\},$$

$$E_-(q_1, q_2, p) = \{s \in E(q_1, q_2, p) : s < s_0\}.$$

For a fixed v at most one $s \geqslant s_0$ with $Y(s) = v$ exists, so $E_+(q_1, q_2, p)$ is an interval; similarly for $E_-(q_1, q_2, p)$.

Write $p_j = p_0 - j$ $(j = 1, 2, \ldots)$. If

$$p_j - \lambda(q) < Y(s) < p_j + \lambda(q),$$

then
$$Y_0 - Y > Y_0 - p_j - \lambda(q) > Y_0 - Y_0 + j - \lambda(q) - \tfrac{1}{2}$$

$$> \tfrac{1}{4}j.$$

By Lemma 9, $|Y'(s)| \geqslant 3\tfrac{1}{2}c^{\frac{1}{2}}(qj/4)^{\frac{1}{2}}$, and an application of Lemma 7 yields

$$|E_+(q_1, q_2, p_j)| \leqslant 2\lambda(q)\,2c^{-\frac{1}{2}}j^{-\frac{1}{2}}q^{-\frac{1}{2}}.$$

The same estimate applies for $|E_-(q_1, q_2, p_j)|$. This proves (13) for $p < p_0$, and a similar argument applies when $p > p_0$.

LEMMA 11. $E(q_1, q_2, p_0)$ *is the union of two intervals, and*

$$|E(q_1, q_2, p_0)| \leqslant c^{-\frac{1}{2}}\min(6\lambda(q)\,q^{-\frac{1}{2}}\|Y_0\|^{-\frac{1}{2}}, 2.3^{-\frac{1}{4}}\lambda^{\frac{1}{2}}(q)\,q^{-\frac{1}{2}}).$$

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Proof. Suppose the first term in the last bracket is the smaller. Then $\|Y_0\| \geqslant 2\lambda(q)$. If $p_0-\lambda(q) < Y < p_0+\lambda(q)$, then

$$|Y_0-Y| = |Y_0-p_0+p_0-Y| > \|Y_0\|-\lambda(q) \geqslant \|Y_0\|/2.$$

By Lemma 9, $|Y'(s)| \geqslant 3^{\frac{1}{2}}c^{\frac{1}{2}}(q\|Y_0\|/2)^{\frac{1}{2}}$. Hence

$$|E_+(q_1,q_2,p_0)| \leqslant 2\lambda(q)\,2^{\frac{1}{2}}c^{-\frac{1}{2}}q^{-\frac{1}{2}}\|Y_0\|^{-\frac{1}{2}}.$$

Similarly for $|E_-(q_1,q_2,p_0)|$, so

$$|E(q_1,q_2,p_0)| \leqslant 6c^{-\frac{1}{2}}\lambda(q)\,q^{-\frac{1}{2}}\|Y_0\|^{-\frac{1}{2}}.$$

On the other hand, we always have the following inequality.

$$\begin{aligned}|E_+(q_1,q_2,p_0)| &\leqslant \int_{p_0-\lambda(q)}^{p_0+\lambda(q)} (3^{\frac{1}{2}}qc|Y-Y_0|)^{-\frac{1}{2}}\,dY\\ &\leqslant 3^{-\frac{1}{4}}c^{-\frac{1}{2}}q^{-\frac{1}{2}}\int_{-\lambda(q)}^{\lambda(q)} Y^{-\frac{1}{2}}\,dY = 3^{-\frac{1}{4}}c^{-\frac{1}{2}}q^{-\frac{1}{2}}\lambda^{\frac{1}{2}}(q);\end{aligned}$$

similarly for $|E_-(q_1,q_2,p_0)|$, and the lemma follows.

4. *Proof of Theorem* 1. In this section let $\epsilon > 0$ be a numerical constant, assumed 'sufficiently small'. Write

$$A = \{s\in(a,b): k(s) \neq 0\}.$$

A is a countable union of closed intervals in each of which $0 < |k(s)| \leqslant \lambda$. Because the complement of A in $[a,b]$ has measure zero, we may restrict attention to one of these intervals. Thus we assume $0 < |k(s)| \leqslant \lambda$ $(a \leqslant s \leqslant b)$.

Let B denote the set of s for which either $|x'(s)| = |y'(s)|$, or $x'(s) = 0$, or $y'(s) = 0$. Because $0 < |k(s)| \leqslant \lambda$, distinct members s_1, s_2 of B satisfy $|s_2-s_1| \geqslant \pi/(4\lambda)$. We can thus express $[a,b]$ as the union of a finite set and countably many closed intervals in each of which

$$|x'(s)|-|y'(s)| \neq 0, \quad x'(s) \neq 0, \quad y'(s) \neq 0. \tag{14}$$

Again, we may assume (14) holds for $a \leqslant s \leqslant b$ in proving the theorem.

Replacing x by $-x$, y by $-y$ if necessary (this does not affect the solubility of (1) or the hypotheses on Γ made up to now) we may assume

$$x'(s) > 0, \quad y'(s) > 0, \quad x'(s)-y'(s) \neq 0 \quad (a \leqslant s \leqslant b).$$

Similarly, we may replace (x,y) by (y,x) if need be, and so we may assume

$$0 < y'(s) < x'(s) \quad (a \leqslant s \leqslant b). \tag{15}$$

After a decomposition of $[a,b]$ into finitely many subintervals, we may assume that (11) and (12) hold, where

$$0 < c \leqslant d \leqslant \lambda, \quad dc^{-1} \leqslant 1+\epsilon.$$

Because of the identity

$$|(x+u)q_1+(y+v)q_2-p-uq_1-vq_2| = |xq_1+yq_2-p|,$$

a translation of Γ by a lattice point does not affect the solubility of (1) or the hypotheses on Γ made up to now. In view of (12), then, we may assume that

$$|x(s)| \leqslant \tfrac{2}{3} \quad (a \leqslant s \leqslant b). \tag{16}$$

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Because s denotes arc length, (9) holds and writing $y_1 = \inf_{[a,b]} y'(s) > 0$, $x_1 = \inf_{[a,b]} x'(s)$, (15) implies $x_1 \geqslant 2^{-\frac{1}{2}}$.

Let E denote the set of s in $[a,b]$ for which the inequalities (1) have integer solutions for all sufficiently large N. It suffices to show that E has density $\leqslant 1-\epsilon$ at any interior point a_0 of $[a,b]$, that is

$$|E \cap [a_0-\delta, a_0+\delta]| \leqslant 2(1-\epsilon)\delta \tag{17}$$

for all sufficiently small $\delta > 0$. We write $[a^*, b^*]$ for $[a,b]$ and define $U, \alpha_1 = \alpha_1(\delta)$, $\alpha_2 = \alpha_2(\delta)$, α_1^*, α_2^* as in section 3.

Write $\lambda(n) = \chi N^{-2}$ $(n = 1, 2, \ldots)$ and define S, $E(q_1, q_2, p)$ as in section 3. We have

$$E \cap [a_0-\delta, a_0+\delta] \subset \bigcup_{M=1}^{\infty} \bigcap_{N=M}^{\infty} \bigcup_{\substack{(q_1,q_2)\in S \\ q\leqslant N}} \bigcup_{p=-\infty}^{\infty} E(q_1, q_2, p).$$

(Here we use the fact that if $(q_1, q_2) \in M_0$ and $(q_1, q_2) \notin S$ then $(-q_1, -q_2) \in S$.) Thus to obtain (17) it suffices to prove that for small enough δ,

$$\sum_{\substack{(q_1,q_2)\in S \\ q\leqslant N}} \sum_{p=-\infty}^{\infty} |E(q_1, q_2, p)| \leqslant 2(1-\epsilon)\delta \tag{18}$$

for all sufficiently large N.

Write $\Sigma'(N)$ for the part of the sum (18) for which

$$|q_1 x'(a_0) + q_2 y'(a_0)| > \tfrac{7}{6} d\tilde{q}\delta \tag{19}$$

and $\Sigma''(N)$ for the remaining part of the sum (18).

LEMMA 12. *$\Sigma'(N) \leqslant (80+58\epsilon)\chi\delta$ for all sufficiently large N.*

Proof. Since $d/ds(\sin(\alpha(s)-\beta)) = \cos(\alpha(s)-\beta)k(s)$, (19) implies

$$\begin{aligned} |q_1 x'(s) + q_2 y'(s)| &= \tilde{q}|\sin(\alpha(s)-\beta)| > \tilde{q}|\sin(\alpha(a_0)-\beta)| - d\tilde{q}\delta \\ &= |q_1 x'(a_0) + q_2 x'(a_0)| - d\tilde{q}\delta > \tfrac{1}{7}|q_1 x'(a_0) + q_2 y(a_0)| \end{aligned} \tag{20}$$

throughout U. Moreover, if $s \in U$,

$$|q_1 x(s) + q_2 y(s) - p - (q_1 x(a_0) + q_2 y(a_0) - p)| \leqslant |q_1 x'(a_0) + q_2 y'(a_0)|\delta + \tfrac{1}{2} d\tilde{q}\delta^2$$

so that if $E(q_1, q_2, p)$ is non-empty,

$$\begin{aligned} |q_1 x(a_0) + q_2 y(a_0) - p| &< \chi N^{-2} + |q_1 x'(a_0) + q_2 y'(a_0)|\delta + \tfrac{1}{2} d\tilde{q}\delta^2 \\ &< (\tfrac{10}{7} + \epsilon)|q_1 x'(a_0) + q_2 y'(a_0)|\delta \end{aligned}$$

for all sufficiently large N, in view of (19). For fixed q_1, q_2, then, there are at most

$$(20/7 + 2\epsilon)|q_1 x'(a_0) + q_2 y'(a_0)|\delta + 1$$

non-zero $|E(q_1, q_2, p)|$, and each of these is

$$\leqslant 14|q_1 x'(a_0) + q_2 y'(a_0)|^{-1}\chi N^{-2}$$

by Lemma 7 and (20). Note that there are $\frac{1}{2}((2N+1)^2 - 1)$ pairs in S. Thus

$$\Sigma'(N) \leqslant 14\chi N^{-2} \sum \frac{((20/7) + 2\epsilon)|q_1 x'(a_0) + q_2 y'(a_0)|\delta + 1}{|q_1 x'(a_0) + q_2 y'(a_0)|},$$

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where the sum extends over at most $2N^2+2N$ pairs q_1, q_2, for each of which (19) holds. The contribution of the term 1 in the numerator tends to 0 as $N\to\infty$, since

$$\frac{1}{N^2}\sum_{q_1,q_2}\frac{1}{|q_1x'(a_0)+q_2y'(a_0)|} \ll \frac{1}{N^2}\sum_{q_1}\left(\frac{1}{\delta\max(|q_1|,1)}+\log N\right)$$
$$\ll N^{-2}(\delta^{-1}\log N + N\log N).$$

The expression that remains is

$$(40+28\epsilon)\chi N^{-2}\delta\sum_{q_1,q_2}1 \leqslant (80+57\epsilon)\chi\delta$$

for all sufficiently large N, and the lemma follows.

Next, we observe that if (19) is violated and $(q_1, q_2)\in S$, then $(q_1, q_2)\in S'$, provided that

$$0<\delta<\tfrac{1}{2}d^{-1}\sin(\min(\alpha_1-\alpha_1^*,\alpha_2^*-\alpha_2)). \tag{21}$$

For we have, in the notation of section 3,

$$\tilde{q}\,|\sin(\beta-\alpha(a_0))| = |q_1x'(a_0)+q_2y'(a_0)| < 2d\tilde{q}\delta$$
$$< \tilde{q}\sin(\min(\alpha_1-\alpha_1^*,\alpha_2-\alpha_2^*)),$$

hence $|\beta-\alpha(a_0)| < \min(\alpha_1^*-\alpha_1,\alpha_2-\alpha_2^*)$ and $\beta\in[\alpha_1^*,\alpha_2^*]$. Of course α_1 and α_2 depend on δ, but (21) is true if $0<\delta<\delta_1$ (say), and we assume this. Thus s_0, Y_0, p_0 are defined for any pair q_1, q_2 occurring in the sum $\Sigma''(N)$.

Write $\Sigma'(N)=\Sigma_1(N)+\Sigma_2(N)$, where $\Sigma_1(N)$ is the part of $\Sigma''(N)$ for which

$$p\neq p_0(q_1,q_2).$$

LEMMA 13. *There is a $\delta_2>0$ such that*

$$\Sigma_1(N)<\epsilon\delta$$

for all sufficiently large N, provided $\delta<\delta_2$.

Proof. Suppose (19) is violated for given q_1, q_2 and $E(q_1, q_2, p)$ is non-empty, then clearly

$$|q_1x(a_0)+q_2y(a_0)-p| < \chi N^{-2}+\delta|q_1x'(a_0)+q_2y'(a_0)|+\tfrac{1}{2}d\tilde{q}\delta^2$$
$$< 2d\tilde{q}\delta^2$$

provided N is sufficiently large. Thus there are at most $4d\tilde{q}\delta^2+1$ values of $p\neq p_0$ for which $|E(q_1,q_2,p)|>0$, and by Lemma 10 these contribute to $\Sigma_1(N)$ at most

$$2\sum_{j=1}^{4dq^2\delta^2+2}8c^{-\frac{1}{2}}\chi N^{-2}q^{-\frac{1}{2}}j^{-\frac{1}{2}} \ll (q\delta^2+2)^{\frac{1}{2}}N^{-2}q^{-\frac{1}{2}} \ll N^{-2}(\delta+q^{-\frac{1}{2}}).$$

Moreover, for given q_1, there are $\ll q_1\delta+1$ values of q_2 for which (19) is violated and $(q_1,q_2)\in S$. (Note that $\tan\beta=|q_1/q_2|\in[y_1,1]$.) Thus

$$\Sigma_1(N)\ll\sum_{q_1=1}^{N}N^{-2}(\delta+q_1^{-\frac{1}{2}})(q_1\delta+1)$$
$$\ll N^{-2}(N^2\delta^2+N\delta+N^{\frac{3}{2}}\delta+N^{\frac{1}{2}})$$

and Lemma 13 follows easily from this estimate.

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Estimation of $\Sigma_2(N)$ depends on the results of section 2.

LEMMA 14. *For* $\frac{1}{3} \leqslant \nu \leqslant \frac{3}{2}$ *write* $A_\nu(h)$ *for the number of* $(q_1, q_2) \in S'$ *with* $0 < q \leqslant h$, $\|Y_0\| < q^{-\nu}$. *Then*

$$A_\nu(h) \ll \begin{cases} h^{\frac{3}{2}+(\frac{3}{2}-\nu)/7} & (1 < \nu \leqslant \frac{3}{2}), \\ h^{\frac{4}{3}} & (\frac{1}{3} \leqslant \nu \leqslant 1). \end{cases} \tag{22}$$

Write $B(h)$ *for the number of* $(q_1, q_2) \in S$ *with* $0 < q \leqslant h$ *for which* (19) *is violated and* $\|Y_0\| < q^{-\frac{13}{8}}$. *Then*

$$B(h) \leqslant 1414 c^{-\frac{1}{2}} d\delta h^{\frac{3}{2}} \tag{23}$$

for all sufficiently large h.

Proof. Let $X(t)$, $Y(t)$ be defined as in section 3 on the interval $t_1 \leqslant t \leqslant t_2$. Then if we write

$$g(x, y) = yX(y/x) - xY(y/x),$$

we have $$g_y(x, y) = X(y/x) + y/xX'(y/x) - Y'(y/x) = X(y/x).$$

Now

$$Y_0 = q_1 x(s_0) + q_2 y(s_0) = q_1 X(\tan\beta) + q_2 Y(\tan\beta)$$

$$= q_1 X(-q_1/q_2) + q_2 Y(-q_1/q_2) = g(-q_2, q_1).$$

Write $q_1 = y, q_2 = -x$. Since $q^{-\nu} \leqslant x^{-\nu}$, $A_\nu(h)$ is at most the number of integral solutions of $\|g(x, y)\| < x^{-\mu}$ in the region $1 \leqslant x \leqslant h$, $t_1 x \leqslant y \leqslant t_2 x$. Now (22) follows from Theorem C and Lemma 6. (Note that we can replace g by $-g$ if $X'(t)$ is negative.)

Write $\theta = x'(a_0)$, $\phi = y'(a_0)$. Clearly $B(h)$ is at most the number of integral solutions of $\|g(x, y)\| < x^{-\frac{13}{8}}$ for which $1 \leqslant x \leqslant h$, $y_1 \leqslant y/x \leqslant 1$,

$$|x\phi - y\theta| \leqslant \tfrac{7}{6} d(x^2 + y^2)^{\frac{1}{2}} \delta.$$

But these inequalities imply

$$x(\phi\theta^{-1} - C\delta) \leqslant y \leqslant x(\phi\theta^{-1} + C\delta),$$

where $C = \frac{7}{6} d . 2^{\frac{1}{2}} \theta^{-1} \leqslant 7d/3$. Write

$$\lambda_1 = t_1, \quad \lambda_2 = t_2, \quad \nu_1 = \max(\lambda_1, \phi\theta^{-1} - C\delta), \quad \nu_2 = \min(\lambda_2, \phi\theta^{-1} + C\delta), \quad l(t) = X(t),$$

so that $|l(t)| \leqslant \frac{2}{3}$ by (16) and $|l'(t)| = x'^3(s(t))\,|k^{-1}(s(t))| \leqslant c^{-1}$. Now we may apply Proposition 1 with $\kappa = 2/3$, $\kappa_1 = c^{-1}$. For sufficiently large h,

$$B(h) \leqslant M_{13/8}(h) < 151(2 + \kappa)^{\frac{3}{2}} \kappa_1^{\frac{1}{2}} (\nu_2 - \nu_1) h^{\frac{3}{2}}$$

$$\leqslant 302 c^{-\frac{1}{2}} 2C\delta h^{\frac{3}{2}},$$

which proves (23).

Completion of the proof of Theorem 1. We subdivide $\Sigma_2(N)$ into sums S_{jk} ($j = 6, 7, \ldots, 13$, $k = 1, \ldots, [\log N/\log 2]$) where S_{jk} is that part of the sum $\Sigma_2(N)$ for which

$$2^k \leqslant q < 2^{k+1}, \tag{24}$$

$$q^{-(j+1)/8} \leqslant \|Y_0\| < q^{-j/8}. \tag{25}$$

(The left inequality of (25) is to be disregarded when $j = 13$, and the right inequality when $j = 6$.)

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For a summand of S_{6k}, Lemma 11 yields

$$|E(q_1, q_2, p_0)| \ll N^{-2} q^{-\frac{1}{2}} q^{\frac{7}{16}} = N^{-2} q^{-\frac{1}{16}},$$

so that

$$S_{6k} \ll N^{-2} 2^{k(2-\frac{1}{16})}$$

and

$$\sum_{k=1}^{[\log N/\log 2]} S_{6k} \to 0 \quad \text{as} \quad N \to \infty. \tag{26}$$

Similarly, if $7 \leqslant j \leqslant 12$, terms of S_{jk} satisfy

$$|E(q_1, q_2, p)| \ll N^{-2} 2^{-\frac{1}{2}k + \frac{1}{2}k(j+1)/8}$$

and the number of these terms is, by Lemma 14,

$$\ll \begin{cases} 2^{5k/3} & \text{if} \quad j = 7 \text{ or } 8 \\ 2^{3k/2 + (\frac{3}{2} - j/8)k/7} & \text{if} \quad 9 \leqslant j \leqslant 12. \end{cases}$$

Thus

$$S_{jk} \ll N^{-2} 2^{k \max(\frac{17}{14} + \frac{1}{16} + \frac{5}{112} j, \frac{5}{3} + \frac{1}{16})}$$

$$\ll N^{-2} 2^{26k/14}$$

and

$$\sum_{k=1}^{[\log N/\log 2]} S_{jk} \to 0 \quad \text{as} \quad N \to \infty \quad (7 \leqslant j \leqslant 12). \tag{27}$$

For a summand of $S_{13,k}$, Lemma 11 yields

$$|E(q_1, q_2, p)| \leqslant 2 . 3^{-\frac{1}{4}} c^{-\frac{1}{2}} \chi^{\frac{1}{2}} N^{-1} 2^{-\frac{1}{2}k}$$

while by Lemma 14 the number of $(q_1, q_2) \in S$ for which $2^k \leqslant q < 2^{k+1}$, (19) is violated and

$$\|Y_0\| < q^{-\frac{13}{8}}$$

is at most

$$1414 c^{-\frac{1}{8}} d\delta (2^{k+1})^{\frac{3}{2}} \leqslant 4040 c^{-\frac{1}{8}} d\delta 2^{\frac{3}{2}k}$$

for $k \geqslant k_0$, say. Thus

$$\sum_{k=k_0}^{[\log N/\log 2]} S_{13,k} \leqslant 8080 . 3^{-\frac{1}{4}} c^{-\frac{5}{8}} d\delta \chi^{\frac{1}{2}} N^{-1} \sum_{k=1}^{[\log N/\log 2]} 2^k$$

$$\leqslant 12440 d^{\frac{1}{8}} \chi^{\frac{1}{2}} \delta$$

if ϵ is suitably small. In view of (26) and (27),

$$\Sigma_2(N) \leqslant 12441 d^{\frac{1}{8}} \chi^{\frac{1}{2}} \delta \tag{28}$$

for all sufficiently large N. Provided $\delta < \delta_2$, and ϵ is suitably small, (28) combined with Lemmas 12 and 13 yields

$$\sum_{\substack{(q_1, q_2) \in S \\ q \leqslant N}} \sum_{p=-\infty}^{\infty} |E(q_1, q_2, p)| \leqslant (80\chi + 12442 d^{\frac{1}{8}} \chi^{\frac{1}{2}}) \delta$$

for all sufficiently large N, and in view of the value of χ this implies (18). This proves Theorem 1.

5. *Proof of Theorem* 2. By arguments given in section 4, we may assume that $[a, b] = U$, where all the conditions of section 3 are fulfilled in an interval $[a^*, b^*]$ with $U = [a_0 - \delta, a_0 + \delta]$ in its interior.

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We may assume $\psi(q) < \frac{1}{4}$ $(q \geqslant 1)$. Define now $\lambda(q) = \psi^2(q)$ $(q = 1, 2, \ldots)$. In the notation of section 3, it suffices to prove

$$\sum_{(q_1, q_2) \in S} \sum_{p=-\infty}^{\infty} |E(q_1, q_2, p)| < \infty. \tag{29}$$

For if F denotes the set of s in U for which (2) has infinitely many solutions then

$$F \subset \bigcup_{\substack{(q_1, q_2) \in S \\ q \geqslant k}} \bigcup_{p=-\infty}^{\infty} E(q_1, q_2, p)$$

$(k = 1, 2, \ldots)$ and thus F can be covered by intervals of arbitrarily small total length.

LEMMA 15.
$$\sum_{(q_1, q_2) \in \tilde{S}} \sum_{p=-\infty}^{\infty} |E(q_1, q_2, p)| < \infty.$$

Proof. Let $(q_1, q_2) \in \tilde{S}$, then by Lemma 8,

$$|E(q_1, q_2, p)| \ll \psi^2(q)\, q^{-1}.$$

Moreover, since $\{(x(s), y(s)) : s \in U\}$ is a bounded set, there are $\ll q$ integers p for which $E(q_1, q_2, p)$ is non-empty. Hence

$$\sum_{p=-\infty}^{\infty} |E(q_1, q_2, p)| \ll \psi^2(q),$$

$$\sum_{\substack{2^k \leqslant q < 2^{k+1} \\ (q_1, q_2) \in \tilde{S}}} \sum_{p} |E(q_1, q_2, p)| \ll \psi^2(2^k)\, 2^{2k} \ll \psi(2^k)\, 2^k,$$

since ψ is monotonic decreasing and $\Sigma\psi(q)$ converges. Lemma 15 follows at once.

LEMMA 16.
$$\sum_{(q_1, q_2) \in S'} \sum_{p \neq p_0} |E(q_1, q_2, p)| < \infty.$$

Proof. Let $(q_1, q_2) \in S'$. Again, there are $\ll q$ integers p for which $E(q_1, q_2, p)$ is non-empty. Hence Lemma 10 yields

$$\sum_{\substack{p=-\infty \\ p \neq p_0}}^{\infty} |E(q_1, q_2, p)| \ll \psi^2(q)\, q^{-\frac{1}{2}} \sum_{1 \leqslant h \ll q} h^{-\frac{1}{2}} \ll \psi^2(q),$$

and the proof now follows Lemma 15.

To complete the proof of Theorem 2 we only require

$$\sum_{(q_1, q_2) \in S'} |E(q_1, q_2, p_0)| < \infty. \tag{30}$$

We partition the sum as in section 4, using the inequalities (24) and (25). (Terms with

$$\|Y_0\| > q^{-1}$$

satisfy

$$|E(q_1, q_2, p_0)| \ll \psi^2(q) \tag{31}$$

by Lemma 11.) If (24) and (25) hold for some j $(8 \leqslant j \leqslant 12)$, then similarly

$$|E(q_1, q_2, p_0)| \ll \psi^2(q)\, 2^{-\frac{1}{2}k + \frac{1}{2}k(j+1)/8} \ll \psi^2(2^k)\, 2^{-\frac{1}{2}k + \frac{1}{2}k(j+1)/8}.$$

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As in section 4 the contribution of these terms to the sum (30) is

$$\ll \psi^2(2^k) \max\left(2^{k(\frac{1}{4}\frac{7}{6}+\frac{1}{16}+\frac{5}{11\cdot 8}j)}, 2^{\frac{5}{3}k+\frac{1}{16}k}\right)$$

$$\ll \psi^2(2^k)\, 2^{2k} \ll \psi(2^k)\, 2^k. \tag{32}$$

Similarly, for $j = 13$ we have

$$|E(q_1, q_2, p_0)| \ll \psi(2^k)\, 2^{-\frac{1}{2}k}$$

if (24) and the right-hand inequality of (25) hold. The corresponding contribution to the sum (30) is

$$\ll \psi(2^k)\, 2^{-\frac{1}{2}k}\, 2^{\frac{3}{2}k} = \psi(2^k)\, 2^k \tag{33}$$

by Lemma 14. Now (30) follows at once from the estimates (31), (32), (33). This proves Theorem 2.

6. *Proofs of Theorems* 3 *and* 4. We recall some definitions related to Hausdorff dimension. Let g be a monotonic increasing positive function defined for $t > 0$. Let E be a real set and let $\delta > 0$. Write

$$g_\delta(E) = \inf \sum_j g(d(I_j))$$

where the infimum is taken over all sequences of sets $I_1, I_2, \ldots$ of diameter $d(I_j) \leqslant \delta$ that cover E. Let

$$g(E) = \sup_{\delta > 0} g_\delta(E).$$

If $g(t) = t^\alpha$ we write $m(\alpha, E)$ instead of $g(E)$. The *Hausdorff dimension* of E is

$$\dim E = \inf\{\alpha > 0 : m(\alpha, E) = 0\}.$$

In proving Theorem 3 we may again assume that $[a, b] = U$ where all conditions of section 3 are satisfied in an interval $[a^*, b^*]$ with U in its interior. To see this, write

$$A = \{s \in (a, b) : k(s) \neq 0\}.$$

Then A is the union of a countable set and a sequence of intervals $U_1, U_2, \ldots$ where each U_i is interior to an interval $[a_i^*, b_i^*]$ and, in $[a_i^*, b_i^*]$, $k(s)$, $x'(s)$, $y'(s)$ are all non-zero. This is essentially proved in section 4. Since the complement of A in $[a, b]$ has dimension zero, it suffices to prove

$$\dim X(\Gamma_i, \mu) = \dim X'(\Gamma_i, \mu) = 3/(\mu + 1),$$

where Γ_i is the curve $x(s)$, $y(s)$ $(s \in U_i)$. After some further simple reductions, discussed at the beginning of section 4, we see that it is enough to consider the interval U of section 3 rather than U_i.

Let $\mu > 2$. First of all we easily establish that

$$\dim X(\Gamma, \mu) \leqslant 3/(\mu + 1). \tag{34}$$

Write $\lambda(q) = \frac{1}{9} q^{-\mu}$ $(q = 1, 2, \ldots)$, and for $(q_1, q_2) \in S$ define $E(q_1, q_2, p)$ as in section 3. It is enough to show that

$$\sum_{(q_1, q_2) \in S} \sum_{p=-\infty}^{\infty} |E(q_1, q_2, p)|^\gamma < \infty \tag{35}$$

for all γ such that

$$3/(\mu + 1) < \gamma < 1. \tag{36}$$

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For $E(q_1, q_2, p) = I(q_1, q_2, p) \cup J(q_1, q_2, p)$ where I and J are intervals (see Lemmas 8, 10, 11). If $\mu < \mu'$ and $w(x(s), y(s)) \geqslant \mu'$, the inequality

$$|q_1 x(s) + q_2 y(s) - p| < \tfrac{1}{9} q^{-\mu}$$

has infinitely many solutions in integers, hence the intervals

$$I(q_1, q_2, p), \quad J(q_1, q_2, p) \quad (q \geqslant k)$$

cover $X(\Gamma, \mu')$ for $k = 1, 2, \ldots$. Since as $h \to \infty$

$$\sum_{\substack{(q_1, q_2) \in S \\ q \geqslant k}} \sum_{p=-\infty}^{\infty} (|I(q_1, q_2, p)|^\gamma + |J(q_1, q_2, p)|^\gamma) \to 0,$$

whenever (36) holds, we see that

$$m(\gamma, X(\Gamma, \mu')) = 0$$

whenever $2 < \mu < \mu', \gamma > 3/(\mu + 1)$. Obviously this implies (34).

The proof of (35) follows the same lines as the proof of (29). If $(q_1, q_2) \in \tilde{S}$, the argument of Lemma 15 yields

$$\sum_p |E(q_1, q_2, p)|^\gamma \ll q^{1-\gamma(\mu+1)}. \tag{37}$$

If $(q_1, q_2) \in S'$, then by the argument of Lemma 16,

$$\sum_{p \neq p_0} |E(q_1, q_2, p)|^\gamma \ll q^{-\gamma\mu - \frac{1}{2}\gamma} \sum_{1 \leqslant k \ll q} k^{-\frac{1}{2}\gamma} \ll q^{1-\gamma(\mu+1)}. \tag{38}$$

If $(q_1, q_2) \in S'$ and $\|Y_0\| > q^{-1}$, then as in the proof of (30),

$$|E(q_1, q_2, p_0)|^\gamma \ll q^{-\mu\gamma}. \tag{39}$$

If (24) and (25) hold where $8 \leqslant j \leqslant 12$ then

$$|E(q_1, q_2, p_0)|^\gamma \ll 2^{-(\mu+\frac{1}{2})k\gamma + \frac{1}{2}k(j+1)\gamma/8}$$

and the contribution of these terms to (29) is

$$\begin{aligned} &\ll \max\left(2^{k(5/3+9/16-(\mu+\frac{1}{2})\gamma)}, 2^{-(\mu+\frac{1}{2})k\gamma + 3k/2 + \frac{1}{2}k(j+1)\gamma/8 + (3/2 - j/8)k/7}\right) \\ &\ll 2^{k(5/2-\gamma(\mu+\frac{1}{2}))}. \end{aligned} \tag{40}$$

Finally for $j = 13$ the corresponding contribution is

$$2^{-(\frac{1}{2}\mu+\frac{1}{2})\gamma k + 3k/2} = 2^{\frac{1}{2}k(3-\gamma(\mu+1))}. \tag{41}$$

Since all the series

$$\sum_{q_1, q_2} q^{1-\gamma(\mu+1)}, \quad \sum_{q_1, q_2} q^{-\mu\gamma},$$

$$\sum_k 2^{k(5/2-\gamma(\mu+\frac{1}{2}))}, \quad \sum_k 2^{\frac{1}{2}k(3-\gamma(\mu+1))}$$

converge for $\gamma > 3/(\mu + 1)$, the estimates (37), (38), (39), (40) and (41) together imply (35). This proves (34).

The proof of the reverse inequality is more difficult, and requires new definitions. We call a countable set of real numbers Λ together with a positive valued function $\mathcal{N}$ defined on Λ a *regular system* $(\Lambda, \mathcal{N})$ *in an interval* I, if for every interval $J \subset I$ there is a number $K = K(J)$ with the following property.

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For all $K \geqslant K(J)$ there are elements $\gamma_1, \ldots, \gamma_t$ of Λ such that for $1 \leqslant j < k \leqslant t$,

$$\gamma_j \in J, \quad \mathscr{N}(\gamma_j) \leqslant K, \quad |\gamma_j - \gamma_k| \geqslant K^{-1}$$

and moreover

$$t \geqslant c_0 |J| K,$$

where $c_0 = c_0(\Lambda, \mathscr{N}) > 0$.

For any regular system $(\Lambda, \mathscr{N})$ in I and any positive function $f(x)$ defined for $x > 0$ we write $(\Lambda, \mathscr{N}; f)$ for the set of all $s \in I$ for which

$$|s - \gamma| < f(\mathscr{N}(\gamma))$$

for infinitely many γ of Λ.

LEMMA 17. *Let $f(t)$, $g(t)$ be positive functions defined for $t > 0$ such that $f(t)$ decreases and $f(t) \leqslant 1/2t$ for large t, $g(t)$ and $t/g(t)$ both increase and tend to zero with t, and*

$$tg(\tfrac{1}{2}f(t)) \to \infty \quad as \quad t \to \infty.$$

Then for any regular system $(\Lambda, \mathscr{N})$ in I we have

$$g((\Lambda, \mathscr{N}; f)) > 0.$$

Proof. This is almost identical with lemma 1 of (1) and can be proved by simply replacing $J_0 = [0, 1]$ by $J_0 = I$ in the proof of that lemma.

Let $U = [a_0 - \delta, a_0 + \delta]$, as in section 3. Define Λ_U to consist of all numbers γ in U for which

$$q_1 x(\gamma) + q_2 y(\gamma) - p = 0 \tag{42}$$

for some integers q_1, q_2, p not all zero. Clearly Λ_U is countable since for fixed q_1, q_2, p there are at most two solutions γ of (42) by the arguments of section 3. Define

$$\mathscr{N}(\gamma) = q^3/(\log q)^{11} \quad (\gamma \in \Lambda_U),$$

where $q = q(\gamma)$ takes the smallest value for which an equation (36) holds.

Let J denote any subinterval of U and let $R_J(H)$ denote the set of all s in J for which there exists γ in $\Lambda_U \cap J$ with $q(\gamma) \leqslant H$ and

$$|s - \gamma| < (\log H)^{11} H^{-3}.$$

LEMMA 18. $|R_J(H)| \to |J|$ *as* $H \to \infty$.

Proof. Let $H \geqslant 3$. Write $\psi(H) = H^{-1}(\log H)^{-2}$, and let $S(H)$ be the set of all s in U for which

$$|q_1 x(s) + q_2 y(s) - p| < \psi^2(q)$$

for some integers q_1, q_2, p with $q \geqslant H$. Theorem 2 shows that $|S(H)| \to 0$ as $H \to \infty$.

Let s be any number in J. By Minkowski's linear forms theorem we may select integers $q_1(s)$, $q_2(s)$, $p(s)$ not all zero such that

$$|q_1 x(s) + q_2 y(s) - p| \leqslant (\log H)^5 H^{-2}, \tag{43}$$

$$|q_1| \leqslant H, \quad |q_2| \leqslant H/(\log H)^5,$$

For large H, $0 < q \leqslant H$. After replacing q_1, q_2, p by $-q_1$, $-q_2$, $-p$ if necessary we may assume $(q_1, q_2) \in \mathcal{S}$. For the rest of the proof we understand $q_1(s)$, $q_2(s)$ by q_1, q_2.

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Let $T(H) = \{s \in J : q < \log H\}$. Clearly

$$T(H) \subset \bigcup_{\substack{(r_1, r_2) \in S \\ r < \log H}} \bigcup_{p=-\infty}^{\infty} E(r_1, r_2, p)$$

in the notation of section 3, where $\lambda(r)$ now denotes $2(\log H)^5 H^{-2}$ ($r = 1, 2, \ldots$). It is easy to deduce from Lemmas 8, 10 and 11 that $|T(H)| \to 0$ as $H \to \infty$.

Let $V(H) = \{s \in J : q \geqslant \log H,\ |q_1 x'(s) + q_2 y'(s)| \leqslant H(\log H)^{-5}\}$. Clearly if $s \in V(H)$ then $q \ll H/(\log H)^5$ whence, if H is sufficiently large, (2) holds. Thus

$$V(H) \subset S(\log H)$$

and $|V(H)| \to 0$ as $H \to \infty$.

Write $J = [a_1, b_1]$,

$$W(H) = \{s \in [a_1 + H^{-\frac{1}{2}}, b_1 - H^{-\frac{1}{2}}] : s \notin T(H) \cup V(H)\}.$$

To prove the lemma it suffices to show that

$$W(H) \subset R_J(H), \tag{44}$$

for all sufficiently large H. Now let $s \in W(H)$; then

$$|q_1 x'(s) + q_2 y'(s)| > H(\log H)^{-5}$$

and, since $k(s) \leqslant d$ in U, and $q \leqslant H$,

$$|q_1 x'(u) + q_2 y'(u)| > \tfrac{1}{2} H(\log H)^{-5} \tag{45}$$

for all u in $(s - H^{-\frac{1}{2}}, s + H^{-\frac{1}{2}})$. In view of (43), $q_1 x(\gamma) + q_2 y(\gamma) - p = 0$ for some γ in the latter interval. By (45),

$$\begin{aligned} |s - \gamma| &\leqslant |q_1 x(s) + q_2 y(s) - p| \, 2(\log H)^5 H^{-1} \\ &\leqslant 2(\log H)^{10} H^{-3} \end{aligned}$$

Moreover, $q(\gamma) \leqslant q \leqslant H$. Thus $s \in R_J(H)$. This proves (44).

Completion of the proof of Theorem 3. Let J be any subinterval of U, then

$$|R_J(H)| \geqslant \tfrac{1}{2}|J|$$

for all sufficiently large H. Write $K = H^3(\log H)^{-11}$. Denote by $\gamma_1, \ldots, \gamma_t$ a maximal subset of elements of Λ_U in J with $q(\gamma_j) \leqslant H$ and $|\gamma_j - \gamma_k| \geqslant K^{-1}$ for all j, k with $j \neq k$. Then $\mathcal{N}(\gamma_j) \leqslant K$ ($j = 1, \ldots, t$). Moreover, any γ in $\Lambda_U \cap J$ with $q(\gamma) \leqslant H$ has distance $\leqslant K^{-1}$ from a γ_j, from which it follows that

$$R_J(H) \subset \bigcup_{j=1}^{t} (\gamma_j - 2K^{-1}, \gamma_j + 2K^{-1}),$$

and $\frac{1}{2}|J| \leqslant |R_J(H)| \leqslant 4tK^{-1}$. We have proved that $(\Lambda_U, \mathcal{N})$ is a regular system in U (with $c_0 = \frac{1}{8}$).

Apply Lemma 17 to this regular system with $g(t) = t^{3/(\mu+1)}$ ($\mu > 2$) and f defined so that f is monotonic decreasing for $t > 0$ and, for large t,

$$f(t) = t^{-(\mu+1)/3} \log t.$$

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All the conditions on f and g imposed in Lemma 17 are satisfied. Thus

$$g((\Lambda_U, \mathscr{N}; f)) > 0.$$

Let $s \in (\Lambda_U, \mathscr{N}; f)$, then for infinitely many γ in Λ_U,

$$\begin{aligned} |s-\gamma| &< f(\mathscr{N}(\gamma)) \\ &\ll \mathscr{N}(\gamma)^{-(\mu+1)/3} \log \mathscr{N}(\gamma) \\ &\ll q(\gamma)^{-(\mu+1)} (\log q(\gamma))^{4\mu+5}. \end{aligned} \tag{46}$$

Let (42) be an equation in which $q = q(\gamma)$, then (46) implies

$$\begin{aligned} |q_1 x(s) + q_2 y(s) - p| &= |q_1(x(s)-x(\gamma)) + q_2(y(s)-y(\gamma))| \\ &\leqslant 2q|s-\gamma| \\ &\ll q^{-\mu}(\log q)^{4\mu+5}. \end{aligned}$$

Since this holds for infinitely many q_1, q_2, p, we see that $s \in X(\Gamma, \mu)$, in other words

$$(\Lambda_U, \mathscr{N}; f) \subset X(\Gamma, \mu),$$

and so
$$g(X(\Gamma, \mu)) > 0 \quad (g(t) = t^{3/(\mu+1)}). \tag{47}$$

This proves (in view of (34)) that $\dim X(\Gamma, \mu) = 3/(\mu+1)$.

Suppose if possible that for some $\mu > 2$,

$$m(3/(\mu+1), X'(\Gamma, \mu)) = 0. \tag{48}$$

Now
$$X(\Gamma, \mu) = X'(\Gamma, \mu) \cup \bigcup_{r=1}^{\infty} X(\Gamma, \mu + r^{-1})$$

and clearly for $r = 1, 2, \ldots$

$$m(3/(\mu+1), X(\Gamma, \mu+r^{-1})) = 0$$

since $\dim X(\Gamma, \mu+r^{-1}) < 3/(\mu+1)$. Thus

$$m(3/(\mu+1), X(\Gamma, \mu)) = 0.$$

This contradicts (47), so (48) is false, and it follows that $\dim X'(\Gamma, \mu) = 3/(\mu+1)$ for all $\mu > 2$. This completes the proof of Theorem 3.

Proof of Theorem 4. Write $A = \{\theta \in \mathbb{C}: \theta^{n-m} \text{ is real}\}$ and write for $k \geqslant 1$

$$Y_k = \{\theta \in Y(\mu): |\theta| \leqslant k, \theta \notin A\}.$$

Let $\theta \in Y_k$. Let $K \geqslant 1$. Given $\epsilon > 0$, there is an infinite sequence of $\mathbf{q}$ for which

$$|q_1 \theta^n + q_2 \theta^m - p| < q^{-\mu+\epsilon} \tag{49}$$

(p integer). Choose an infinite subsequence such that

$$\left(\frac{q_1}{q}, \frac{q_2}{q}, \frac{p}{q}\right) \to (\beta_1, \beta_2, \gamma)$$

for some real β_1, β_2, γ. Clearly $\max(|\beta_1|, |\beta_2|) = 1$. Thus, writing

$$f(x) = \beta_1 x^n + \beta_2 x^m - \gamma,$$

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we have $$f'(\theta) = n\beta_1\theta^{n-1} + m\beta_2\theta^{m-1} \neq 0$$

as $\theta \notin A$. The polynomials $P(x) = P(\mathbf{q}, x) = q_1\theta^n + q_2\theta^m - p$ appearing in (49) obviously satisfy

$$|P'(\mathbf{q}, \theta)| > \tfrac{1}{2}q|f'(\theta)|$$

for sufficiently large q. Using

$$\left|\frac{P'(x)}{P(x)}\right| = \left|\sum_{j=1}^{n}\frac{1}{x-\alpha_j}\right| \leqslant \frac{n}{\min\limits_{1\leqslant j\leqslant n}|x-\alpha_j|},$$

($\alpha_1, \ldots, \alpha_n$ are the zeros of P), we deduce that

$$|\theta - \alpha(\mathbf{q})| < 2nq^{\epsilon-\mu-1}/|f'(\theta)|$$

for a zero $\alpha(\mathbf{q})$ of some $P(\mathbf{q}, x)$ with $|\mathbf{q}| \geqslant K$.

Now (49) implies

$$|p| < 1 + 2qk^n$$

and so the number of such $\alpha(\mathbf{q})$ for which q takes a prescribed value is

$$\leqslant 2n(2q+1)(4qk^n+3).$$

This shows that the set $$Y_{k,v} = \{\theta \in Y_k : |f'(\theta)| \geqslant v^{-1}\}$$

can be covered by discs $D_1, D_2, \ldots$ whose diameters $d_1, d_2, \ldots$ satisfy

$$\sum_{j=1}^{\infty} d_j^{\alpha} \leqslant C(n,k,v)\sum_{q=K}^{\infty} q^{2-\alpha(\mu+1-\epsilon)}.$$

If $\alpha > 3/(\mu+1-\epsilon)$ the last sum tends to 0 as $K \to \infty$ and $m(\alpha, Y_{k,v}) = 0$. It follows easily that

$$\dim(Y(\mu)\backslash A) \leqslant 3/(\mu+1). \tag{50}$$

The set $A \cap Y(\mu)$ can be decomposed into finitely many sets

$$A(\alpha) = \{re^{i\alpha} : r \text{ real}\} \cap Y(\mu)$$

where $\cos(n-m)\alpha = 1$. If $re^{i\alpha} \in A(\alpha)$ then

$$|q_1 r^n \cos n\alpha + q_2 r^m \cos m\alpha - p| < q^{-\mu+\epsilon}$$

and

$$|q_1 r^n \sin n\alpha + q_2 r^m \sin m\alpha| < q^{-\mu+\epsilon}$$

for infinitely many q. Now $\cos n\alpha \cos m\alpha$, $\sin n\alpha \sin m\alpha$ cannot both be 0 (their sum is 1). Say $\cos n\alpha \neq 0$, $\cos m\alpha \neq 0$. Then the bound

$$\dim A(\alpha) \leqslant 3/(\mu+1-\epsilon)$$

follows from Theorem 3 (or is trivial if $\frac{1}{2} < \mu \leqslant 2+\epsilon$). We now easily deduce that

$$\dim A \cap Y(\mu) \leqslant 3/(\mu+1). \tag{51}$$

A fairly straightforward adaptation to the plane of Lemmas 17 and 18 is required in order to prove

$$m(3/(\mu+1),\ Y(\mu)) > 0 \quad (\mu > \tfrac{1}{2}) \tag{52}$$

and we omit the details. Theorem 4 follows readily from (50), (51) and (52).

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