PROCEEDINGS
OF THE
Cambridge Philosophical Society.

An integral equation. By Professor G. H. HARDY and E. C. TITCHMARSH.

[*Received* 11 January, *read* 8 February, 1932.]

1. The integral equation

$$f(x) = \frac{\lambda}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} f(y)\, dy, \tag{1·1}$$

where x, $f(x)$, and λ are real and α positive, may be regarded as a differential equation of order α. Suppose for example that α is a positive integer p, that $f(x)$ tends to 0, when $x \to \infty$, with sufficient rapidity, and that

$$f_1(x) = \int_x^\infty f(y)\, dy, \quad f_2(x) = \int_x^\infty f_1(y)\, dy, \ldots.$$

Then, if we integrate repeatedly by parts, and write z for $f_p(x)$, (1·1) becomes

$$\frac{d^p z}{dx^p} = (-1)^p \lambda z.$$

The only solutions are finite combinations of exponentials.

The general equation (1·1) also has exponential solutions. The conditions that $y = e^{-ax}$ should be a solution are that $\Re(a) > 0$ and $\lambda = a^\alpha$, where a^α means $e^{\alpha \log a}$ and $\log a$ has its principal value. If $\lambda > 0$, a may have any of the values

$$\lambda^{1/\alpha} e^{2r\pi i/\alpha} \qquad (r = 0, \pm 1, \ldots)$$

for which $|2r\pi/\alpha| < \frac{1}{2}\pi$. Appropriate linear combinations of these solutions will give real solutions, and it is natural to ask if they are the only solutions. If $\alpha \leqslant 4$, and in particular if $\alpha < 1$, the only admissible value of a is $\lambda^{1/\alpha}$. The question is then whether a solution of (1·1) is necessarily a multiple of e^{-ax}.

We take this last question as typical, and answer it by proving the following theorem.

THEOREM A. *If* (i) $\lambda > 0$, $0 < \alpha < 1$;

(ii) $f(x)$ *is integrable, in the sense of Lebesgue, in any finite interval of positive values of* x;

(iii) *the integral in* (1·1) *exists, as*

$$\lim_{X \to \infty,\, \delta \to 0} \int_{x+\delta}^{X},$$

for all positive x;

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(iv) (1·1) *is true for all positive* x; *then*

$$f(x) = Ce^{-ax}, \tag{1·2}$$

where $a = \lambda^{1/\alpha}$ *and* C *is constant.*

In other words, the exponential solutions are the only solutions for which (1·1) is significant, in the most natural sense, and true for all positive x.

2. There are two natural methods for the solution of (1·1).

(i) Suppose that α is a rational number p/q, and that

$$\int^{\infty} x^n |f(x)| \, dx < \infty$$

for all n; and write (1·1) in the form

$$f = \lambda J^{\alpha} f.$$

The operator J^{α} satisfies

$$J^{\alpha} J^{\beta} f = J^{\alpha+\beta} f,$$

and so, if we iterate (1·1) q times, we obtain

$$f = \lambda^q J^p f,$$

which has only exponential solutions. Apart from the restriction on α, the method demands rather heavy restrictions on f (though not so stringent as the condition actually stated).

(ii) We may introduce the Mellin transform

$$\phi(s) = \int_0^{\infty} x^{s-1} f(x) \, dx \tag{2·1}$$

of $f(x)$, and translate (1·1) into a functional equation for $\phi(s)$. We have formally

$$\begin{aligned} \phi(s) &= \frac{\lambda}{\Gamma(\alpha)} \int_0^{\infty} x^{s-1} dx \int_x^{\infty} (y-x)^{\alpha-1} f(y) \, dy \\ &= \frac{\lambda}{\Gamma(\alpha)} \int_0^{\infty} f(y) \, dy \int_0^{y} x^{s-1} (y-x)^{\alpha-1} dx \\ &= \frac{\lambda \Gamma(s)}{\Gamma(s+\alpha)} \phi(s+\alpha). \end{aligned} \tag{2·2}$$

If

$$\chi(s) = \lambda^{s/\alpha} \frac{\phi(s)}{\Gamma(s)}, \tag{2·3}$$

then (2·2) becomes

$$\chi(s+\alpha) = \chi(s), \tag{2·4}$$

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so that $\chi(s)$ is periodic. If we can show that $\chi(s)$ also satisfies appropriate conditions of regularity and order, we shall be able to deduce that

$$\chi(s) = \Sigma a_n e^{2n\pi i s/a}, \tag{2·5}$$

a finite sum of exponentials, and our conclusion will then follow from (2·5), (2·3), and Mellin's inversion formula.

The method is familiar in principle, and we have used it repeatedly ourselves. The manner of its application in the present instance is however not quite obvious, since the conditions of Theorem A are by no means sufficient to justify the transformations as they stand.

3. We begin by proving that $f(x)$ *is an integral function of exponential type*†.

Suppose that $c > 1$ and $0 < x \leqslant \frac{1}{2}c$, and write

$$f(x) = \frac{\lambda}{\Gamma(\alpha)}\int_x^c f(y)(y-x)^{\alpha-1}\,dy + \frac{\lambda}{\Gamma(\alpha)}\int_c^\infty f(y)(y-x)^{\alpha-1}\,dy$$
$$= \lambda f_\alpha^*(x) + \lambda g(x),$$

say. Then

$$f_\alpha^*(x) = \frac{\lambda}{\Gamma(\alpha)}\int_x^c \{f_\alpha^*(y) + g(y)\}(y-x)^{\alpha-1}\,dy = \lambda f_{2\alpha}^*(x) + \lambda g_\alpha^*(x),$$

since
$$(f_\alpha^*)_\beta^* = f_{\alpha+\beta}^*.^{\ddagger}$$

Hence
$$f(x) = \lambda g(x) + \lambda^2 g_\alpha^*(x) + \lambda^2 f_{2\alpha}^*(x).$$

Repeating the argument, we obtain

$$f(x) = \lambda g(x) + \lambda^2 g_\alpha^*(x) + \ldots + \lambda^n g_{(n-1)\alpha}^*(x) + \lambda^n f_{n\alpha}^*(x).$$

But
$$\lambda^n f_{n\alpha}^*(x) = \frac{\lambda^n}{\Gamma(n\alpha)}\int_x^c f(y)(y-x)^{n\alpha-1}\,dy \to 0,$$

when $n \to \infty$, and so

$$f(x) = \sum_0^\infty \lambda^{n-1} g_{n\alpha}^*(x) \quad (0 < x \leqslant \tfrac{1}{2}c). \tag{3·1}$$

4. So far x is real and positive: we now use the series (3·1) to define $f(x)$ for complex x.

† $f(z)$ is 'of exponential type' if $|f(z)| < Ke^{K|z|}$ for some K.

‡ This is

$$\frac{1}{\Gamma(\beta)}\int_x^c f_\alpha^*(y)(y-x)^{\beta-1}\,dy = \frac{1}{\Gamma(\alpha+\beta)}\int_x^c f(y)(y-x)^{\alpha+\beta-1}\,dy,$$

and is familiar when all the integrals involved are absolutely convergent. The formula is proved, under conditions much more general than those required here, by Bosanquet (2).

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We write

$$F(x)=\int_x^\infty y^{\alpha-1}f(y)\,dy \tag{4·1}$$

(at present for real positive x). By condition (iii) of Theorem A, $F(x)$ is bounded for $x>1$, and tends to zero when $x\to\infty$. We denote the upper bound of $|F(x)|$ by M, and use $K=K(\alpha)$ to denote a positive number depending on α only.

It is plain that $g(x)$ is an analytic function of x regular for $|x|<c$. Also

$$\begin{aligned} g(x) &= -\int_c^\infty \left(\frac{y-x}{y}\right)^{\alpha-1} F'(y)\,dy \\ &= \left(\frac{c-x}{c}\right)^{\alpha-1} F(c)+\int_c^\infty F(y)\frac{d}{dy}\left(\frac{y-x}{y}\right)^{\alpha-1} dy. \end{aligned} \tag{4·2}$$

From (4·2) it follows (i) that $|g(x)|$ has an upper bound of the type KM for $|x|\leqslant\frac{1}{2}c$, and (ii) that

$$|g(x)|\leqslant 2M\left(\frac{c-x}{c}\right)^{\alpha-1}$$

for real x between $\frac{1}{2}c$ and c.

We define $g^*_{n\alpha}(x)$, where $n>0$, x is complex, and $|x|\leqslant\frac{1}{2}c$, by

$$g^*_{n\alpha}(x)=\frac{1}{\Gamma(n\alpha)}\int_x^{\frac{1}{2}c}(y-x)^{n\alpha-1}g(y)\,dy+\frac{1}{\Gamma(n\alpha)}\int_{\frac{1}{2}c}^{c}(y-x)^{n\alpha-1}g(y)\,dy,$$

each integration being rectilinear. The second integral is majorised by

$$\frac{2Mc^{1-\alpha}}{\Gamma(n\alpha)}\int_{\frac{1}{2}c}^{c}|y-x|^{n\alpha-1}(c-y)^{\alpha-1}dy<\frac{KMc^{n\alpha}}{\Gamma(n\alpha)}.$$

It is therefore regular for $|x|\leqslant\frac{1}{2}c$; and the first integral is plainly regular and has a majorant of the same form. Hence $g^*_{n\alpha}(x)$ is regular and has a similar majorant.

It follows that the series (3·1) is uniformly convergent for $|x|\leqslant\frac{1}{2}c$. Hence $f(x)$ is an analytic function regular for $|x|\leqslant\frac{1}{2}c$, and therefore an integral function. Further,

$$|f(x)|\leqslant KM+KM\sum_1^\infty\frac{\lambda^{n-1}c^{n\alpha}}{\Gamma(n\alpha)}<KMe^{K\lambda c}$$

for $|x|\leqslant\frac{1}{2}c$, so that $f(x)$ is of exponential type.

5. We prove next that

$$f(x)=o(x^{1-\alpha}), \tag{5·1}$$

as $x\to\infty$, uniformly in any finite half-strip parallel to the real axis. For this we consider $F(x)$ for complex x, defining it by (4·1) and taking the integral in the first instance from x to a point in the

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real axis and then along the real axis to infinity. $F(x)$ is regular except at the origin, is of exponential type, and tends to zero along the positive real axis. Hence* it tends to zero uniformly in the half-strip, and therefore

$$F'(x) = -x^{\alpha-1}f(x)$$

does the same. This proves (5·1).

It now follows from Cauchy's theorem that (1·1) is true for complex x, and that the integration may be taken along any path (of sufficient regularity) from x to infinity inside the half-strip, and in particular along a line parallel to the real axis.

6. We now introduce the function $\phi(s)$ defined, when

$$0 < \sigma = \Re(s) < \alpha,$$

by (2·1). It is essential for our purpose to use a more extensive representation of $\phi(s)$, and we select the representation

$$\phi(s) = \frac{i}{2\sin s\pi}\int_C (-x)^{s-1}f(x)\,dx, \tag{6·1}$$

where

$$(-x)^{s-1} = e^{(s-1)\log(-x)},$$

the logarithm being real for $x<0$, and C is a loop surrounding the positive real axis in the counter-clockwise direction. It follows from (6·1), in the classical manner, that $\phi(s)$ is regular for $\sigma<\alpha$, except perhaps for simple poles at $s=0, -1, -2, \ldots$, and that

$$\psi(s) = \frac{\phi(s)}{\Gamma(s)}$$

is regular for $\sigma<\alpha$.

We have formally

$$\begin{aligned}\phi(s) &= \frac{i}{2\sin s\pi}\int_C (-x)^{s-1}dx\,\frac{\lambda}{\Gamma(\alpha)}\int_x^\infty (y-x)^{\alpha-1}f(y)\,dy\\ &= \frac{i}{2\sin s\pi}\int_C (-x)^{s-1}dx\,\frac{\lambda}{\Gamma(\alpha)}\int_0^\infty u^{\alpha-1}f(u+x)\,du\\ &= \frac{\lambda}{\Gamma(\alpha)}\,\frac{i}{2\sin s\pi}\int_0^\infty u^{\alpha-1}\,du\int_C (-x)^{s-1}f(u+x)\,dx.\end{aligned}$$

If here we write w for $u+x$, and denote by C' a loop like C, but passing round $w=u$ instead of $w=0$, we obtain

$$\phi(s) = \frac{\lambda}{\Gamma(\alpha)}\,\frac{i}{2\sin s\pi}\int_0^\infty u^{\alpha-1}\,du\int_{C'}(u-w)^{s-1}f(w)\,dw.$$

* See, for example, Hardy and Titchmarsh (4), Lemma 8. We may prove first that $F(x)$ is bounded, and then use Montel's theorem. Theorems of this type are not affected by a possible singularity at the origin.

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Here we may replace C' by C; and if we do this, and again invert the order of integration, we obtain

$$\phi(s) = \frac{\lambda}{\Gamma(\alpha)} \frac{i}{2\sin s\pi} \int_C f(w)\, dw \int_0^\infty u^{\alpha-1}(u-w)^{s-1}\, du.$$

Finally, since

$$\int_0^\infty u^{\alpha-1}(u-w)^{s-1}\, du = (-w)^{\alpha+s-1} \frac{\Gamma(\alpha)\,\Gamma(1-s-\alpha)}{\Gamma(1-s)},$$

we deduce

$$\psi(s) = \frac{\phi(s)}{\Gamma(s)} = \lambda \frac{i}{2\Gamma(s+\alpha)\sin(s+\alpha)\pi} \int_C (-w)^{\alpha+s-1} f(w)\, dw$$
$$= \lambda\psi(s+\alpha),$$

which is (2·4). Our deduction is so far formal. We proceed to justify our formal analysis for values of s whose real part σ is negative and sufficiently large. It is sufficient to consider real values of s.

7. Suppose that $s < -B < 0$. An examination of the transformations of § 6, in the light of what was proved in § 5, shows that there are two steps only which require justification, viz. the two inversions of the order of integration.

(1) We have to show that

$$\int_C (-x)^{s-1}\, dx \int_0^\infty u^{\alpha-1} f(u+x)\, du = \int_0^\infty u^{\alpha-1}\, du \int_C (-x)^{s-1} f(u+x)\, dx.$$

It is plain from considerations of uniform convergence that the inversion would be valid if the limit ∞ were replaced by a finite U; and it is therefore sufficient to prove that

$$\int_C (-x)^{s-1}\, dx \int_U^\infty u^{\alpha-1} f(u+x)\, du$$

exists and tends to zero when $U \to \infty$. This will certainly be so if

$$\left| \int_U^\infty u^{\alpha-1} f(u+x)\, du \right| < k(|x|), \quad \int_C |(-x)^{s-1}|\, k(|x|)\, |dx| < \infty.$$

In particular, since B is at our disposal, it will be sufficient to prove that we may take for $k(|x|)$ a function of the type $K|x|^K$. Further, we need only consider the distant part of C, on which $\mathfrak{R}(x)$ is positive and large.

Now $$J = \int_U^\infty u^{\alpha-1} f(u+x)\, du = \int_{x+U}^\infty (w-x)^{\alpha-1} f(w)\, dw,$$

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the w-path being parallel to the real axis. If $U < |x|$, then

$$J = \int_{x+U}^{x+|x|} + \int_{x+|x|}^{\infty} = J_1 + J_2,$$

say. It is plain from (5·1) that J_1 has a majorant of the type required, and

$$J_2 = -\int_{x+|x|}^{\infty} \left(\frac{w-x}{w}\right)^{\alpha-1} F'(w)\,dw$$
$$= \left(\frac{|x|}{x+|x|}\right)^{\alpha-1} F(x+|x|) + \int_{x+|x|}^{\infty} F(w) \frac{d}{dw}\left(\frac{w-x}{w}\right)^{\alpha-1} dw$$

is bounded. If $U \geqslant |x|$ the argument is simpler, no decomposition of the integral being required. In any case J has a majorant $K|x|^K$, and the inversion is justified. It is in fact sufficient that $B \geqslant 1$.

8. (2) The second inversion is

$$\int_0^{\infty} u^{\alpha-1}\,du \int_C (u-w)^{s-1} f(w)\,dw = \int_C f(w)\,dw \int_0^{\infty} u^{\alpha-1}(u-w)^{s-1}\,du.$$

It is sufficient to show that

$$\int_C f(w)\,dw \int_U^{\infty} u^{\alpha-1}(u-w)^{s-1}\,du$$

exists and tends to zero when $U \to \infty$. Further, this would plainly be true if C were replaced by any finite part of itself. It is therefore sufficient to prove that, if C_ϖ is the part of C on which $\mathfrak{R}(w) > \varpi$, then

$$(8{\cdot}1) \qquad |J| = \left| \int_{C_\varpi} f(w)\,dw \int_U^{\infty} u^{\alpha-1}(u-w)^{s-1}\,du \right| < \epsilon$$

for $\varpi \geqslant \varpi_0(\epsilon)$ and all U in question. We need only consider the upper half of C_ϖ, which we may suppose defined by

$$w = i + W, \quad W > \varpi.$$

9. Integrating twice by parts, we have

$$\int_U^{\infty} u^{\alpha-1}(u-w)^{s-1}\,du = -U^{\alpha-1}\frac{(U-w)^s}{s} + (\alpha-1)\,U^{\alpha-2}\frac{(U-w)^{s+1}}{s(s+1)}$$
$$+ \frac{(\alpha-1)(\alpha-2)}{s(s+1)} \int_U^{\infty} u^{\alpha-3}(u-w)^{s+1}\,du = j_1 + j_2 + j_3,$$

say. We denote the contributions of these three terms to J by J_1, J_2, and J_3 respectively.

(i) We estimate j_3 on the hypothesis that $U < \frac{1}{2}W$. The result is the same, and the argument simpler, in the contrary case. We suppose $B \geqslant 3$, so that $s < -3$.

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Then

$$|j_3| < K\int_U^\infty u^{\alpha-3}\{(u-W)^2+1\}^{\frac{1}{2}(s+1)}\,du = K\left(\int_U^{\frac{1}{2}W} + \int_{\frac{1}{2}W}^{\frac{3}{2}W} + \int_{\frac{3}{2}W}^\infty\right)$$

$$< K\left(\int_U^{\frac{1}{2}W} u^{\alpha-3}W^{s+1}\,du + \int_{\frac{1}{2}W}^{\frac{3}{2}W} W^{\alpha-3}\{(u-W)^2+1\}^{\frac{1}{2}(s+1)}\,du + \int_{\frac{3}{2}W}^\infty u^{\alpha-3}W^{s+1}\,du\right).$$

The first and last terms are $O(W^{s+1})$ and the second is $O(W^{\alpha-3})$. Since $f(w) = o(W^{1-\alpha})$, we deduce

$$J_3 = o\left(\int_\varpi^\infty W^{2-\alpha+s}\,dW + \int_\varpi^\infty W^{-2}\,dW\right) = o(1).$$

It follows that (8·1) is true if J is replaced by J_3.

(ii) In estimating J_1 and J_2 we suppose that $\varpi < \frac{1}{2}U$. The argument is again simpler in the contrary case.

Since $f(w) = o(W^{1-\alpha})$, we have

$$J_1 = o\left[U^{\alpha-1}\int_\varpi^\infty W^{1-\alpha}\{(U-W)^2+1\}^{\frac{1}{2}s}\,dW\right]$$

$$= o\left\{U^{\alpha-1}\left(\int_\varpi^{\frac{1}{2}U} + \int_{\frac{1}{2}U}^{\frac{3}{2}U} + \int_{\frac{3}{2}U}^\infty\right)\right\}$$

$$= o\left\{U^{\alpha-1}\int_\varpi^{\frac{1}{2}U} W^{1-\alpha}(U-W)^s\,dW\right\}$$

$$+ o\left[U^{\alpha-1}\int_{\frac{1}{2}U}^{\frac{3}{2}U} U^{1-\alpha}\{(U-W)^2+1\}^{\frac{1}{2}s}\,dW\right]$$

$$+ o\left\{U^{\alpha-1}\int_{\frac{3}{2}U}^\infty W^{1-\alpha}(W-U)^s\,dW\right\}$$

$$= o(U^{s+1}) + o(1) + o(U^{s+1}) = o(1).$$

Here it is only necessary to suppose $B \geqslant 2$. The corresponding argument with J_2 demands that $B \geqslant 3$, and leads to

$$J_2 = o(U^{s+1}) + o(U^{-1}) + o(U^{s+1}) = o(1).$$

From these results (and the corresponding results when $\varpi > \frac{1}{2}U$) it follows that (8·1) is true if J is replaced by J_1 or J_2; and this completes the justification of the inversion.

It follows that (2·4) is true for real $s < -3$, and so wherever $\chi(s)$ is regular.

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10. It is now easy to complete the proof of Theorem A. $\chi(s)$ is regular for $\sigma < a$, and has period a, so that it is an integral function. Also, since the argument of $-x$ on C does not exceed π,

$$\chi(s) = \lambda^{s/a}\frac{\phi(s)}{\Gamma(s)} = \lambda^{s/a}\frac{i\Gamma(1-s)}{2\pi}\int(-x)^{s-1}f(x)\,dx$$

is $O(e^{\frac{1}{2}\pi|t|})$ on any parallel to the imaginary axis. It follows from classical theorems that $\chi(s)$ is a constant*.

Hence $$\phi(s) = A\lambda^{-s/a}\Gamma(s),$$

where A is constant. In particular this is true in the strip $0 < \sigma < a$ in which $\phi(s)$ is defined by (2·1). Inverting by Mellin's formula†, we find that $f(x)$ is a multiple of e^{-ax}, where $a^a = \lambda$.

The proof would of course been very much simpler and shorter if we had imposed more restrictions on $f(x)$, but such restrictions would have destroyed the interest of the theorem, which lies almost entirely in its generality.

We may observe in conclusion that our equation is included in the general type

$$f(x) = \lambda\int_{-\infty}^{\infty} K(y-x)f(y)\,dy, \tag{10·1}$$

with $K(u) = u^{a-1}$ for positive and $K(u) = 0$ for negative u. It has been proved by Bochner that, in certain circumstances, (10·1) has only exponential solutions; but his conditions are much too stringent for application to the problem considered here‡.

* A function $F(s)$ which is integral, has period a, and is $O(e^{A|t|})$ for some A, is necessarily a finite sum

$$\Sigma a_n e^{2n\pi is/a}.$$

Here $A = \frac{1}{2}\pi < 2\pi/a$, so that $n = 0$ is the only admissible value.

† Appropriate conditions will be found in Hardy (3).

‡ Bochner (1). Bochner supposes that $K(u)$ tends to zero, when $u \to \infty$, more rapidly than any power of u, and that $|f(x)| < a(1+|x|)^b$ for some constant a, b.

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