Forum of Mathematics, Sigma (2026), **Vol. 14**:e77 1–37
doi:10.1017/fms.2026.10219

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Polymatroids and moduli of points in flags

Patricio Gallardo[1], Javier González-Anaya[2,3] and Jose Gonzalez[4]

[1]Department of Mathematics, University of California, Riverside, CA, USA; E-mail: pgallard@ucr.edu.
[2]Department of Mathematics and Computer Science, Santa Clara University, Santa Clara, CA, USA;
E-mail: jgonzalezanaya@scu.edu.
[3]Department of Mathematics, Harvey Mudd College, Claremont, CA, USA.
[4]Department of Mathematics, University of California, Riverside, CA, USA;
E-mail: joselg@ucr.edu (Corresponding author).

Received: 21 May 2025; **Revised:** 23 December 2025; **Accepted:** 2 February 2026

2020 Mathematical Subject Classification: *Primary* – 14J10, 05B35, 14M25; *Secondary* – 14D06, 52C35, 52B12

Abstract

We introduce and study different compactifications of the moduli space of n distinct weighted labeled points in a flag of affine spaces. We construct these spaces via the weighted and generalized Fulton-MacPherson compactifications of Routis and Kim-Sato. For certain weights, our compactifications are toric and isomorphic to the polypermutohedral and polystellahedral varieties, which arise in the work of Crowley-Huh-Larson-Simpson-Wang and Eur-Larson on polymatroids, a generalization of matroids. Moreover, we show that these toric compactifications have a fibration structure, with fibers isomorphic to the Losev-Manin space, and are related to each other via a geometric quotient.

Contents

1 Introduction **2**
2 Preliminaries **4**
2.1 Notation . . . 4
2.2 Polymatroids, polypermutohedra and polystellahedra . . . 4
2.3 Wonderful compactifications . . . 7
2.4 Fulton-MacPherson compactifications and related constructions . . . 8
3 Moduli of points in a flag and polypermutohedral varieties **9**
3.1 Weighted Fulton-MacPherson compactification of points in a flag . . . 9
3.2 Moduli of n weighted points in a flag up to translation and scaling . . . 12
3.3 The Losev-Manin compactification and the polypermutohedral variety . . . 17
3.4 Pullback of building sets and nontrivial toric fibrations . . . 20
3.5 Proof of Theorem 1.1 . . . 22
3.6 Pullback of building sets and refinements . . . 23
4 Generalized Fulton-MacPherson compactification and polystellahedral varieties **24**
4.1 Generalized Fulton-MacPherson compactification . . . 24
4.2 The polystellahedral variety as a blow-up of a product of projective spaces . . . 26
4.3 The polypermutohedral variety as a quotient of the polystellahedral variety . . . 35
4.4 Proof of Theorem 1.2 . . . 36
References **37**

© The Author(s), 2026. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution-NonCommercial-ShareAlike licence (https://creativecommons.org/licenses/by-nc-sa/4.0), which permits non-commercial re-use, distribution, and reproduction in any medium, provided the same Creative Commons licence is used to distribute the re-used or adapted article and the original article is properly cited. The written permission of Cambridge University Press or the rights holder(s) must be obtained prior to any commercial use.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

1. Introduction

Our work presents novel morphisms between two types of spaces, one originating in moduli theory, and the other in combinatorics, specifically in the theory of polymatroids.

From the moduli perspective, we study generalizations of the Hassett spaces of weighted pointed stable rational curves. On the combinatorial side, we encounter as special cases the polypermutohedral and polystellahedral varieties of [CHL+22, EL24], which arise from polytopes that generalize the permutohedron and stellahedron.

We begin by describing the moduli-theoretic side.Recall that the Fulton-MacPherson (FM) compactification $X[n]$, introduced in 1994, is a smooth normal crossings compactification of the configuration space of n distinct labeled points in a smooth variety X [FM94]. It has since been widely used in moduli theory; see, for example, [CGK09, GR17, Pan95, KKO14]. In 2014, Routis [Rou14] generalized $X[n]$ to a weighted version $X_{\mathbf{w}}[n]$, allowing certain coincidences between points prescribed by a weight vector $\mathbf{w} \in \mathcal{D}_n^{\mathrm{FM}}$ (Definition 2.17). The moduli spaces constructed via the FM compactifications in the previous examples, as well as those we construct here, inherit a "universal" flat family from the FM space $X[n]^+ \to X[n]$. The fibers of the family are either the configuration of distinct points on X, or degenerations of X marked by n points; see Section 2.4.

Given an n-tuple of positive integers $\mathbf{a} = (a_1, \ldots, a_n)$ with $a_{i+1} \leq a_i$, we use the weighted Fulton-MacPherson space to construct compactifications $\mathbb{C}_{\mathbf{w}}^{[\mathbf{a}]}$ of the open locus $\mathcal{U}^{\mathbf{a}} \subsetneq \mathbb{C}^{\mathbf{a}} := \prod \mathbb{C}^{a_i}$ parametrizing n distinct weighted labeled points p_i within a flag $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$, where $p_i \in \mathbb{C}^{a_i}$ for all i. We then leverage $\mathbb{C}_{\mathbf{w}}^{[\mathbf{a}]}$ to define a compactification $T_{\mathbf{w}}^{\mathbf{a}}$ of the moduli space of n weighted points in the flag, up to translation along the $\mathbb{C}^{a_n}$-coordinates and global scaling by a $\mathbb{G}_m$ factor; see Section 3.2. We say the above compactifications are *geometric* because they inherit a "universal" family from the FM space. When $\mathbf{a} = (d, \ldots, d)$, the space $T_{\mathbf{w}}^{\mathbf{a}}$ recovers the moduli spaces of weighted pointed trees of projective spaces from [CGK09, GR17] as special cases. In particular, when $d = 1$, $T_{\mathbf{w}}^{\mathbf{a}}$ specializes to the moduli space of weighted pointed stable rational curves [Has03].

On the combinatorial side, we have the polypermutohedral varieties, introduced in 2022 in [CHL+22]. These varieties arise from studying the intersection theory of polymatroids, which are combinatorial abstractions of subspace arrangements, in the same way that matroids are combinatorial abstractions of hyperplane arrangements [EL24]. The inner normal fan of the polypermutohedron coincides with the Bergman fan of its associated polymatroid, and these varieties play a crucial role in establishing the Kähler package for the Chow ring of polymatroids. While their construction is not initially related to moduli theory, they can be seen as a generalization of the permutohedral variety.

In our first result, we relate the above algebro-geometric and combinatorial constructions. We generalize the well-known fact that the permutohedral variety is isomorphic to the toric compactification of a moduli space of weighted points known as the Losev-Manin compactification [LM00]. Its proof is given in Section 3.

Theorem 1.1. *Consider an n-tuple $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}_{>0}^n$ with $a_{i+1} \leq a_i$, and a weight vector $\mathbf{w} \in \mathcal{D}_n^{\mathrm{T}}$ (Definition 3.10). There exists a smooth, normal crossings, geometric compactification $T_{\mathbf{w}}^{\mathbf{a}}$ of the moduli space of n distinct labeled points in the flag $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$, up to scaling and translations preserving the flag, such that:*

(i). *If $\mathbf{w} = (\varepsilon_1, \ldots, \varepsilon_{n-1}, 1)$ with $\sum_{i=1}^{n-1} \varepsilon_i \leq 1$, then $T_{LM}^{\mathbf{a}} := T_{\mathbf{w}}^{\mathbf{a}}$ is isomorphic to the polypermutohedral toric variety associated to $\mathbf{a}^- := (a_1, \ldots, a_{n-1})$. We refer to this as the Losev-Manin compactification of this moduli problem.*

(ii). *Let $T^{\mathbf{a}}$ denote the compactification corresponding to the weight $\mathbf{w} = (1, \ldots, 1)$. Just as in Kapranov's construction of $\overline{M}_{0,n}$, there is a sequence of smooth blow-ups*

$$T^{\mathbf{a}} \longrightarrow T_{LM}^{\mathbf{a}} \longrightarrow \mathbb{P}^{a_1+\ldots+a_{(n-1)}-1}.$$

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

(iii). $T^{\mathbf{a}}_{LM}$ *is a nontrivial, Zariski locally trivial fibration over* $\prod_{i=1}^{n-1}\mathbb{P}^{a_i-1}$*, with fiber isomorphic to the standard* $(n-2)$*-dimensional Losev-Manin moduli compactification* $\overline{M}^{LM}_{0,n+1}$.

Our next result is motivated by the following moduli considerations: If $T^{\mathbf{a}}_{LM}$ is a toric compactification of the moduli space of n points in the flag $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$, up to translation and scaling, then we expect that there exists another space parametrizing configurations of points without considering equivalence classes. One would anticipate that this configuration space recovers $T^{\mathbf{a}}_{LM}$ as a quotient, since this behavior is observed for the moduli space of points in the line [HK00, Theorem 3.4]. Moreover, one would expect such space to be modular and admit a birational map to the product of projective spaces $\prod_{i=1}^{n}\mathbb{P}^{a_i}$, as this latter space is a natural compactification of n points in the flag $\mathbb{C}^{a_n} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$.

The following theorem shows that all our expectations are realized for arbitrary dimensions and numbers of points, and its proof is given in Section 4. On the geometric side, we construct the moduli spaces $\mathbb{P}^{[\mathbf{a}]}_H$ via the generalized Fulton-MacPherson compactification, introduced in [KS09] and denoted $X^{[n]}_D$. In this generalization, one fixes a nonsingular proper subvariety $D \subseteq X$ and parametrizes configurations of points away from D. As in the classical case, $X^{[n]}_D$ carries a "universal" flat family parametrizing degenerations of X marked by n points, which are always away from D but not necessarily distinct. On the combinatorial side, we encounter the polystellahedral variety $PS_{\mathbf{a}}$ of an n-tuple $\mathbf{a}$; this variety plays a central role in constructing the augmented Chow rings of polymatroids [EL24]. As in our first theorem, the geometric and combinatorial constructions are independent of each other, and a priori, there was no reason to expect they were the same.

Theorem 1.2. *Consider an* n*-tuple* $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ *with* $a_{i+1} \leq a_i$*. There exists a smooth, normal crossings, geometric compactification* $\mathbb{P}^{[\mathbf{a}]}_H$ *of the configuration space of* n *not necessarily distinct points in the flag* $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$ *such that:*

(i). *The variety* $\mathbb{P}^{[\mathbf{a}]}_H$ *is constructed as an iterated blow-up of* $\prod_{i=1}^{n}\mathbb{P}^{a_i}$ *along torus-invariant subvarieties. In particular, it is a toric variety itself.*

(ii). $\mathbb{P}^{[\mathbf{a}]}_H$ *is isomorphic to the polystellahedral variety associated to* $\mathbf{a}$.

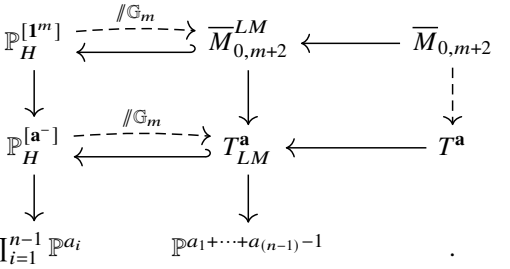
(iii). *Let* $\mathbf{a}^- := (a_1, \ldots, a_{n-1})$*. There exists an open* $\left(\mathbb{P}^{[\mathbf{a}^-]}_H\right)^\circ \subsetneq \mathbb{P}^{[\mathbf{a}^-]}_H$ *and a geometric quotient such that*

$$T^{\mathbf{a}}_{LM} \cong \left(\mathbb{P}^{[\mathbf{a}^-]}_H\right)^\circ /\!\!/ \mathbb{G}_m.$$

Along the way, we study pullbacks and refinements of combinatorial building sets (Sections 3.4 and 3.6), produce associated toric fibrations (Section 3.4), give combinatorial interpretations to some toric fans (Section 4.2), and obtain a quotient presentation of the exceptional divisor of a toric blow-up (Section 4.3). These can all be of independent interest.

We conclude with some remarks about our results. Theorems 1.1 and 1.2 provide evidence that our moduli spaces are combinatorially rich objects that generalize both the permutohedral and stellahedral varieties. Moreover, by [EL24, Proposition 2.2], given any n-tuple $\mathbf{a}$, there is a map $\mathbb{P}^{[\mathbf{1}^m]}_H \to \mathbb{P}^{[\mathbf{a}^-]}_H$, where $\mathbf{a}^- = (a_1, \ldots, a_{n-1})$, $m = \sum_{i=1}^{n-1} a_i$, and $\mathbf{1}^m = (1, \ldots, 1) \in \mathbb{Z}^m_{>0}$. Analogously, we prove in Proposition 3.43 that there is a map $\overline{M}^{LM}_{0,m+2} \to T^{\mathbf{a}}_{LM}$. Currently, there is no moduli interpretation for these maps. Altogether, we can summarize the morphisms described thus far in the following diagram:

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press



Here, besides the two vertical arrows described above, the remaining vertical solid arrows, horizontal solid arrows in the second row, and dashed arrows are obtained from Theorems 1.1 and 1.2. Lastly, the horizontal inclusion maps indicate the fact that $T_{LM}^{\mathbf{a}}$ embeds as a torus-invariant divisor inside $\mathbb{P}_H^{[\mathbf{a}^-]}$, as already presented in the discussion before Theorem 1.5 in [EHL23].

Many fundamental questions about the birational geometry of $\overline{M}_{0,n}$ naturally extend to the setting of $T^{\mathbf{a}}$. For example, the study of the birational geometry of $T^{\mathbf{a}}$, and the values of $\mathbf{a}$ for which $T^{\mathbf{a}}$ is a Mori dream space—a topic that has drawn substantial attention in the algebraic geometry community. While the classification of values of n for which $\overline{M}_{0,n}$ is a Mori dream space is now largely resolved (see [CT15, GK16, HKL18]), it would be interesting to answer the analogous question of determining for which n-tuples $\mathbf{a}$ the space $T^{\mathbf{a}}$ is a Mori dream space; see, for example, [GGAGR25, Theorem 1.3].

2. Preliminaries

In this section we introduce the notation and preliminary results needed for our work. Throughout this article we work over the field of complex numbers $\mathbb{C}$.

2.1. Notation

Define $[n] := \{1, \ldots, n\}$ for all $n \in \mathbb{Z}_{>0}$. Given a finite set E, consider the lattice $\mathbb{Z}^E$ and the vector space $\mathbb{R}^E = \mathbb{Z}^E \otimes_{\mathbb{Z}} \mathbb{R}$, both with distinguished basis $\{e_i \mid i \in E\}$. If $I \subseteq E$, define

$$e_I := \sum_{i \in I} e_i \in \mathbb{Z}^E \subseteq \mathbb{R}^E.$$

Given a point $\mathbf{x} = \sum_{i \in E} x_i e_i \in \mathbb{R}^E$ and a subset $S \subseteq E$, define $\mathbf{x}_S = \sum_{i \in S} x_i$. We denote the equivalence class of $e_I \in \mathbb{R}^E$ in $\mathbb{R}^E/\mathbb{R}e_E$ by $\overline{e}_I$. The associated projective space to E, denoted as $\mathbb{P}^E$, is the toric variety constructed from the fan Σ_E in $\mathbb{R}^E/\mathbb{R}e_E$ given by the collection of all cones of the form

$$\operatorname{cone}(\overline{e}_{i_1}, \ldots, \overline{e}_{i_k} \mid I = \{i_1, \ldots, i_k\} \subsetneq E),$$

as I ranges over all proper subsets of E. Notice that if $|E| = n$, then $\mathbb{P}^E$ has dimension $n-1$. A vector $\mathbf{x} = \sum_{i \in E} x_i e_i$ has homogeneous coordinates $[x_i \mid i \in E] \in \mathbb{P}^E$.

Notation 2.1. Given a fan Σ supported in $\mathbb{R}^E/\mathbb{R}e_E$, we denote its corresponding toric variety by $X(\Sigma)$.

Here we always consider the star subdivision of a fan Σ along a cone $\tau \in \Sigma$ to be its barycentric subdivision along τ. In other words, the subdivision of Σ by the vector $v_\tau = v_1 + \cdots + v_r$, where the v_i are the primitive lattice vectors generating the rays of τ.

2.2. Polymatroids, polypermutohedra and polystellahedra

Next, we discuss the necessary combinatorial background on polymatroids, which are generalizations of matroids. Just as in the theory of matroids, polymatroids admit several equivalent definitions. Here we

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

will present their definition in terms of rank functions, and then prove the equivalence of this presentation with the one using independence and base polytopes. From the latter perspective, polymatroids are precisely the integral generalized permutohedra from [Pos09]. Polymatroids were originally studied by Edmonds in connection with combinatorial optimization problems [Edm70]. In recent years, there has been increased interest in extending central results from matroid theory to polymatroid theory. For example, many authors have focused on generalizing key concepts and properties, such as defining the Bergman fan of a polymatroid [CHL+22], their augmented Chow rings [EL24], and even their Hodge theory [PP23]. Additionally, polymatroids have found applications in the realm of commutative algebra and algebraic geometry; see, for example, [CCRL+20].

Definition 2.2. A function $f : 2^E \to \mathbb{R}$ is called a *set function*. A set function f is:

- *Normalized*, if $f(\emptyset) = 0$.
- *Monotone* or *nondecreasing*, if $f(S) \leq f(T)$ whenever $S \subseteq T \subseteq E$.
- *Submodular*, if $f(I \cup J) + f(I \cap J) \leq f(I) + f(J)$.

There are two important polytopes associated with normalized submodular set functions, which can be thought of as generalizing the notions of base and independence polytopes of matroids.

Definition 2.3. Let $f : 2^E \to \mathbb{R}$ be a normalized submodular set function.

1. We denote the *generalized permutohedron* associated to f as
$$GP(f) = \left\{\mathbf{x} \in \mathbb{R}^E \mid \mathbf{x}_E = f(E) \text{ and } \mathbf{x}_S \leq f(S) \text{ for all } S \subseteq E\right\}.$$
2. If f is further assumed to be monotone, then we refer to its corresponding generalized permutohedron as the *base polytope* corresponding to f. We denote it by $B(f)$.
3. Suppose that f is monotone, then we define the *independence polytope* corresponding to f as
$$I(f) = \left\{\mathbf{x} \in \mathbb{R}^E_{\geq 0} \mid \mathbf{x}_S \leq f(S) \text{ for all } S \subseteq E\right\}.$$

In particular, note that $B(f)$ is the face of $I(f)$ whose supporting hyperplane has normal vector $e_E = (1, \ldots, 1) \in \mathbb{R}^E$.

Definition 2.4 (cf. [EL24, Definition 1.1]). A *cage* over $[n]$ is an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of nonnegative integers. If $a_1 = \cdots = a_n$ we say the cage is *constant*. A *caging* of a finite set E is a surjective map of finite sets $\pi : A \to E$. We assume without loss of generality that $E = [n] = \{1, \ldots, n\}$, in which case the cage corresponding to this caging is $\mathbf{a} = (a_1, \ldots, a_n)$, where $a_i = |\pi^{-1}(i)|$.

For simplicity, in the sequel we shall consider only cages $\mathbf{a}$ over $[n]$ such that $a_1 \geq \cdots \geq a_n$. This constitutes no loss of generality and simplifies the presentation significantly.

Definition 2.5. A *polymatroid* $P = (E, f)$ is a pair of a finite set E and a normalized, submodular, monotone set function $f : 2^E \to \mathbb{R}$. The function f is often called the *rank function* of the polymatroid.

- The *independence polytope* of P is $I(P) := I(f) \subseteq \mathbb{R}^E_{\geq 0}$.
- The *base polytope* of P is $B(P) := B(f) \subseteq \mathbb{R}^E_{\geq 0}$. It is the face of $I(P)$ defined by the vector $e_E = (1, \ldots, 1) \in \mathbb{R}^E_{\geq 0}$.

We say that the polymatroid has cage $\mathbf{a} = (a_1, \ldots, a_n)$ if $f(i) \leq a_i$ for all $i \in [n]$. In particular, a *matroid* is a polymatroid with cage $\mathbf{a} = (1, \ldots, 1)$.

A polymatroid P over a set E is equivalently defined by its rank function, its independence polytope or its base polytope. In order to see this let us recall the following characterization result. For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^E_{\geq 0}$, write $\mathbf{u} \geq \mathbf{v}$ if $\mathbf{u} - \mathbf{v} \in \mathbb{R}^E_{\geq 0}$.

Proposition 2.6 [Sch03, Theorem 44.5]. *A nonempty polytope $Q \subseteq \mathbb{R}^E_{\geq 0}$ is the independence polytope of a polymatroid if and only if:*

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

1. *Given* $\mathbf{v} \in \mathbb{R}^E_{\geq 0}$, *if* $\mathbf{u} \geq \mathbf{v}$ *for some* $\mathbf{u} \in Q$, *then* $\mathbf{v} \in Q$.
2. *Given* $\mathbf{v} \in \mathbb{R}^E_{\geq 0}$, *every maximal element of* $\mathbf{Q} \cap \{\mathbf{u} \leq \mathbf{v}\}$ *has the same sum of coordinates* $\mathbf{u}_E$.

In particular, if Q *is the independence polytope of a polymatroid, then the rank function* f_Q *of the polymatroid is*

$$f_Q(S) = \max\{\mathbf{x}_S \mid \mathbf{x} \in Q\}, \quad \textit{for all } S \subseteq E.$$

A polytope $R \subseteq \mathbb{R}^E_{\geq 0}$ is the base polytope of a polymatroid if the set

$$Q = \{\mathbf{u} \in \mathbb{R}^E_{\geq 0} \mid \text{ there exists } \mathbf{v} \in R \text{ such that } \mathbf{v} - \mathbf{u} \in \mathbb{R}^E_{\geq 0}\}$$

is the independence polytope of a polymatroid. When this is the case, the second item in Proposition 2.6 guarantees R is the face of Q whose supporting hyperplane has normal vector $e_E = (1, \ldots, 1)$.

Example 2.7 (Permutohedra and stellahedra as polymatroids). Let $E = [n]$ and we identify $\mathbb{R}^n = \mathbb{R}^E$. The *permutohedron* in $\mathbb{R}^n$ is the polytope

$$\underline{\Pi}_E = \text{Conv}\{w \cdot (1, 2, \ldots, n) \mid w \text{ is a permutation of } [n]\} \subseteq \mathbb{R}^n.$$

It is the base polytope of the polymatroid (E, f) with $E = [n]$ and

$$f(S) = \max\{\mathbf{x}_S \mid \mathbf{x} \in \underline{\Pi}_E\} = n + (n-1) + \cdots + (n - |S| + 1).$$

The *permutohedral variety* $\underline{X}_E$ is the smooth projective toric variety whose fan is the inner normal fan of the image of $\underline{\Pi}_E$ in $\mathbb{R}^E/\mathbb{R}e_E$.

On the other hand, the *stellahedron* Π_E is the independence polytope corresponding to this polymatroid. Alternatively, as a consequence of Proposition 2.6, we have that

$$\Pi_E = \{\mathbf{u} \in \mathbb{R}^E_{\geq 0} \mid \text{ there exists } \mathbf{v} \in \underline{\Pi}_E \text{ such that } \mathbf{v} - \mathbf{u} \in \mathbb{R}^E_{\geq 0}\}.$$

The inner normal fan of the stellahedron is called the *stellahedral fan*. It is a unimodular simplicial fan with respect to the lattice $\mathbb{Z}^E \subseteq \mathbb{R}^E$ [EHL23]. The *stellahedral variety* X_E is the projective toric variety whose fan is the stellahedral fan.

Definition 2.8 [EL24, Definition 2.6]. Let $\pi : A \to E$ be a caging with cage $\mathbf{a}$, and let $P = (E, f)$ be a polymatroid. The *expansion* of P with respect to π is the polymatroid $\pi^*(P)$ on A with rank function $f \circ \pi$. Equivalently, if $p_\pi : \mathbb{R}^A \to \mathbb{R}^E$ is the linear map induced by π, we have that

$$I(\pi^*(P)) = p_\pi^{-1}(I(P)) \cap \mathbb{R}^A_{\geq 0}.$$

We can now define the main polytopes that we study in this paper.

Definition 2.9. Let $\pi : A \to E$ be a caging with cage $\mathbf{a}$. Consider the polymatroid $P = (E, f)$ from Example 2.7 and its expansion $\pi^*(P)$. Then,

- The *polypermutohedron with cage* $\mathbf{a}$ *is the base polytope of* $\pi^*(P)$.
- The *polystellahedron with cage* $\mathbf{a}$ *is the independence polytope of* $\pi^*(P)$.

Both of these polytopes live in $\mathbb{R}^A$. However, we often consider the polypermutohedron as a polytope in $\mathbb{R}^A/\mathbb{R}e_A$. Let us now define the toric varieties corresponding to these polytopes.

- The *polypermutohedral variety with cage* $\mathbf{a}$ is the toric variety defined by the inner normal fan of the polypermutohedron with cage $\mathbf{a}$ in $\mathbb{R}^A/\mathbb{R}e_A$. We denote it by $PP_\mathbf{a}$.
- The *polystellahedral variety with cage* $\mathbf{a}$ is the toric variety defined by the inner normal fan of the polystellahedron with cage $\mathbf{a}$ in $\mathbb{R}^A$. We denote it by $PS_\mathbf{a}$.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

In particular, the polypermutohedron and polystellahedron with cage $\mathbf{a} = (1, \ldots, 1)$ coincide with the permutohedron and stellahedron, respectively.

As already noted in [EL24, Section 6], for any cage $\mathbf{a}$ the polypermutohedral variety $PP_{\mathbf{a}}$ admits an embedding $PP_{\mathbf{a}} \hookrightarrow PS_{\mathbf{a}}$ as a torus-invariant divisor in the polystellahedral variety $PS_{\mathbf{a}}$. This is because the polypermutohedron with cage $\mathbf{a}$ is the facet of the polystellahedron with cage $\mathbf{a}$ corresponding to the ray spanned by $e_A = (1, \ldots, 1) \in \mathbb{Z}^A \subseteq \mathbb{R}^A$.

2.3. Wonderful compactifications

Next, we introduce the necessary tools from Li's theory of wonderful compactifications [Li09]. Throughout this subsection we fix Y to be a nonsingular variety over $\mathbb{C}$.

Definition 2.10 [Li09, Section 5.1]. Let $A, A_1, \ldots, A_k, B$ be smooth subvarieties of Y. Then,

- The intersection of A and B is said to be *clean* if the set-theoretic intersection $A \cap B$ is smooth, and we have the following relation between tangent spaces:

$$T_{A\cap B,y} = T_{A,y} \cap T_{B,y}, \text{ for all } y \in A \cap B.$$

- The intersection of $A_1, \ldots, A_k$ is said to be *transversal*, if either $k = 1$, or for all $y \in Y$

$$\operatorname{codim}\left(\bigcap_{i=1}^{k} T_{A_i,y}, T_y\right) = \sum_{i=1}^{k} \operatorname{codim}_y(A_i, Y).$$

  Here $\operatorname{codim}_y(A_i, Y) = \operatorname{codim}(A_i, Y)$ if $y \in A_i$ and 0 if $y \notin A_i$.

Definition 2.11 [Li09, Definition 2.1]. An *arrangement* of subvarieties of Y is a finite collection $\mathcal{S} = \{S_i\}$ of properly contained smooth subvarieties $S_i \subsetneq Y$ such that for all $i \neq j$, S_i and S_j intersect cleanly, and $S_i \cap S_j$ is either equal to some S_k or empty. In particular, $\mathcal{S}$ is closed under intersections.

Definition 2.12 [Li09, Definition 2.2]. Let $\mathcal{S}$ be an arrangement of subvarieties of Y.

- A subset $\mathcal{G} \subseteq \mathcal{S}$ is called a *building set of* $\mathcal{S}$ if, for all $S \in \mathcal{S}$, the minimal elements of $\{G \in \mathcal{G} \mid S \subseteq G\}$ intersect transversally, and their intersection is S. In particular, this condition is automatically satisfied if $S \in \mathcal{G}$.
- The collection $\mathcal{G}$ is called a *building set* if the set $\mathcal{S}$ of all possible intersections of elements of $\mathcal{G}$ is an arrangement, and furthermore $\mathcal{G}$ is a building set of $\mathcal{S}$.

We arrive at the main definition in our subsection

Definition 2.13 [Li09, Definition 1.1]. Let $\mathcal{G}$ be a nonempty building set of subvarieties of Y, and $Y^\circ = Y \setminus \bigcup_{G\in\mathcal{G}} G$. Then, the closure of the image of the locally closed embedding

$$Y^\circ \hookrightarrow \prod_{G\in\mathcal{G}} \operatorname{Bl}_G Y$$

is called the *wonderful compactification* of $\mathcal{G}$ on Y. It is denoted $Y_{\mathcal{G}}$.

Definition 2.14 [Li09, Definition 2.7]. Let Z be a nonsingular subvariety of a nonsingular variety Y, and $\pi : \operatorname{Bl}_Z Y \to Y$ be the blow-up of Y along Z. For any subvariety V of Y, we define the *dominant transform of* V to be the strict transform of V if $V \not\subseteq Z$, and to be the scheme-theoretic inverse $\pi^{-1}(V)$ if $V \subseteq Z$.

Proposition 2.15 [Li09, Theorem 1.2 and Proposition 2.13]. *Let Y be a nonsingular variety and $\mathcal{G} = \{S_1, S_2, \ldots, S_n\}$ be a building set of an arrangement of subvarieties of Y. Then,*

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

(1) *The wonderful compactification $Y_{\mathcal{G}}$ is a nonsingular variety. Moreover, for every $S_i \in \mathcal{G}$ there exists an irreducible smooth divisor D_{S_i} such that:*
   (i) *If $Y^\circ = Y \setminus \bigcup S_i$, then $Y_{\mathcal{G}} \setminus Y^\circ = \bigcup D_{S_i}$.*
   (ii) *Any set of divisors D_{S_i} intersect transversely.*

(2) *$Y_{\mathcal{G}}$ is isomorphic to an iterated blow-up $\mathrm{Bl}_{\mathcal{G}} Y$ of Y along the dominant transforms of the elements in $\mathcal{G}$.*

As Li observes, the order in which the blow-ups are performed in part (2) of the previous Proposition is flexible. We have the following:

Proposition 2.16 [Li09, Theorem 1.3]. *Let Y be a nonsingular variety and $\mathcal{G} = \{S_1, S_2, \ldots, S_n\}$ be a building set of an arrangement of subvarieties of Y. Suppose the elements of $\mathcal{G}$ are ordered in such a way that the first k terms $S_1, S_2, \ldots, S_k$ form a building set for all $1 \leq k \leq n$. Then, the wonderful compactification $Y_{\mathcal{G}}$ is isomorphic to an iterated blow-up $\mathrm{Bl}_{\mathcal{G}} Y$ of Y along the dominant transforms of the elements of $\mathcal{G}$ in this chosen order.*

2.4. Fulton-MacPherson compactifications and related constructions

Our compactifications will depend on a weight vector.

Definition 2.17. The domain of admissible weights for the weighted compactification of n points in the flag $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$, is the set

$$\mathcal{D}_n^{\mathrm{FM}} = \{(w_1, \ldots, w_n) \in \mathbb{Q}^n \mid 0 < w_i \leq 1, \forall i \in [n]\ \}.$$

We refer to the elements of these sets as *weight vectors*.

Let X be a nonsingular variety. For all $I \subseteq [n]$ we define the I-diagonal

$$\Delta_I = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j,\ \forall i, j \in I\} \subseteq X^n.$$

Following the convention in [FM94], throughout this article, we will refer to the flat families over our moduli spaces as their "universal" families. These families are flat and parametrize the desired degenerations. The use of quotation marks follows from the fact that the Fulton-MacPherson space represents a functor of so-called screens [FM94, Theorem 4], but it is unknown if the Fulton-MacPherson spaces represent a functor of pointed degenerations of varieties.

Definition 2.18 [Rou14, Theorems 2 and 3]. Let X be a nonsingular variety of dimension d. Given $\mathbf{w} = (w_1, \ldots, w_n) \in \mathcal{D}_n^{\mathrm{FM}}$, define the building set $\mathcal{K}_{\mathbf{w}}$ on X as

$$\mathcal{K}_{\mathbf{w}} := \left\{ \Delta_I \subseteq X^n \;\middle|\; I \subseteq [n],\ |I| \geq 2 \text{ and } w_I := \sum_{i \in I} w_i > 1 \right\}.$$

Then, the weighted Fulton-MacPherson compactification $X_{\mathbf{w}}[n]$ is the wonderful compactification of the open set

$$X^n \setminus \bigcup_{\Delta_I \in \mathcal{K}_{\mathbf{w}}} \Delta_I.$$

Moreover, there is a smooth "universal" family $\phi_{\mathbf{w}} : X_{\mathbf{w}}[n]^+ \to X_{\mathbf{w}}[n]$ equipped with n sections $\sigma_i : X_{\mathbf{w}}[n] \to X_{\mathbf{w}}[n]^+$ whose images lie in the relative smooth locus of $\phi_{\mathbf{w}}$. The map $\phi_{\mathbf{w}}$ is a flat morphism between nonsingular varieties whose fibers are the n-pointed $\mathbf{w}$-stable degenerations of X.

In particular, if $\mathbf{w} = (1, \ldots, 1)$, the space $X_{\mathbf{w}}^{[n]}$ coincides with the classical Fulton-MacPherson compactification $X^{[n]}$ of [FM94].

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

3. Moduli of points in a flag and polypermutohedral varieties

In this section we prove Theorem 1.1. The proof is carried out in stages: First, in Subsection 3.1, we construct a compactification of the configuration space of n labeled weighted points in an affine flag. We then use this construction in Subsection 3.2 to define the moduli space $T^{\mathbf{a}}_{\mathbf{w}}$. The toric compactification $T^{\mathbf{a}}_{LM}$ and its isomorphism with the polypermutohedral variety are described in Subsection 3.3. The nontrivial fibration of $T^{\mathbf{a}}_{\mathrm{LM}}$ is established in Subsection 3.4. The proof of the theorem relies on the previous subsections, and is given in Subsection 3.5. The section concludes with Subsection 3.6, where we prove the existence of a toric map from the standard Losev-Manin space to the compactification $T^{\mathbf{a}}_{LM}$.

Throughout this section we assume all cages $\mathbf{a} = (a_1, \dots, a_n)$ over $[n]$ are such that $a_1 \geq \cdots \geq a_n$; see Definition 2.4. This constitutes no loss of generality and simplifies the presentation significantly.

3.1. Weighted Fulton-MacPherson compactification of points in a flag

Let

$$\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$$

be a flag of linear subspaces, where $\mathbb{C}^{a_{i+1}}$ is the a_{i+1}-dimensional subspace of $\mathbb{C}^{a_i}$ generated by the first a_{i+1} coordinates. We will constantly use the fact that given $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{d} = (a_1, \dots, a_1)$, we have a closed immersion

$$i_{\mathbf{a}} : \mathbb{C}^{\mathbf{a}} = \prod_{i=1}^{n} \mathbb{C}^{a_i} \hookrightarrow \mathbb{C}^{\mathbf{d}} = (\mathbb{C}^{a_1})^n$$

given by the componentwise inclusion of each $\mathbb{C}^{a_i}$ into $\mathbb{C}^{a_1}$ as the subspace spanned by the first a_i coordinates.

Definition 3.1. Consider the cages $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{d} = (a_1, \dots, a_1)$. Suppose $I \subseteq [n]$ is such that $|I| \geq 2$. The *I-diagonal of the* $\mathbf{a}$*-flag* is

$$\Delta^{\mathbf{a}}_I = \{(x_1, \dots, x_n) \in (\mathbb{C}^{a_1})^n \mid x_i \in \mathbb{C}^{a_i} \text{ and } x_i = x_j \text{ for all } i, j \in I\}.$$

When $a_1 = \cdots = a_n$, the I-diagonal $\Delta^{\mathbf{a}}_I$ is the usual I-diagonal Δ_I in $(\mathbb{C}^{a_1})^n$.

Lemma 3.2. *Let* $\mathbf{w} = (w_1, \dots, w_n)$ *be a weight vector in* $\mathcal{D}^{\mathrm{FM}}_n$*; see Definition 2.17. The set*

$$\mathcal{K}^{\mathbf{a}}_{\mathbf{w}} := \left\{\Delta^{\mathbf{a}}_I \mid \mathbf{w}_I > 1\right\}.$$

is a building set in $\mathbb{C}^{\mathbf{a}}$ *in the sense of Li [Li09]; see Definition 2.12.*

We point out that the case $a_1 = \cdots = a_n$ is a particular case of Routis' results in [Rou14].

Proof of Lemma 3.2. Let $\langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle$ be the set of all possible scheme-theoretic intersections of the elements $\mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$. Since $\Delta^{\mathbf{a}}_I \cap \Delta^{\mathbf{a}}_J = \Delta^{\mathbf{a}}_{I \cup J}$ if and only if $I \cap J \neq \emptyset$, we have the following decomposition:

$$\langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle = \left\{\Delta^{\mathbf{a}}_{I_1} \cap \cdots \cap \Delta^{\mathbf{a}}_{I_r} \;\middle|\; r \geq 2,\ \Delta^{\mathbf{a}}_{I_j} \in \mathcal{K}^{\mathbf{a}}_{\mathbf{w}},\ \text{and } I_j \cap I_k = \emptyset \text{ for all } j \neq k\right\} \cup \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}.$$

Notice that all elements in $\langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle$ are smooth varieties. Moreover, the set $\langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle$ is an arrangement of subvarieties in the sense of Li (see Definition 2.11), that is, any two elements intersect cleanly in $\mathbb{C}^{\mathbf{a}}$. To see this, recall that two nonsingular subvarieties $A, B \subseteq Y$ of a variety Y are said to intersect cleanly if their intersection is smooth and their tangent spaces satisfy $T_{A,y} \cap T_{B,y} = T_{A \cap B, y}$ for all $y \in A \cap B$. In the case at hand every element of $\langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle$ is a linear subspace of $\mathbb{C}^{\mathbf{a}}$, in which case this condition is immediate, since each one of the subvarieties can be identified with its tangent space.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

It only remains to prove that $\mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ is a building set for the arrangement $\langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle$. By definition, this amounts to proving that given any $S \in \langle \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \rangle \setminus \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$, the minimal elements of $\{\Delta^{\mathbf{a}}_I \in \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} : \Delta^{\mathbf{a}}_I \supseteq S\}$ intersect transversely and their intersection is S. This is a direct dimension count.

Let us once again identify each one of the diagonals with their tangent spaces. By [Li09, Section 5.1.2] a collection $\Delta^{\mathbf{a}}_{I_1}, \dots, \Delta^{\mathbf{a}}_{I_r}$ of diagonals with $\{I_1, \dots, I_r\}$ pairwise disjoint intersects transversely if and only if for each $p \in \mathbb{C}^{\mathbf{a}}$

$$\sum_{j=1}^{r} \operatorname{codim}_p\left(\Delta^{\mathbf{a}}_{I_j}, \mathbb{C}^{\mathbf{a}}\right) = \operatorname{codim}_p\left(\bigcap_{j=1}^{r} \Delta^{\mathbf{a}}_{I_j}, \mathbb{C}^{\mathbf{a}}\right). \tag{3.1}$$

Since a subcollection of $\Delta^{\mathbf{a}}_{I_1}, \dots, \Delta^{\mathbf{a}}_{I_r}$ is also a collection of diagonals with disjoint indices, we may assume without loss of generality that $p \in \bigcap_{j=1}^{r} \Delta^{\mathbf{a}}_{I_j}$. Define i_j to be the largest element of I_j. Then, a direct dimension count shows that

$$\operatorname{codim}(\Delta^{\mathbf{a}}_{I_j}, \mathbb{C}^{\mathbf{a}}) = \sum_{k \in I_j} a_k - a_{i_j},$$

so that the left hand side of Equation (3.1) becomes $\sum_{k \in \cup I_j} a_k - a_{i_1} - \cdots - a_{i_r}$. On the other hand, computing the right-hand side by directly counting the codimension of $\Delta^{\mathbf{a}}_{I_1} \cap \cdots \cap \Delta^{\mathbf{a}}_{I_r}$ in $\mathbb{C}^{\mathbf{a}}$ shows that it is also equal to $\sum_{k \in \cup I_j} a_k - a_{i_1} - \cdots - a_{i_r}$. □

Definition 3.3. The *weighted compactification* $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ of $\mathbb{C}^{\mathbf{a}} \setminus \bigcup_{\Delta^{\mathbf{a}}_I \in \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}} \Delta^{\mathbf{a}}_I$ is the wonderful compactification of the building set $\mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ in $\mathbb{C}^{\mathbf{a}}$. We refer to $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ as the *weighted Fulton-MacPherson compactification of points in a flag.*

Example 3.4. If $\mathbf{d} = (d, \dots, d)$ is a constant cage, then the compactifications $\mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}}$ specialize to Routis' weighted Fulton-MacPherson for configurations of points in $\mathbb{C}^d$; see [Rou14, Definition 2.6].

The wonderful compactification $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ is endowed with a structural map $\psi_{\mathbf{a}} : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathbb{C}^{\mathbf{a}}$, which will be important for the following results. For each $I_0 \subseteq [n]$ such that $|I_0| \geq 2$ and $\Delta^{\mathbf{a}}_{I_0} \in \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$, the morphism $\psi_{\mathbf{a}} : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathbb{C}^{\mathbf{a}}$ is given by the composition

$$\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to X_{\mathbf{a}} := \prod_{\Delta^{\mathbf{a}}_I \in \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}} \operatorname{Bl}_{\Delta^{\mathbf{a}}_I} \mathbb{C}^{\mathbf{a}} \to \operatorname{Bl}_{\Delta^{\mathbf{a}}_{I_0}} \mathbb{C}^{\mathbf{a}} \to \mathbb{C}^{\mathbf{a}}, \tag{3.2}$$

where the map on the left is the inclusion $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \subseteq X_{\mathbf{a}}$ in the definition of $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$, the middle map is the projection on the indicated factor and the map on the right is the indicated blow-up. The morphism $\psi_{\mathbf{a}} : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathbb{C}^{\mathbf{a}}$ is independent of the choice of I_0, since when restricted to the open dense subset $\mathcal{U}^{\mathbf{a}} \subseteq \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ it agrees with the inclusion $\mathcal{U}^{\mathbf{a}} \hookrightarrow \mathbb{C}^{\mathbf{a}}$. In particular, $\psi_{\mathbf{a}}$ is birational and it is the identity when restricted to $\mathcal{U}^{\mathbf{a}} \subseteq \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ on the source and $\mathcal{U}^{\mathbf{a}} \subseteq \mathbb{C}^{\mathbf{a}}$ on the target. Moreover, [Li09, Theorem 1.3] shows that $\psi_{\mathbf{a}} : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathbb{C}^{\mathbf{a}}$ is an iterated blow-up along the dominant transforms of the varieties $\Delta^{\mathbf{a}}_I \in \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ in any order such that any initial sub-list of the ordered list is a building set (see Proposition 2.16).

The following lemma shows how a smooth subvariety interacts with the wonderful compactification of a larger ambient space, assuming that its intersections with the ambient building set form a building set.

Lemma 3.5. *Let X be a smooth variety, Y be a smooth subvariety of X and $\mathcal{G}$ be a building set on X. Suppose that the collection of the scheme-theoretic intersections with Y of the varieties in $\mathcal{G}$ forms a building set $\mathcal{H}$ in Y. Then, there exists a unique closed immersion $j : Y_{\mathcal{H}} \hookrightarrow X_{\mathcal{G}}$ which makes the*

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

following diagram commute:

$$\begin{array}{ccc} Y_{\mathcal{H}} & \xhookrightarrow{j} & X_{\mathcal{G}} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xhookrightarrow{i} & X, \end{array}$$

where $f : X_{\mathcal{G}} \to X$ and $g : Y_{\mathcal{H}} \to Y$ are the wonderful compactification morphisms and $i : Y \hookrightarrow X$ is the inclusion. In particular, $Y_{\mathcal{H}}$ is the strict transform of Y under the morphism $f : X_{\mathcal{G}} \to X$ considered as an iterated blow-up.

Proof. Fix any subvariety Z in $\mathcal{G}$. By [Har13, Corollary 7.15], the closed immersion $i : Y \hookrightarrow X$ induces a closed immersion $i_Z : \mathrm{Bl}_{Z\cap Y} Y \hookrightarrow \mathrm{Bl}_Z X$ such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Bl}_{Z\cap Y} Y & \xhookrightarrow{i_Z} & \mathrm{Bl}_Z X \\ \downarrow & & \downarrow \\ Y & \xhookrightarrow{i} & X. \end{array}$$

Then, if we let Z vary and we take the product over all $Z \in \mathcal{G}$, then we get a closed immersion $\widetilde{Y} \hookrightarrow \widetilde{X}$ where

$$\widetilde{Y} := \prod_{Z\in\mathcal{G}} \mathrm{Bl}_{Z\cap Y} Y \longrightarrow \widetilde{X} := \prod_{Z\in\mathcal{G}} \mathrm{Bl}_Z X.$$

Let $U_X = X \setminus \bigcup_{Z\in\mathcal{G}} Z$ and $U_Y = Y \setminus \bigcup_{Z\in\mathcal{G}}(Z \cap Y)$. From the construction of the wonderful compactifications, the morphisms f and g are isomorphisms over U_X and U_Y. We notice that $U_Y = Y \cap U_X$ and hence also $U_Y = \widetilde{Y} \cap U_X$. Therefore, we get a commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \xhookrightarrow{} & \widetilde{X} \\ \uparrow & & \uparrow \\ U_Y & \xhookrightarrow{} & U_X \end{array}$$

where the vertical maps are open immersions and the horizontal maps are closed immersions. This diagram induces an identification between the closure of U_Y in $\widetilde{Y}$, that is $Y_{\mathcal{H}}$, and the closure of U_Y in $\widetilde{X}$. The diagram also induces a closed immersion from the closure of U_Y in $\widetilde{X}$ into the closure of U_X in $\widetilde{X}$, that is $X_{\mathcal{G}}$. Then we obtain a closed immersion $j : Y_{\mathcal{H}} \hookrightarrow X_{\mathcal{G}}$.

If we consider the diagram in the statement using the closed immersion j that we constructed as the top morphism, the diagram commutes because by construction it commutes over the dense open subset U_Y of $Y_{\mathcal{H}}$.

Now, notice that there can be at most one morphism $Y_{\mathcal{H}} \hookrightarrow X_{\mathcal{G}}$ making the diagram in the statement commute because such a morphism has prescribed values over the dense open subset U_Y of $Y_{\mathcal{G}}$. Indeed, by the commutativity of the diagram, when the morphism $Y_{\mathcal{H}} \hookrightarrow X_{\mathcal{G}}$ is restricted to U_Y it agrees with the composition

$$(f|_{U_X})^{-1} \circ i \circ (g|_{U_Y}).$$

Then, the existence and uniqueness of the closed immersion j making the diagram commute follows. Finally, the image of $Y_{\mathcal{H}}$ under j is a subvariety of $X_{\mathcal{G}}$ with the same dimension as Y and which maps birationally onto Y under f, but this characterizes the strict transform of Y under f, and this completes the proof. □

In the case of the weighted compactification $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$, we arrive at the following result.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Lemma 3.6. *Given positive integers $a_1 \geq \cdots \geq a_n$, let $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{d} = (a_1, \ldots, a_1)$, and let $\mathbf{w} \in \mathcal{D}_n^{\mathrm{FM}}$ be a set of admissible weights. Then, there exists a unique closed immersion $j_{\mathbf{a}} : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \hookrightarrow \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}}$ which makes the following diagram commute:*

$$\begin{array}{ccc} \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} & \xhookrightarrow{j_{\mathbf{a}}} & \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}} \\ \downarrow{\scriptstyle \psi_{\mathbf{a}}} & & \downarrow{\scriptstyle \psi_{\mathbf{d}}} \\ \mathbb{C}^{\mathbf{a}} & \xhookrightarrow{i_{\mathbf{a}}} & \mathbb{C}^{\mathbf{d}} \end{array}$$

where $\psi_{\mathbf{a}}$ and $\psi_{\mathbf{d}}$ are the wonderful compactification morphisms (see Equation (3.2)), and $i_{\mathbf{a}}$ is the inclusion.

Proof. This result follows as a corollary of Lemma 3.5. Indeed, let $X = \mathbb{C}^{\mathbf{d}}$, $Y = \mathbb{C}^{\mathbf{a}}$, and let $\mathcal{G} = \mathcal{K}^{\mathbf{d}}_{\mathbf{w}}$ be the set of all diagonals $\Delta^{\mathbf{d}}_I$ associated with $\mathbf{w}$. Consider any $I_0 \subseteq [n]$ such that $|I_0| \geq 2$. Notice that the I_0-diagonal $\Delta^{\mathbf{a}}_{I_0}$ is in $\mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ if and only if the I_0-diagonal $\Delta^{\mathbf{d}}_{I_0}$ is in $\mathcal{K}^{\mathbf{d}}_{\mathbf{w}}$. Also, the I_0-diagonal $\Delta_{I_0} = \Delta^{\mathbf{d}}_{I_0} \subseteq \mathbb{C}^{\mathbf{d}}$ intersects $\mathbb{C}^{\mathbf{a}}$, since even the deepest diagonal $\Delta_{[n]} \subseteq \Delta_{I_0}$ clearly intersects $\mathbb{C}^{\mathbf{a}}$. Moreover, $\Delta^{\mathbf{a}}_{I_0} = \mathbb{C}^{\mathbf{a}} \cap \Delta^{\mathbf{d}}_0$ scheme-theoretically. Then, the intersections of the diagonals in $\mathcal{G} = \mathcal{K}^{\mathbf{d}}_{\mathbf{w}}$ with $Y = \mathbb{C}^{\mathbf{a}}$ form the corresponding building set $\mathcal{H} = \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ of all diagonals $\Delta^{\mathbf{a}}_I$ in $\mathbb{C}^{\mathbf{a}}$ associated with $\mathbf{w}$. By applying Lemma 3.5 in this setting, we obtain the existence and uniqueness of the desired closed immersion $j_{\mathbf{a}} : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \hookrightarrow \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}}$, making the diagram commute. □

To conclude this series of lemmas let us construct the "universal" family of the configuration spaces $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$. Routis constructs the "universal" family of the weighted Fulton-MacPherson compactification when the cage is constant; see Example 3.4. In light of the previous Lemma, this gives a "universal" family for our compactifications $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ for all cages $\mathbf{a}$.

Lemma 3.7 [Rou14, Theorem 3]. *Let $\mathbf{d}$ be a constant cage. There exists a "universal" family $(\mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}})^+ \to \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}}$ equipped with n sections $\sigma_i : \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}} \to (\mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}})^+$ whose images give the marked points. The "universal" family is a flat morphism between nonsingular varieties, whose fibers are $\mathbf{w}$-stable degenerations of $\mathbb{C}^{\mathbf{d}}$.*

Definition 3.8. Consider the cages $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{d} = (a_1, \ldots, a_1)$, and let $\mathbf{w} \in \mathcal{D}_n^{\mathrm{FM}}$. Define $(\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}})^+$, the "universal" family of the configuration spaces $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$, to be the pullback in the following Cartesian diagram:

$$\begin{array}{ccc} (\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}})^+ & \hookrightarrow & (\mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}})^+ \\ \downarrow & & \downarrow \\ \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} & \hookrightarrow & \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}}, \end{array} \tag{3.3}$$

where the bottom horizontal arrow is the closed immersion from Lemma 3.6. The map $(\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}})^+ \to \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ is equipped with n sections $\sigma_i : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to (\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}})^+$, $i \in [n]$, whose images are disjoint, obtained from the universal property of the Cartesian product via the identity map of $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ and the composition

$$\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \hookrightarrow \mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}} \xrightarrow{\sigma_i} (\mathbb{C}^{[\mathbf{d}]}_{\mathbf{w}})^+.$$

3.2. Moduli of n weighted points in a flag up to translation and scaling

The goal of this section is to give a direct construction of the moduli space $T^{\mathbf{a}}_{\mathbf{w}}$ of points in the flag, up to scaling and translation by the $\mathbb{C}^{a_n}$-coordinates, as a wonderful compactification, and to construct its "universal" family.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Definition 3.9. Consider a cage $\mathbf{a}$ and a collection of weight data $\mathbf{w} \in \mathcal{D}_n^{\mathrm{FM}}$ such that $\mathbf{w}_{[n]} > 1$. Let $D^{\mathbf{a}}_{[n]} = \pi^{-1}(\Delta^{\mathbf{a}}_{[n]}) \subseteq \mathbb{C}^{\mathbf{a}}_{\mathbf{w}}$ be the exceptional divisor of $\pi : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathbb{C}^{\mathbf{a}}$ above the deepest diagonal $\Delta^{\mathbf{a}}_{[n]}$. We define the variety

$$T^{\mathbf{a}}_{\mathbf{w}} := (\pi|^{\mathbf{a}}_{D_{[n]}})^{-1}(0)$$

as the fiber above $0 \in \Delta^{\mathbf{a}}_{[n]} \subseteq \mathbb{C}^{\mathbf{a}}$. In particular, every square in the following diagram is Cartesian:

$$\begin{array}{ccccc} T^{\mathbf{a}}_{\mathbf{w}} & \hookrightarrow & D^{\mathbf{a}}_{[n]} & \hookrightarrow & \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \\ \downarrow & & \downarrow{\scriptstyle \pi|_{D_{[n]}}} & & \downarrow{\scriptstyle \pi} \\ 0 & \hookrightarrow & \Delta^{\mathbf{a}}_{[n]} & \hookrightarrow & \mathbb{C}^{\mathbf{a}}. \end{array} \tag{3.4}$$

The above construction is similar to the one for the case of n points in affine space; see [CGK09, Definition 3.1.1] and [GR17, Definition 4.10]. We highlight that it's independent of the point $x \in \Delta^{\mathbf{a}}_{[n]}$ used. However, we make this specific choice for convenience. Indeed, every element in the building set $\mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ is invariant under the action of $\Delta^{\mathbf{a}}_{[n]}$ on $\mathbb{C}^{\mathbf{a}}$ by translation. In particular, for all $x \in \Delta^{\mathbf{a}}_{[n]}$ this action defines an isomorphism $T^{\mathbf{a}}_{\mathbf{w}} \cong (\pi|_{D^{\mathbf{a}}_{[n]}})^{-1}(x)$.

The requirement $\mathbf{w}_{[n]} > 1$ guarantees that the building set of $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ is not empty. This motivates the following.

Definition 3.10. The domain of admissible weights for the moduli space of n points in the flag $\mathbb{C}^{\mathbf{a}}$ is the set

$$\mathcal{D}_n^{\mathrm{T}} = \{\mathbf{w} = (w_1, \ldots, w_n) \in \mathbb{Q}^n \mid 0 < w_i \leq 1,\ \forall i \in [n],\ \mathbf{w}_{[n]} > 1\}.$$

Consider cages $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{d} = (a_1, \ldots, a_1)$ over $[n]$, and let $\mathbf{w} \in \mathcal{D}_n^{\mathrm{T}}$. For ease of notation let us denote the deepest diagonal as $\Delta := \Delta^{\mathbf{a}}_{[n]}$. Suppose $\mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ (Definition 3.2) is endowed with a total order satisfying the condition of Proposition 2.16, and such that Δ is its minimal element (e.g., an order with nondecreasing dimensions). Under this order, the space $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ is an iterated blow-up $\pi : \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathbb{C}^{\mathbf{a}}$, its first step being the blow-up $\pi_1 : \mathrm{Bl}_{\Delta}\, \mathbb{C}^{\mathbf{a}} \to \mathbb{C}^{\mathbf{a}}$. The fiber of $0 \in \mathbb{C}^{\mathbf{a}}$ under π_1 is $\mathbb{P}((N_{\Delta|\mathbb{C}^{\mathbf{a}}})^{\vee})|_0$, where $N_{\Delta|\mathbb{C}^{\mathbf{a}}}$ is the normal bundle of $\Delta \subseteq \mathbb{C}^{\mathbf{a}}$. The conormal bundle $(N_{\Delta|\mathbb{C}^{\mathbf{a}}})^{\vee}$ has constant fiber $(\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee}$ over every point of $\Delta \cong \mathbb{C}^{a_n}$, so in particular this central fiber is isomorphic to the projectivization of this vector space. Therefore, it is possible to coordinatize this fiber via an isomorphism

$$\mathbb{P}((\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee}) \cong \mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}. \tag{3.5}$$

The points in $\mathbb{P}((\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee})$ are in correspondence with the nonzero vectors in $\mathbb{C}^{\mathbf{a}}/\Delta$, up to scaling. In turn, the nonzero vectors in $\mathbb{C}^{\mathbf{a}}/\Delta$, up to scaling, are in correspondence with equivalence classes of configurations $(q_1, \ldots, q_n) \in \mathbb{C}^{\mathbf{a}} \setminus \Delta$, up to scaling and translation by elements in Δ. Each such equivalence class has a unique representative of the form $(p_1, \ldots, p_{n-1}, 0) \in \mathbb{C}^{\mathbf{a}} \setminus \{0\}$, up to scaling. Then, we fix the identification in (3.5) as the unique isomorphism that maps the point of $\mathbb{P}((\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee})$ associated to $(p_1, \ldots, p_{n-1}, 0)$ to the point

$$[x_{11} : \cdots : x_{1a_1} : x_{21} : \cdots : x_{2a_2} : \cdots : x_{(n-1)1} : \cdots : x_{(n-1)a_{n-1}}] \in \mathbb{P}^{a_1+\cdots+a_{n-1}-1},$$

where $p_i = (x_{i1}, \ldots, x_{ia_i})$, for each $i \in [n-1]$.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

For all $I \subsetneq [n]$, the image $\delta_I^{\mathbf{a}}$ of the I-diagonal $\Delta_I^{\mathbf{a}}$ under this isomorphism is

$$\delta_I^{\mathbf{a}} = \begin{cases} \bigcap_{i \in I \setminus \{n\}} V(x_{i1}, \ldots, x_{ia_i}), \text{ if } n \in I; \\ \bigcap_{i,j \in I, i<j} V(x_{i1} - x_{j1}, \ldots, x_{ia_j} - x_{ja_j}, x_{i(a_j+1)}, \ldots, x_{ia_i}), \text{ if } n \notin I. \end{cases} \tag{3.6}$$

Example 3.11. Consider the cage $\mathbf{a} = (3, 2, 2)$. Then, the possible nonsingleton subsets $I \subsetneq [3]$ are $\{1, 2\}, \{1, 3\}$ and $\{2, 3\}$. In the previous notation, the exceptional divisor above 0 is an iterated blow-up of $\mathbb{P}^{3+2-1} = \mathbb{P}^4$, with coordinates $[x_{11} : x_{12} : x_{13} : x_{21} : x_{22}]$. In addition, we have that

$$\delta_{12} = V(x_{11} - x_{21}, x_{12} - x_{22}, x_{13}), \quad \delta_{13} = V(x_{11}, x_{12}, x_{13}), \quad \delta_{23} = V(x_{21}, x_{22}).$$

Remark 3.12. For a general cage $\mathbf{a} = (a_1, a_2, a_3)$ the resulting variety is the blow-up of a projective space along two toric subvarieties and one nontoric. The techniques introduced in [GGAGR25] to study the Mori Dream Space property for toric varieties blown-up at the closure of a toric orbit applies in this case, and provides an interesting line for future inquiry.

Lemma 3.13. *Let $\mathbb{C}^m$ be an affine space and let $V \subsetneq W$ be linear subspaces of $\mathbb{C}^m$. Consider the blow-up $\mathrm{Bl}_V \mathbb{C}^m \to \mathbb{C}^m$, with exceptional divisor E and let $\widetilde{W}$ be the strict transform of W. Then any fiber of $E \to V$ can be identified with $\mathbb{P}((\mathbb{C}^m/V)^\vee)$, and under this identification the intersection of $\widetilde{W}$ with that fiber of $E \to V$ is the image of $\mathbb{P}((W/V)^\vee)$ under the closed immersion $\mathbb{P}((W/V)^\vee) \to \mathbb{P}((\mathbb{C}^m/V)^\vee)$ induced by the inclusion $V \subsetneq W$.*

Proof. The exceptional divisor E is the projective bundle $E = \mathbb{P}((N_{V|\mathbb{C}^m})^\vee)$ over V. Identifying the tangent space of a linear subspace with the linear subspace itself, we see that the bundle $(N_{V|\mathbb{C}^m})^\vee$ has constant fiber $(\mathbb{C}^m/V)^\vee$.

To describe the intersection of the strict transform $\widetilde{W}$ with each fiber of $E \to V$, we start by identifying this intersection set-theoretically. Such an intersection is a variety (reduced and irreducible) and hence is determined by its underlying set. Fix any element of V, which we can assume to be 0 by homogeneity. We then have a Cartesian diagram representing the fiber of $E \to V$ over $0 \in V$ as follows:

$$\begin{array}{ccc} \mathbb{P}((\mathbb{C}^m/V)^\vee) & \longrightarrow & E = \mathbb{P}((N_{V|\mathbb{C}^m})^\vee) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & V. \end{array}$$

The intersection of $\widetilde{W}$ with this fiber is obtained set-theoretically by projectivizing the image of the injective linear map $W/V \to \mathbb{C}^m/V$. This injective linear map induces a surjective linear map $(\mathbb{C}^m/V)^\vee \to (W/V)^\vee$, and therefore a closed immersion $\mathbb{P}((W/V)^\vee) \to \mathbb{P}((\mathbb{C}^m/V)^\vee)$. It follows that the intersection of the strict transform $\widetilde{W}$ with any fiber of $E \to V$ is the image of $\mathbb{P}((W/V)^\vee)$ under this closed immersion, as desired. □

Proposition 3.14. *Consider a cage $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{w} \in \mathcal{D}_n^{\mathrm{T}}$. Then, $T_{\mathbf{w}}^{\mathbf{a}}$ is the wonderful compactification of $\mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$ with respect to the building set*

$$\mathcal{H}_{\mathbf{w}}^{\mathbf{a}} = \{\delta_I^{\mathbf{a}} \mid I \subsetneq [n], \mathbf{w}_I > 1\}.$$

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Proof. From Definition 3.9 we get the following commutative diagram consisting of Cartesian squares:

$$\begin{array}{ccc} T^{\mathbf{a}}_{\mathbf{w}} & \longrightarrow & \mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \\ \downarrow & & \downarrow \\ \mathbb{P}((\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee}) \cong \mathbb{P}^N & \longrightarrow & \mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}} \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{C}^{\mathbf{a}}, \end{array} \tag{3.7}$$

where $N := a_1 + \cdots + a_{(n-1)} - 1$. Let us recall that $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ is the wonderful compactification of the building set $\mathcal{G} := \mathcal{K}^{\mathbf{a}}_{\mathbf{w}}$ in $\mathbb{C}^{\mathbf{a}}$. Since Δ is minimal in $\mathcal{G}$ by [Li09, Proposition 2.8] the dominant transforms to $\mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$ of the elements in $\mathcal{G}$ form a building set $\widetilde{\mathcal{G}}$ in $\mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$. From the inductive construction of wonderful compactifications in [Li09, Definition 2.12], we know that $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ is the wonderful compactification of the building set $\widetilde{\mathcal{G}}$ in $\mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$.

Moreover, the morphism $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$ is the iterated blow-up of the dominant transforms of the elements in $\widetilde{\mathcal{G}}$ in any order compatible with inclusion of subvarieties. Since the exceptional divisor $E^{\mathbf{a}}_{[n]}$ of the blow-up $\mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}} \to \mathbb{C}^{\mathbf{a}}$ is maximal in $\widetilde{\mathcal{G}}$, we can order the iterated blow-up process so that the last blow-up step is that of a dominant transform of $E^{\mathbf{a}}_{[n]}$. Such dominant transform is a Cartier divisor, and then that last blow-up step can be omitted without changing the resulting variety $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$. Hence we can consider the building set for the wonderful compactification $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$ to be $\widetilde{\mathcal{G}}' = \widetilde{\mathcal{G}} \setminus \{E^{\mathbf{a}}_{[n]}\}$. Moreover, since Δ is contained in all elements in $\mathcal{G}$, then all the dominant transforms that yield the elements in $\widetilde{\mathcal{G}}'$ from elements in $\mathcal{G} \setminus \{\Delta\}$ are actually strict transforms.

Since $\mathbb{C}^{\mathbf{a}}$ is an affine space and all elements in $\mathcal{G} \setminus \{\Delta\}$ are linear subspaces containing Δ, then Lemma 3.13 computes their intersections with $\mathbb{P}((\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee}) \cong \mathbb{P}^N$ seen as the fiber over $0 \in \Delta$ of the exceptional divisor of $\mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}} \to \mathbb{C}^{\mathbf{a}}$. For any $I \subsetneq [n]$ with $|I| \geq 2$ such that $\Delta^{\mathbf{a}}_I$ is in $\mathcal{G} \setminus \{\Delta\} = \mathcal{K}^{\mathbf{a}}_{\mathbf{w}} \setminus \{\Delta\}$, the intersection of the strict transform $\widetilde{\Delta^{\mathbf{a}}_I}$ of $\Delta^{\mathbf{a}}_I$ in $\mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$ with $\mathbb{P}((\mathbb{C}^{\mathbf{a}}/\Delta)^{\vee}) \cong \mathbb{P}^N$ is precisely $\delta^{\mathbf{a}}_I$ by Lemma 3.13.

Therefore, the set of intersections of the elements of the building set $\widetilde{\mathcal{G}}'$ with $\mathbb{P}^N = \mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$ is precisely the building set $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}} = \{\delta^{\mathbf{a}}_I \mid I \subsetneq [n],\ \mathbf{w}_I > 1\}$ in $\mathbb{P}^N$. From the diagram (3.7) with Cartesian squares, we have that $T^{\mathbf{a}}_{\mathbf{w}}$ is the strict transform of $\mathbb{P}^N$ in $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}}$ under the iterated blow-up $\mathbb{C}^{[\mathbf{a}]}_{\mathbf{w}} \to \mathrm{Bl}_{\Delta}\,\mathbb{C}^{\mathbf{a}}$. Therefore by Lemma 3.5, $T^{\mathbf{a}}_{\mathbf{w}}$ is equal to the wonderful compactification of the building set $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}} = \{\delta^{\mathbf{a}}_I \mid I \subsetneq [n],\ \mathbf{w}_I > 1\}$ in $\mathbb{P}^N = \mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$, and this completes the proof. □

Corollary 3.15. *Consider a cage* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{w} \in \mathcal{D}^{\mathrm{T}}_n$. *Then,* $T^{\mathbf{a}}_{\mathbf{w}}$ *is the iterated blow-up of* $\mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$ *along the (strict transform of the) elements of* $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$ *in any order compatible with inclusion of subvarieties. In particular,* $T^{\mathbf{a}}_{\mathbf{w}}$ *is a toric variety if* $\mathbf{w}$ *is such that* $\mathbf{w}_{[n-1]} \leq 1$.

Proof. By Proposition 3.14, we know that $T^{\mathbf{a}}_{\mathbf{w}}$ is a wonderful compactification. So, the claim about the iterated blow-up follows from the general theory of Li [Li09, Theorem 1.3]. The claim about the case $\mathbf{w}_{[n-1]} \leq 1$ follows from the fact that such condition implies $n \in I$ if $\delta^{\mathbf{a}}_I \in \mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$. By (3.6), such locus $\delta^{\mathbf{a}}_I$ is a torus-invariant subvariety of $\mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$ under the standard toric structure of projective space. □

Example 3.16. Consider the cage $\mathbf{a} = (3, 2, 2)$ as in Example 3.11. Then, $T^{\mathbf{a}}_{\mathbf{w}}$ is the iterated blow-up of $\mathbb{P}^4$, with coordinates $[x_{11} : x_{12} : x_{13} : x_{21} : x_{22}]$ along a subcollection of the diagonals δ_{12}, δ_{13} and δ_{23} depending on the weight vector $\mathbf{w}$. For example:

- If $\mathbf{w} = (1, 1, 1)$, then $T^{\mathbf{a}}_{\mathbf{w}}$ is the iterated blow-up of $\mathbb{P}^4$ along (the strict transform of) δ_{12}, δ_{13} and δ_{23} in any order.
- If $\mathbf{w} = (\varepsilon, \varepsilon, 1)$, then $T^{\mathbf{a}}_{\mathbf{w}}$ is the iterated blow-up of $\mathbb{P}^4$ along the disjoint toric subvarieties δ_{13} and δ_{23}.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Lemma 3.17. *Consider the cages* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{d} = (a_1, \dots, a_1)$*, and let* $\mathbf{w} \in \mathcal{D}_n^{\mathrm{T}}$*. Then, there exists a unique closed immersion* $f_{\mathbf{a}} : T^{\mathbf{a}}_{\mathbf{w}} \hookrightarrow T^{\mathbf{d}}_{\mathbf{w}}$ *which makes the following diagram commute:*

$$\begin{array}{ccc} T^{\mathbf{a}}_{\mathbf{w}} & \xhookrightarrow{f_{\mathbf{a}}} & T^{\mathbf{d}}_{\mathbf{w}} \\ \downarrow{\scriptstyle \theta_{\mathbf{a}}} & & \downarrow{\scriptstyle \theta_{\mathbf{d}}} \\ \mathbb{P}^{a_1+\cdots+a_{(n-1)}-1} & \xhookrightarrow{g_{\mathbf{a}}} & \mathbb{P}^{a_1(n-1)-1} \end{array}$$

where $\theta_{\mathbf{a}}$ *and* $\theta_{\mathbf{d}}$ *are the wonderful compactification morphisms and* $g_{\mathbf{a}}$ *is the inclusion.*

Proof. We define $N := a_1 + \cdots + a_{(n-1)} - 1$ and recall our convention that $a_1 \geq \cdots \geq a_n$. Let us recall the description of the inclusion $g_{\mathbf{a}} : \mathbb{P}^N \hookrightarrow \mathbb{P}^{a_1(n-1)-1}$. We have inclusions $\mathbb{C}^{a_i} \hookrightarrow \mathbb{C}^{a_1}$ as the span of the first a_i coordinates, for each i. By taking the product, we get an inclusion of affine spaces $\mathbb{C}^{a_1+\cdots+a_{n-1}} \hookrightarrow \mathbb{C}^{a_1(n-1)}$, which induces the closed immersion $g_{\mathbf{a}} : \mathbb{P}^N \hookrightarrow \mathbb{P}^{a_1(n-1)-1}$ between their projectivizations.

Consider any $I_0 \subsetneq [n]$. Notice that $\delta^{\mathbf{a}}_{I_0}$ is in $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$ if and only if $\delta^{\mathbf{d}}_{I_0}$ is in $\mathcal{H}^{\mathbf{d}}_{\mathbf{w}}$. Also, $\delta_{I_0} = \delta^{\mathbf{d}}_{I_0} \subseteq \mathbb{P}^{a_1(n-1)-1}$ intersects $\mathbb{P}^N$, since even $\delta_{[n]} \subseteq \delta_{I_0}$ clearly intersects $\mathbb{P}^N$. Moreover, $\delta^{\mathbf{a}}_{I_0} = \mathbb{P}^N \cap \delta^{\mathbf{d}}_0$ scheme-theoretically.

Then, the intersections with $Y = \mathbb{P}^N$ of the varieties $\delta^{\mathbf{a}}_I$ in the building set $\mathcal{G} = \mathcal{H}^{\mathbf{d}}_{\mathbf{w}}$ in $X = \mathbb{P}^{a_1(n-1)-1}$ form the building set $\mathcal{H} = \mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$ in $\mathbb{P}^N$. By applying Lemma 3.5 in this setting, we obtain the existence and uniqueness of the desired closed immersion $f_{\mathbf{a}} : T^{\mathbf{a}}_{\mathbf{w}} \hookrightarrow T^{\mathbf{d}}_{\mathbf{w}}$, making the diagram commute. □

The variety $T^{\mathbf{a}}_{\mathbf{w}}$ acquires a "universal" family from its embedding into $T^{\mathbf{d}}_{\mathbf{w}}$.

Definition 3.18. The "universal" family $(T^{\mathbf{a}}_{\mathbf{w}})^+$ of $T^{\mathbf{a}}_{\mathbf{w}}$ is the pullback of the "universal" family $\phi : (T^{\mathbf{w}}_{d,n})^+ \to T^{\mathbf{d}}_{\mathbf{w}}$ along the closed immersion $T^{\mathbf{a}}_{\mathbf{w}} \hookrightarrow T^{\mathbf{d}}_{\mathbf{w}}$. That is, the following diagram is Cartesian:

$$\begin{array}{ccc} (T^{\mathbf{a}}_{\mathbf{w}})^+ & \longrightarrow & (T^{\mathbf{d}}_{\mathbf{w}})^+ \\ \downarrow & & \downarrow \\ T^{\mathbf{a}}_{\mathbf{w}} & \longrightarrow & T^{\mathbf{d}}_{\mathbf{w}}. \end{array}$$

In particular, $(T^{\mathbf{a}}_{\mathbf{w}})^+ \to T^{\mathbf{a}}_{\mathbf{w}}$ is flat and it is equipped with n sections $\sigma_i : T^{\mathbf{a}}_{\mathbf{w}} \to (T^{\mathbf{a}}_{\mathbf{w}})^+$ whose images give the marked points.

For the description of the fibers in the "universal" family $(T^{\mathbf{d}}_{\mathbf{w}})^+ \to T^{\mathbf{d}}_{\mathbf{w}}$, we refer the reader to [GR17, Section 2.4]. Let us remark that there is a Zariski open $\mathcal{V}^{\mathbf{a}}_{\mathbf{w}}$ such that the geometric fibers of $(T^{\mathbf{a}}_{\mathbf{w}})^+ \to T^{\mathbf{a}}_{\mathbf{w}}$ over $\mathcal{V}^{\mathbf{a}}_{\mathbf{w}}$ are all isomorphic to a flag of projective spaces $\mathbb{P}^{a_n} \subseteq \cdots \subseteq \mathbb{P}^{a_1}$ with a flag of hyperplanes $H_n \subseteq \cdots \subseteq H_1$ and n points away from such hyperplanes – that is, all points are in a flag of affine linear spaces $\mathbb{C}^{a_n} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$. This configuration of n points is defined up to global scaling and translation by a vector in $\mathbb{C}^{a_n}$. The study of the degenerations of these fibers will be part of forthcoming work.

Remark 3.19. Here is an alternative interpretation of the moduli problem that we study in this section. If we consider $\mathbb{P}^{a_1} \supseteq \mathbb{C}^{a_1}$, it is straightforward to verify that the group of automorphisms of $\mathbb{P}^{a_1}$ that fix the vector spaces in the flag $\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}$ as sets and the hyperplane at infinity $\mathbb{P}^{a_1} \setminus \mathbb{C}^{a_1}$ pointwise, is the same as the subgroup of the automorphisms of $\mathbb{C}^{a_1}$ generated by scalings and translations by the elements of $\mathbb{C}^{a_n}$. Then, $T^{\mathbf{a}}_{\mathbf{w}}$ provides a compactification of the space of weighted configurations of distinct points in a flag of affine spaces up to either of these equivalent equivalence relations.

Remark 3.20. In the case of a constant cage $\mathbf{d} = (d, \dots, d)$, the moduli spaces $T^{\mathbf{d}}_{\mathbf{w}}$ are the moduli spaces of pointed, stable rooted trees studied for weights equal to one in [CGK09] and for general weights in [GR17]. Let us mention how our notation relates to these works. The space $T^{\mathbf{a}}_{\mathbf{w}}$ with $\mathbf{w} = (1, \dots, 1)$ and $\mathbf{a} = (d, \dots, d)$ is denoted as $T_{d,n}$ in [CGK09]. The space $T^{\mathbf{a}}_{\mathbf{w}}$ with $\mathbf{w} \in \mathcal{D}_n^T$ and $\mathbf{a} = (d, \dots, d)$ is denoted as $T^{\mathbf{w}}_{d,n}$ in [GR17].

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

3.3. The Losev-Manin compactification and the polypermutohedral variety

Among the compactifications constructed in Section 3.2, there is one that is a higher-dimensional analog to the Losev-Manin compactification of $M_{0,n}$. In this subsection we prove that this generalization, which we denote as $T^{\mathbf{a}}_{LM}$, is isomorphic to the polypermutohedral variety $PP_{\mathbf{a}^-}$ (Definition 2.9) with cage $\mathbf{a}^- = (a_1, \dots, a_{n-1})$.

Definition 3.21. Consider the cage $\mathbf{a} = (a_1, \dots, a_n)$, and let $\mathbf{w} = (w_1, \dots, w_n) \in \mathcal{D}^{\mathrm{T}}_n$ be such that $w_{[n-1]} \leq 1$ and $w_i + w_n > 1$ for all $i \in [n-1]$. Then, we call the space $T^{\mathbf{a}}_{\mathbf{w}}$ the *Losev-Manin compactification* for this moduli problem. We denote it as $T^{\mathbf{a}}_{LM}$.

Theorem 3.22. *Consider the cages* $\mathbf{a} = (a_1, \dots, a_n)$ *and* $\mathbf{a}^- = (a_1, \dots, a_{n-1})$. *Then, the Losev-Manin compactification* $T^{\mathbf{a}}_{LM}$ *is isomorphic to the polypermutohedral variety with cage* $\mathbf{a}^-$. *That is,*

$$\mathrm{T}^{\mathbf{a}}_{LM} \cong PP_{\mathbf{a}^-}.$$

When $\mathbf{a} = \mathbf{1}^n = (1, \dots, 1) \in \mathbb{Z}^n_{>0}$, the polypermutohedron $PP_{\mathbf{a}^-} = PP_{\mathbf{1}^{(n-1)}}$ coincides with the standard permutohedron of dimension $n-2$. On the other hand, the toric variety corresponding to the m-dimensional permutohedron is isomorphic to the Losev-Manin compactification $\overline{M}^{LM}_{0,m+3}$ of $M_{0,m+3}$; see [LM00]. This shows the following.

Corollary 3.23. *Let* $\mathbf{1}^n = (1, \dots, 1) \in \mathbb{Z}^n_{>0}$. *Then, there is an isomorphism*

$$T^{\mathbf{1}^n}_{LM} \cong PP_{\mathbf{1}^{(n-1)}} \cong \overline{M}^{LM}_{0,n+1}.$$

In order to prove Theorem 3.22 let us introduce the necessary results from the theory of combinatorial building sets. We mainly follow [Zel06, PRW08].

Definition 3.24. A *building set* $\mathcal{B}$ on a finite set E is a collection of nonempty subsets of E such that:

1. $\{i\} \in \mathcal{B}$ for all $i \in E$.
2. If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.

Denote by $\mathcal{B}_{max} \subseteq \mathcal{B}$ the set of maximal (with respect to inclusion) elements of $\mathcal{B}$. The building set $\mathcal{B}$ is said to be *connected* if $\mathcal{B}_{max} = \{E\}$.

Definition 3.25. Given a building set $\mathcal{B}$, call a subset $N \subseteq \mathcal{B} - \mathcal{B}_{max}$ *nested* if it satisfies the following condition: for any $k \geq 2$ and any $S_1, \dots, S_k \in N$ such that none of the S_i contains another one, the union $S_1 \cup \cdots \cup S_k$ is not in $\mathcal{B}$.

Let $\mathcal{B}$ be a connected building set on E. The *nested fan associated to* $\mathcal{B}$ is the fan $\Sigma(\mathcal{B})$ in $\mathbb{R}^E/\mathbb{R}e_E$ whose cones are

$$\sigma_N := \{\mathrm{cone}(\overline{e}_{S_1}, \dots, \overline{e}_{S_k}) \mid N = \{S_1, \dots, S_k\} \text{ is a nested set of } \mathcal{B}\}.$$

The fan $\Sigma(\mathcal{B})$ is the normal fan of a smooth polytope $P_{\mathcal{B}}$, known as the *nestohedron* of $\mathcal{B}$; see [Zel06, Theorem 6.1]. In particular, this fan is unimodular with respect to the image lattice $\mathbb{Z}^E \subseteq \mathbb{R}^E$ in $\mathbb{R}^E/\mathbb{R}e_E$; see [Zel06, Theorem 5.1 and Corollary 5.2].

The nested fan associated to a connected building set can be constructed via an iterated process of star subdivision of Σ_E as follows:

Construction 3.26 [FY04, Theorem 4]. Start with the fan Σ_E in $\mathbb{R}^E/\mathbb{R}e_E$ whose rays have been labeled by the singletons i for $i \in E$. Given a nonsingleton set $I \in \mathcal{B}$, define the cone $\sigma_I := \mathrm{cone}(\overline{e}_i \mid i \in I) \in \Sigma_E$. Finally, define a total order of $\mathcal{B} = \{I_1, \dots, I_k\}$ compatible with inclusion, so that larger sets come before smaller sets. Then, define the fan $\Sigma(\mathcal{B})$ as that one obtained from Σ_E after performing iterated star subdivisions along the cones σ_I as I ranges through the elements of $\mathcal{B}$ in the order we defined.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Example 3.27. The building set $\mathcal{B}_{LM} := 2^{[n]} \setminus \emptyset$ is known as the *boolean* building set. Its corresponding nestohedron is the $(n-1)$-dimensional permutohedron.

Example 3.28. A large class of examples of building sets and nestohedra is given by the so-called *graphical building sets* and their corresponding *graph associahedra*, first introduced in [CD06]. Given a connected graph G with vertex set E, its corresponding graphical building set is

$$\mathcal{B}(G) = \{T \subseteq [n] \mid G|_T \text{ is a connected subgraph of } G\}.$$

Examples include permutohedra, when G is a complete graph, and stellahedra, when G is a star graph.

Notation 3.29. Consider a building set $\mathcal{B}$ over E, we denote by $X(\mathcal{B})$ the toric variety constructed from the nested fan $\Sigma(\mathcal{B})$.

Let us now introduce the notion of *pullback* of a combinatorial building set $\mathcal{B}$ over E along a surjective function $\pi : A \to E$ of finite sets. This construction already appears in [CHL+22], and is equivalent to Eur and Larson's definition of *expansion* for base polytopes of a polymatroid. As particular examples, we shall prove that every polypermutohedron is the pullback of the boolean building set $\mathcal{B}_{LM} = 2^{[n]} \setminus \emptyset$.

Definition 3.30. Consider a building set $\mathcal{B}$ on E, and let $\pi : A \to E$ be a surjective function of finite sets. The *pullback of $\mathcal{B}$ along π* is the building set over A defined as

$$\pi^*\mathcal{B} = \{\{i\} \mid i \in A\} \cup \{\pi^{-1}(I) \mid I \in \mathcal{B}\}.$$

The verification that $\pi^*\mathcal{B}$ is a building set is direct, and left to the reader. If π is surjective, then $(\pi^*\mathcal{B})_{max} = \pi^{-1}(\mathcal{B}_{max})$, and hence $\mathcal{B}$ is connected if and only if $\pi^*(\mathcal{B})$ is connected.

Let us compute explicitly the nested sets of the pullback $\pi^*\mathcal{B}$, as well as the cones of its nested fan $\Sigma(\pi^*\mathcal{B})$. Recasting Construction 3.26 in this context shows that $\Sigma(\pi^*\mathcal{B})$ is obtained from the normal fan of the simplex Σ_A by a series of star subdivisions.

Definition 3.31. Consider a building set $\mathcal{B}$ on E, and let $\pi : A \to E$ be a surjective function of finite sets. A *π-pair* (I, N) consists of a subset $I \subsetneq A$, not containing any fibers of π, and a nested set N of $\mathcal{B}$. Here both cases $I = \emptyset$ and $N = \emptyset$ are allowed.

Lemma 3.32. *Let $\pi^*\mathcal{B}$ be the pullback of the building set $\mathcal{B}$ on E along $\pi : A \to E$. Then, there is a bijection between π-pairs and nested sets of $\pi^*\mathcal{B}$. Explicitly, every nested set of $\pi^*\mathcal{B}$ is of the form*

$$N_{(I,N)} = \{\{i\} \mid i \in I\} \cup \{\pi^{-1}(S) \mid S \in N\},$$

for a unique π-pair (I, N). Conversely, every π-pair defines a unique nested set in this manner.

Proof. Consider a nested set M of $\pi^*\mathcal{B}$. If $|M| = 1$, then the result is direct, so let us consider the case $|M| \geq 2$. Then, by definition, $M \subseteq 2^A \setminus \emptyset$ consists of a collection of singletons and sets of the form $\pi^{-1}(S)$ for subsets $S \in \mathcal{B}$. Let I denote the collection of singletons in M that are not full fibers of π, in other words, all singletons excluding those that are a full fiber. If I were to contain any fiber $\pi^{-1}(i)$ for some $i \in E$, this fiber would have size at least two, the collection of all singletons making up this fiber would be disjoint, and their union $\pi^{-1}(i)$ would be an element of $\pi^*\mathcal{B}$ with at least two elements, which contradicts M being a nested set. Therefore, I cannot contain any full fibers of size at least two.

On the other hand, consider any $k \geq 2$ elements $\pi^{-1}(S_1), \ldots, \pi^{-1}(S_k) \in M$, so that in particular $S_1, \ldots, S_k \in \mathcal{B}$. We remark that, in particular, it is possible for some $\pi^{-1}(S_i)$ to be a singleton. Then, $\pi^{-1}(S_i) \subseteq \pi^{-1}(S_j)$ if and only if $S_i \subseteq S_j$. Similarly, $\pi^{-1}(S_1) \cup \cdots \cup \pi^{-1}(S_k) \in \pi^*\mathcal{B}$ if and only if $S_1 \cup \cdots \cup S_k \in \mathcal{B}$. Therefore, the collection of all subsets $S \in \mathcal{B}$ such that $\pi^{-1}(S) \in M$ forms a nested set N of $\mathcal{B}$.

Conversely, given a π-pair (I, N), it is a direct verification to check that $N_{(I,N)} := \{\{i\} \mid i \in I\} \cup \{\pi^{-1}(S) \mid S \in N\}$ defines a nested set of $\pi^*\mathcal{B}$. □

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Corollary 3.33. *Given a π-pair (I,N), the cone corresponding to the nested set $N_{(I,N)}$ in $\Sigma(\pi^*\mathcal{B})$ is*

$$\sigma_{(I,N)} := \operatorname{cone}(\bar{e}_i \mid i \in I) + \operatorname{cone}(\bar{e}_{\pi^{-1}(S)} \mid S \in N) \subseteq \mathbb{R}^A/\mathbb{R}e_A.$$

Moreover, every cone in the nested fan of $\pi^*\mathcal{B}$ is of this form.

Corollary 3.34. *Given a π-pair (I,N), the cone $\sigma_{(I,N)} \in \Sigma(\pi^*\mathcal{B})$ can be decomposed as*

$$\sigma_{(I,N)} = \sigma_{(I,\emptyset)} + \sigma_{(\emptyset,N)}.$$

With these results at hand, we can prove that polypermutohedra are pullbacks of the boolean building set.

Proposition 3.35. *Let $\mathcal{B}_{LM} = 2^{[n]} \setminus \emptyset$ be the boolean building set, and let $\pi : A \to [n]$ be a caging with cage* **a**. *Then, the nested fan $\Sigma(\pi^*\mathcal{B}_{LM})$ is the inner normal fan of the polypermutohedron with cage* **a**.

Proof. It is well-known that nested sets of the boolean building set $\mathcal{B}_{LM}$ over $[n]$ are in bijection with flags of nonempty subsets $S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq [n]$. Indeed, a nested set of $\mathcal{B}_{LM}$ cannot contain two incomparable sets, as their union is also an element of $\mathcal{B}_{LM}$. Therefore, in light of Corollary 3.34, the cones of $\pi^*\mathcal{B}_{LM}$ are

$$\sigma_{(I,N)} = \sigma_{(I,\emptyset)} + \sigma_{(\emptyset,N)},$$

where $N = \{S_1 \subsetneq S_2 \subsetneq \cdots \subsetneq S_k \subsetneq [n]\}$. This is precisely the Bergman fan of the boolean polymatroid with cage π, as introduced in [CHL+22, Definition 1.2]. It is proved in [CHL+22, Appendix A] that this is the inner normal fan of the polypermutohedron. □

Definition 3.36. A polymatroid $P = (A, f)$ is said to be *boolean* if its base polytope is a polypermutohedron. In other words, if its base polytope is the nestohedron corresponding to $\pi^*\mathcal{B}_{LM}$ for some caging $\pi : A \to [n]$.

Proof of Theorem 3.22. Recall that the compactification $T^{\mathbf{a}}_{\mathbf{w}}$ is the iterated blow-up of the projective space $\mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$ along the dominant transforms of the varieties in the geometric building set

$$\mathcal{H}^{\mathbf{a}}_{\mathbf{w}} = \left\{\delta^{\mathbf{a}}_I \mid \mathbf{w}_I > 1\right\}.$$

When $\mathbf{w}$ is such that $\mathbf{w}_{[n-1]} \leq 1$ and $w_i + w_n > 1$ for all $i \in [n-1]$, every subvariety in $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$ is of the form

$$\delta^{\mathbf{a}}_I = \bigcap_{i \in I \setminus \{n\}} V(x_{i1}, \ldots, x_{ia_i}),$$

for some $I \subsetneq [n]$ such that $n \in I$ and $|I| \geq 2$; see Equation (3.6). These subvarieties are invariant under the standard torus action on $\mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$.

On the other hand, let $\mathcal{B}_{LM} = 2^{[n-1]} \setminus \emptyset$ be the boolean building set over $[n-1]$, and $\rho : A^- \to [n-1]$ be a caging with cage $\mathbf{a}^-$. Then, for every I as above, the cone $\sigma_{\rho^{-1}(I\setminus\{n\})} \in \Sigma(\rho^*\mathcal{B}_{LM})$ corresponds to the subvariety $\delta^{\mathbf{a}}_I$ under the toric orbit-cone correspondence. Moreover, every cone in $\Sigma(\rho^*\mathcal{B}_{LM})$ is of this form by definition. Therefore, both constructions can be obtained as iterated blow-ups of $\mathbb{P}^{a_1+\cdots+a_{(n-1)}-1}$ along the same subvarieties, in an order that can be assumed to be the same, so they are the same. □

Example 3.37. Consider the cage $\mathbf{a} = (3,2,2)$. As seen in Example 3.16, the Losev-Manin compactification $T^{\mathbf{a}}_{LM}$ for this cage is the blow-up of $\mathbb{P}^4$ with coordinates $[x_{11} : x_{12} : x_{13} : x_{21} : x_{22}]$ along the toric subvarieties $\delta_{13} = V(x_{11}, x_{12}, x_{13})$ and $\delta_{23} = V(x_{21}, x_{22})$.

Let us construct the polypermutohedron with cage $\mathbf{a}^- = (3,2)$ using the concept of pullback of a building set. For ease of notation, let $A = (11, 12, 13, 21, 22, 23)$ with the caging $\pi : A \to E = [2]$ being the projection onto the first coordinate.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

A π-pair consists of a tuple (I, N) where $I \subsetneq A$ does not contain any fiber of π, and N is a nested set of $\mathcal{B}_{LM} = 2^{[2]} \setminus \emptyset$. Note that in this case the only nested sets are the singletons $\{1\}$ and $\{2\}$.

In order to compute the nested fan of $\pi^*\mathcal{B}_{LM}$, we restrict our attention to its maximal cones. These correspond to π-pairs (I, N) such that $|I| = 3$. All such subsets are:

$$I = \{11, 12, 21\}, \{11, 12, 22\}, \{11, 13, 21\}, \{11, 13, 22\}, \{12, 13, 21\}, \{12, 13, 22\}.$$

In particular, the nested fan will have $2 \times 6 = 12$ maximal cones. For example, the cone corresponding to the π-pair $(I, N) = (\{11, 12, 21\}, \{1\})$ is

$$\operatorname{cone}(\overline{e}_{11}, \overline{e}_{12}, \overline{e}_{21}, \overline{e}_{11} + \overline{e}_{12} + \overline{e}_{13}) = \operatorname{cone}(\overline{e}_{11}, \overline{e}_{12}, \overline{e}_{21}) + \operatorname{cone}(\overline{e}_{\pi^{-1}(1)}).$$

This can be readily seen to be a maximal cone in the fan of $T^{\mathbf{a}}_{LM}$. Indeed, the fan of the 4-simplex in $\mathbb{R}^A/\mathbb{R}e_A$ has maximal cones $\operatorname{cone}(\overline{e}_S \mid S \subsetneq A \text{ and } |S| = 4)$, and blowing up the diagonal δ_{i3} corresponds to taking the barycentric subdivision of this fan along the vector $e_{\pi^{-1}(i)}$ for $i = 1, 2$. The maximal cone shown in the previous display is the cone in the subdivided fan obtained by replacing $\overline{e}_{13}$ by $\overline{e}_{\pi^{-1}(1)}$ in the cone $\operatorname{cone}(\overline{e}_{11}, \overline{e}_{12}, \overline{e}_{13}, \overline{e}_{21})$ of the simplex, as prescribed in the construction of the barycentric subdivision. The same reasoning works for all other maximal cones, illustrating the isomorphism in Theorem 3.22.

3.4. Pullback of building sets and nontrivial toric fibrations

Throughout this section we fix a caging $\pi : A \to [n]$ with cage $\mathbf{a} = (a_1, \ldots, a_n)$. The relation between the nested fan of a building set $\mathcal{B}$ and that one of its pullback $\pi^*\mathcal{B}$ gives rise to interesting combinatorics. A witness of this is the fact that the toric variety constructed from the pullback fan is a nontrivial, Zariski locally trivial fibration over a product of projective spaces with fibers corresponding to the base building set. Define the subsets $A_i := \pi^{-1}(i) \subseteq A$ for all $i = 1, \ldots, n$, so that $|A_i| = a_i$.

Theorem 3.38. *Let $\Sigma(\mathcal{B})$ and $\Sigma(\pi^*\mathcal{B})$ denote the nested fans of $\mathcal{B}$ and $\pi^*\mathcal{B}$, respectively. Then, there is a nontrivial splitting of the fan $\Sigma(\pi^*\mathcal{B})$,*

$$0 \to \Sigma(\mathcal{B}) \to \Sigma(\pi^*\mathcal{B}) \to \Sigma_{A_1} \times \cdots \times \Sigma_{A_n} \to 0,$$

where Σ_{A_i} is the normal fan to the $(a_i - 1)$-dimensional simplex in $\mathbb{R}^{A_i}/\mathbb{R} \cdot e_{A_i}$. In particular, there is a nontrivial, Zariski locally trivial toric fibration of projective toric varieties

$$X(\pi^*\mathcal{B}) \to \mathbb{P}^{A_1} \times \cdots \times \mathbb{P}^{A_n}$$

with fibers isomorphic to $X(\mathcal{B})$.

Let us recall the definition of a splitting of a fan.

Definition 3.39 [CLS11, Definition 3.3.18]. Consider three lattices N_0, N and N' and a short exact sequence

$$0 \longrightarrow N_0 \xrightarrow{\iota} N \xrightarrow{\varphi} N' \longrightarrow 0.$$

Suppose that $\Sigma_0, \Sigma, \Sigma'$ are fans in $(N_0)_{\mathbb{R}}, N_{\mathbb{R}}$ and $N'_{\mathbb{R}}$ and the maps ι and φ are compatible with them. We say that Σ is *split by* Σ_0 *and* Σ', or that the sequence defines a *splitting of* Σ, if there exists a subfan $\widehat{\Sigma}$ of Σ such that

1. Given cones $\widehat{\sigma} \in \widehat{\Sigma}$ and $\sigma_0 \in \Sigma_0$, the sum $\widehat{\sigma} + \sigma_0$ is in Σ, and every cone of Σ arises in this way.
2. $\varphi_{\mathbb{R}}$ maps every cone in $\widehat{\Sigma}$ bijectively to a cone $\Sigma' \in \Sigma'$ such that $\varphi(\widehat{\sigma} \cap N) = \Sigma' \cap N'$. Furthermore, the map $\widehat{\sigma} \mapsto \Sigma'$ defines a bijection between $\widehat{\Sigma}$ and Σ'.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Proposition 3.40 [CLS11, Theorem 3.3.19]. *If Σ is split by Σ_0 and Σ' as in Definition 3.39, then $X(\Sigma)$ is a Zariski locally trivial fibration (fiber bundle) over $X(\Sigma')$ with fiber $X(\Sigma_0)$. In particular, all fibers of $X(\Sigma) \to X(\Sigma')$ are isomorphic to $X(\Sigma_0)$.*

Proof of Theorem 3.38. By Lemma 3.34, every cone of $\Sigma(\pi^*\mathcal{B})$ corresponds to a π-pair (I, N) and has the form

$$\sigma_{(I,N)} = \sigma_{(I,\emptyset)} + \sigma_{(\emptyset,N)},$$

where

$$\sigma_{(I,\emptyset)} = \operatorname{cone}(\overline{e}_i \mid i \in I) \quad \text{and} \quad \sigma_{(\emptyset,N)} = \operatorname{cone}(\overline{e}_{\pi^{-1}(S)} \mid S \in N) \subseteq \mathbb{R}^A/\mathbb{R}e_A,$$

and, by the definition of a π-pair, $I \subsetneq A$ does not contain any fiber of π and N is a nested set of $\mathcal{B}$.

Consider the short exact sequence of vector spaces

$$0 \longrightarrow \mathbb{R}^{[n]}/\mathbb{R}e_{[n]} \xrightarrow{\iota} \mathbb{R}^A/\mathbb{R}e_A \xrightarrow{\varphi} \mathbb{R}^{A_1}/\mathbb{R}e_{A_1} \times \cdots \times \mathbb{R}^{A_n}/\mathbb{R}e_{A_n} \longrightarrow 0.$$

Here, the maps in the sequence are defined as $\iota(\overline{e}_i) = \overline{e}_{A_i}$ for all $i \in [n]$, and if $i \in A_i \subseteq A$, then $\varphi(\overline{e}_i) = (0, \ldots, \overline{e}_i, \ldots, 0)$. One can check directly that these maps are well-defined and that the sequence is exact.

Let us show that ι defines an inclusion of fans. Consider a cone $\sigma_N \in \Sigma(\mathcal{B})$ corresponding to a nested set N of $\mathcal{B}$. Then, $\sigma_N = \operatorname{cone}(\overline{e}_S \mid S \in N)$, and its image under ι is precisely the cone $\sigma_{(\emptyset,N)}$ in $\mathbb{R}^A/\mathbb{R}e_A$. In particular, ι is an inclusion of fans.

Next, let us see that φ defines a surjective map of fans. Consider a cone $\sigma_{(I,N)} = \sigma_{(I,\emptyset)} + \sigma_{(\emptyset,N)} \in \Sigma(\pi^*\mathcal{B})$. By the exactness of the sequence, $\sigma_{(\emptyset,N)}$ is in the kernel of φ. On the other hand, the cone $\sigma_{(I,\emptyset)}$ maps to the cone $\sigma_{I\cap A_1} \times \cdots \times \sigma_{I\cap A_n}$ in $\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$. To show surjectivity note that every cone in the product fan is of the form $\sigma_{J_1} \times \cdots \times \sigma_{J_n}$ for some subsets $\emptyset \subseteq J_i \subsetneq A_i$ (the empty set corresponding to the zero cone). Then, $(I := J_1 \cup \cdots \cup J_n, \emptyset)$ is a π-pair because I does not contain any fiber of π. Moreover, we have that $\varphi(\sigma_{(I,\emptyset)}) = \sigma_{J_1} \times \cdots \times \sigma_{J_n}$.

Let us proceed to show that $\Sigma(\pi^*\mathcal{B})$ is split by $\Sigma(\mathcal{B})$ and $\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$. Condition (1) in Definition 3.39 is an immediate consequence of Corollary 3.34 saying that every cone $\sigma_{(I,N)}$ can be decomposed as the sum $\sigma_{(I,\emptyset)} + \sigma_{(\emptyset,N)}$. To verify condition (2) we define the subfan $\widehat{\Sigma}$ of $\Sigma(\pi^*\mathcal{B})$ to be the fan consisting of all cones of the form $\sigma_{(I,\emptyset)}$ as I ranges over all the subsets of A not containing any fiber of π. Then, it follows directly from our discussion on the surjectivity of φ that φ defines a bijection between the cones of $\widehat{\Sigma}$ and those of $\Sigma_{A_1} \times \cdots \times \Sigma_{A_n}$. Moreover, since φ restricts to a map of lattices, it follows that $\varphi(\sigma_{(I,\emptyset)} \cap \mathbb{Z}^A/\mathbb{Z}e_A) = \varphi(\sigma_{(I,\emptyset)}) \cap \mathbb{Z}^{A_1}/\mathbb{Z}e_{A_1} \times \cdots \times \mathbb{Z}^{A_n}/\mathbb{Z}e_{A_n}$.

To conclude, let us prove that the fibration is not trivial. Consider any (vector space) splitting $\nu : \mathbb{R}^{A_1}/\mathbb{R}e_{A_1} \times \cdots \times \mathbb{R}^{A_n}/\mathbb{R}e_{A_n} \to \mathbb{R}^A/\mathbb{R}e_A$. Then, for the fibration to be trivial we must have that $\nu(\sigma_I) = \sigma_{(I,\emptyset)}$ for all $I = J_1 \cup \cdots \cup J_n$ with $J_i \subsetneq A_i$. However, note that the one-dimensional cones of the form $\sigma_{(I,\emptyset)}$ span all of $\mathbb{R}^A/\mathbb{R}e_A$. This would imply that the image of ν is all of $\mathbb{R}^A/\mathbb{R}e_A$, which is impossible since the image of ν is the complement of $\mathbb{R}^{[n]}/\mathbb{R}e_{[n]} \hookrightarrow \mathbb{R}^A/\mathbb{R}e_A$. □

From Theorem 3.38 together with Proposition 3.40 we recover the following surprising result, initially proved for constant cages $\mathbf{a} = (d, \ldots, d)$, $d \geq 1$, and using geometric methods in [GGAGR25].

Corollary 3.41. *There is a nontrivial, Zariski locally trivial toric fibration of projective toric varieties*

$$T^{\mathbf{a}}_{LM} \to \mathbb{P}^{a_1-1} \times \cdots \times \mathbb{P}^{a_{(n-1)}-1}$$

with fibers isomorphic to $\overline{M}^{LM}_{0,n+1}$, the Losev-Manin compactification of $M_{0,n+1}$.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Proof. Recall from Corollary 3.23 that $\overline{M}^{LM}_{0,n+1}$ is the toric variety corresponding to the $(n-2)$-dimensional permutohedron. This permutohedron is the nestohedron defined by the building set $\mathcal{B}_{LM} = 2^{[n-1]} \setminus \emptyset$, so that $\overline{M}^{LM}_{0,n+1} \cong X(\mathcal{B}_{LM})$.

On the other hand, by Theorem 3.22, we know that $T^{\mathbf{a}}_{LM} \cong PP_{(\mathbf{a}^-)}$, where $\mathbf{a}^- = (a_1, \dots, a_{n-1})$. Therefore, $T^{\mathbf{a}}_{LM}$ is isomorphic to the toric variety $X(\rho^* \mathcal{B}_{LM})$, where $\rho : A^- \to [n-1]$ is any caging with cage $\mathbf{a}^-$. The result then follows from Theorem 3.38. □

Example 3.42. Let us illustrate the fibration for our running example with cage $\mathbf{a} = (3, 2, 2)$. In this case, Corollary 3.41 shows that the rational map

$$\mathbb{P}^4 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^1$$

given by mapping $[x_{11} : x_{12} : x_{13} : x_{21} : x_{22}]$ to $([x_{11} : x_{12} : x_{13}], [x_{21} : x_{22}])$ gets resolved by blowing up the diagonals δ_{13} and δ_{23}. In addition, the corollary proves that the fibers of the locally trivial fibration $T^{\mathbf{a}}_{LM} \to \mathbb{P}^2 \times \mathbb{P}^1$ are isomorphic to the permutohedral variety of dimension 1, that is, isomorphic to $\mathbb{P}^1$.

The splitting of the polypermutohedral fan can be seen explicitly in this example. Consider the maximal cone $\operatorname{cone}(\overline{e}_{11}, \overline{e}_{12}, \overline{e}_{21}, \overline{e}_{11} + \overline{e}_{12} + \overline{e}_{13})$ of the polypermutohedral variety $T^{\mathbf{a}}_{LM}$. This gets mapped to the cone

$$\operatorname{cone}((\overline{e}_{11}, 0), (\overline{e}_{12}, 0), (0, \overline{e}_{21}), (\overline{e}_{11} + \overline{e}_{12} + \overline{e}_{13}, 0))$$

in the fan of $\mathbb{P}^2 \times \mathbb{P}^1$. This cone is the sum of

$$\widehat{\sigma} = \operatorname{cone}((\overline{e}_{11}, 0), (\overline{e}_{12}, 0), (0, \overline{e}_{21})) \quad \text{and} \quad \sigma_0 = \operatorname{cone}((\overline{e}_{11} + \overline{e}_{12} + \overline{e}_{13}, 0)).$$

In particular, the subspace $(N_0)_{\mathbb{R}}$ is the span of $(\overline{e}_{11} + \overline{e}_{12} + \overline{e}_{13}, 0) = (0, -\overline{e}_{21} - \overline{e}_{22})$ in $\mathbb{R}^A / \mathbb{R} e_A$, and the $\mathbb{P}^1$ fibers are the toric varieties with fan

$$\Sigma_0 = \{\pm(\overline{e}_{11} + \overline{e}_{12} + \overline{e}_{13}, 0), \{0\}\}.$$

3.5. *Proof of Theorem 1.1*

The different proofs of our theorem are spread throughout the previous subsections. The following is meant to aid the reader in navigating those results.

Proof. After constructing the compactifications $\mathbb{C}^{\mathbf{a}}_{\mathbf{w}}$ in Definition 3.3, the moduli spaces $T^{\mathbf{a}}_{\mathbf{w}}$ are defined in Subsection 3.2; see also Theorem 3.9. The claim that $T^{\mathbf{a}}_{\mathbf{w}}$ is smooth, and its boundary divisors are normal crossings is a direct consequence of Li's general theory of wonderful compactifications; see Proposition 2.15.

The proof of part (i) of the Theorem is given in Theorem 3.22. Part (iii) is Corollary 3.41. Let us proceed to prove part (ii).

Define $N = a_1 + \cdots + a_{(n-1)} - 1$. Let $\mathbf{w}_{LM}$ be any weight vector such that $w_{[n-1]} < 1$ and $w_i + w_n > 1$ for all $i \in [n-1]$, and let $\mathbf{w} = (1, \dots, 1)$. Then, $T^{\mathbf{a}}_{LM}$ and $T^{\mathbf{a}}$ are the wonderful compactifications of $\mathbb{P}^N$ with respect to the geometric building sets $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}_{LM}}$ and $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$, respectively. In particular, since $\mathbf{w}_{LM} \leq \mathbf{w}$, we have that $\mathcal{H}^{\mathbf{a}}_{\mathbf{w}_{LM}} \subseteq \mathcal{H}^{\mathbf{a}}_{\mathbf{w}}$. For simplicity let us denote these building sets as $\mathcal{H}_{LM}$ and $\mathcal{H}$. Suppose that

$$\mathcal{H} = \{U_1, \dots, U_p, V_1, \dots, V_q\}$$

is such that $\mathcal{H}_{LM} = \{U_1, \dots, U_p\}$ and $\mathcal{H} \setminus \mathcal{H}_{LM} = \{V_1, \dots, V_q\}$. Each of the subvarieties U_k equals a $\delta^{\mathbf{a}}_{I_k}$ for some $I_k \subsetneq [n]$ such that $|I_k| \geq 2$ and $n \in I_k$. Similarly, each V_k is of the form $\delta^{\mathbf{a}}_{J_k}$ for some $J_k \subsetneq [n]$ such that $|J_k| \geq 2$ and $n \notin J_k$; see Equation (3.6). We refer to the sets I_k, J_k as the index sets of U_k and V_k, respectively.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Let us order $\mathcal{H}_{LM}$ so that the subvarieties with larger index sets appear before those with smaller index sets. Similarly, order $\mathcal{H}$ so that the elements of $\mathcal{H}_{LM} \subseteq \mathcal{H}$ come first – appearing in the order we just fixed – and are followed by the subvarieties V_k ordered so that those subvarieties with larger index sets appear before those with smaller index sets.

We claim that these orders for $\mathcal{H}_{LM}$ and $\mathcal{H}$ both satisfy the conditions of Proposition 2.16, so that $T^{\mathbf{a}}_{LM}$ and $T^{\mathbf{a}}$ are both the iterated blow-ups of $\mathbb{P}^N$ along the building sets in the orders we just prescribed for their building sets. This would imply the result, as the pth step of the iterated blow-up process of $T^{\mathbf{a}}$ is precisely that one in the construction of $T^{\mathbf{a}}_{LM}$.

The fact that $\mathcal{H}_{LM}$ in this order satisfies the conditions of Proposition 2.16 is a direct consequence of the fact that, for any $1 \leq r \leq p$, the subset $\{U_1, \dots, U_r\} \subseteq \mathcal{H}_{LM}$ is closed under intersections of its own elements. In particular, every such subset is a building set so the claim follows.

Consider a subset $\mathcal{H}_r := \{U_1, \dots, U_p, V_1, \dots, V_r\} \subseteq \mathcal{H}$ for some $1 \leq r \leq q$, and let $\mathcal{S}_r$ be the induced arrangement of subvarieties of $\mathcal{H}_r$ in $\mathbb{P}^N$. Every element $S \in \mathcal{S}_r$ can be written as an intersection of the form $S = U_I \cap V_{J_1} \cap \cdots \cap V_{J_k}$ for some $0 \leq k \leq r$, $I \subsetneq [n]$ such that $n \in I$ and $|I| \geq 2$, and subsets $J_i \subsetneq [n]$ such that $|J_i| \geq 2$ and $n \notin J_i$. This follows from the fact that $\mathcal{H}_{LM}$ is closed under intersection. Moreover, as in the proof of Lemma 3.2, we can assume that all the subsets $I, J_1, \dots, J_k$ are disjoint.

In order to show $\mathcal{H}_r$ is a building set we prove that the intersection $U_I \cap V_{J_1} \cap \cdots \cap V_{J_k}$ is transverse. As in Lemma 3.2, this is a direct dimension count. Since all the subvarieties $U_I, V_{J_1}, \cdots, V_{J_k}$ are linear, we can identify them with their tangent spaces. Then, transversality will follow from the equality

$$\operatorname{codim}(U_I \cap V_{J_1} \cap \cdots \cap V_{J_k}, \mathbb{P}^N) = \operatorname{codim}(U_I, \mathbb{P}^N) + \sum_{i=1}^{k} \operatorname{codim}(V_{J_i}, \mathbb{P}^N).$$

This verification is left to the reader, as the dimension count is very similar to that one in the proof of Lemma 3.2, using the fact that all the subsets $I, J_1, \dots, J_k$ are disjoint.

Since the previous argument works for all elements S of the arrangement $\mathcal{S}_r$, we obtain that $\mathcal{H}_r$ is a building set. Since the argument holds for all values of r, the conditions of Proposition 2.16 are met, so $T^{\mathbf{a}}$ is the iterated blow-up of $\mathbb{P}^N$ along the subvarieties in $\mathcal{H}$ in the order we described previously. In particular, we have a sequence of blow-up maps $T^{\mathbf{a}} \to T^{\mathbf{a}}_{LM} \to \mathbb{P}^N$, as desired. □

3.6. Pullback of building sets and refinements

The data of a caging $\pi : A \to [n]$ is equivalent to that of a partition of the set A with n elements. Given two cagings $\pi : A \to [n]$ and $\pi' : A \to [n']$, we say that π *refines* π', written $\pi \geq \pi'$, if for every $i \in [n]$ one has $\pi^{-1}(i) \subseteq \pi'^{-1}(i')$ for some $i' \in [n']$. Eur and Larson in [EL24, Proposition 2.2] leverage this observation to prove that given a fixed set A and two cagings π and π' such that $\pi \geq \pi'$, the nested fans of the two corresponding polystellahedra are related by an explicit series of star subdivisions. Here we prove that this phenomenon is also present for pullbacks of building sets. As noted by Eur and Larson, this is a direct consequence of [FM05, Theorem 4.2] and [FY04, Proposition 2].

Lemma 3.43. *Let $\pi : A \to [n]$ and $\pi' : A \to [n']$ be two cagings such that $\pi \geq \pi'$. Consider the building sets $\mathcal{B}_{LM} = 2^{[n]} \setminus \emptyset$ and $\mathcal{B}'_{LM} = 2^{[n']} \setminus \emptyset$ on $[n]$ and $[n']$, respectively. Then, the fan $\Sigma(\pi^* \mathcal{B}_{LM})$ is obtained from $\Sigma(\pi'^* \mathcal{B}'_{LM})$ by a sequence of star subdivisions. Moreover, these subdivisions are unimodular. In particular, the toric variety $X(\pi^* \mathcal{B}_{LM})$ is obtained from $X(\pi'^* \mathcal{B}'_{LM})$ by a sequence of smooth blow-ups along torus-invariant subvarieties.*

Proof. The nonempty poset $\mathcal{L} = 2^A$ is finite and there exists a greatest lower bound for any nonempty subset. Hence, $\mathcal{L}$ is an atomic meet-semilattice (see [FM05, Section 2] for relevant definitions). Since $\pi \geq \pi'$, for every element $S' \in \mathcal{B}'_{LM}$ the subset $S = \pi(\pi'^{-1}(S')) \subseteq [n]$ is such that $\pi^{-1}(S) = \pi'^{-1}(S')$. Therefore, $\pi'^*(\mathcal{B}'_{LM}) \subseteq \pi^*(\mathcal{B}_{LM})$. Then, the result follows from [FM05, Theorem 4.2]. The fact that the resulting fan of these subdivisions is unimodular follows from [FY04, Proposition 2]. □

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Using this result in the context of polypermutohedra we obtain the following.

Proposition 3.44. *Consider a caging $\pi : A \to [n]$ with cage $\mathbf{a} = (a_1, \dots, a_n)$, and let $m = \sum_{i=1}^{n-1} a_i$. Then, there is a toric map*

$$\overline{M}^{LM}_{0,m+2} \to T^{\mathbf{a}}_{LM}$$

which is the composition of blow-ups along smooth torus-invariant subvarieties of $T^{\mathbf{a}}_{LM}$.

Proof. Define $A^- = \pi^{-1}([n-1])$ and $\rho = \pi|_{A^-}$. By Lemma 3.43, there is a reduction map $X(PP_{\mathbf{1}^m}) \to X(PP_{\mathbf{a}^-})$. By Theorem 3.22 and Corollary 3.23, we have that $X(PP_{\mathbf{1}^m}) \cong T^{\mathbf{1}^{(m+1)}}_{LM} \cong \overline{M}^{LM}_{0,m+2}$. On the other hand, $X(PP_{\mathbf{a}^-}) \cong T^{\mathbf{a}}_{LM}$. The result follows. □

4. Generalized Fulton-MacPherson compactification and polystellahedral varieties

This section is dedicated to the proof of Theorem 1.2. We begin with the construction of our moduli spaces in Section 4.1. To establish the isomorphism between the moduli perspective and the combinatorial side, we introduce a novel presentation of the polystellahedral variety as a blow-up of a product of projective spaces, which is achieved in Section 4.2. The presentation of the polypermutohedral variety as a quotient of the polystellahedral variety appears in Subsection 4.3. The proof of the theorem synthesizes the key results from these subsections, and is given in Subsection 4.4.

4.1. Generalized Fulton-MacPherson compactification

Let X be a complex nonsingular algebraic variety, and let D be a nonsingular closed proper subvariety of X. In [KS09] Kim and Sato introduced the following compactification $X^{[n]}_D$ of the configuration space of n labeled points in $X \setminus D$, characterized by allowing the points to coincide, but not to meet D.

Definition 4.1 [KS09, Section 1.1]. The compactification $X^{[n]}_D$ of n points in X away from D is the closure of $X^n \setminus \bigcup_S D_S$ diagonally embedded in

$$X^n \times \prod_{\substack{S \subseteq [n] \\ S \neq \emptyset}} \mathrm{Bl}_{D_S} X^n,$$

where $D_S \subseteq X^n$ is the locus parametrizing the collection of points x whose ith component x_i is in D if $i \in S \subseteq [n]$.

As noted in [KS09, Section 2.2], the collection

$$\mathcal{G}_{X,D} := \{D_S \mid S \subseteq [n], S \neq \emptyset\} \tag{4.1}$$

in X^n is a building set in the sense of Li [Li09]. It follows that $X^{[n]}_D$ is the wonderful compactification of X^n with respect to $\mathcal{G}_{X,D}$.

Theorem 4.2 [KS09, Theorem 1]. *Let X and D be as above, then the following holds:*

1. *The variety $X^{[n]}_D$ is nonsingular.*
2. *There is a "universal" family $X^{[n]+}_D \to X^{[n]}_D$. It is a flat family of stable degenerations of X with n smooth labeled points away from D.*
3. *The boundary $X^{[n]}_D \setminus (X^n \setminus \bigcup_{S \in \mathcal{G}_{X,D}} D_S)$ is a union of divisors $\widetilde{D}_S$ corresponding to D_S, $|S| \geq 1$. Any set of these divisors intersects transversally.*

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Let $\pi : A \to [n]$ be a caging with cage $\mathbf{a} = (a_1, \dots, a_n)$, where $a_i = |\pi^{-1}(i)|$. The main goal of this section is to generalize $X_D^{[n]}$ to the case of configurations of n points in a flag $\mathbb{P}^{a_n} \subseteq \cdots \subseteq \mathbb{P}^{a_1}$, lying away from a subflag $H_n \subseteq \cdots \subseteq H_1$ of hyperplanes $H_i \subseteq \mathbb{P}^{a_i}$.

Consider the cages $\mathbf{a}$ and $\mathbf{d} = (d, \cdots, d)$, where $d = a_1$ and $a_1 \geq \cdots \geq a_n$. Define $\mathbb{P}^{\mathbf{a}} := \prod_{i=1}^n \mathbb{P}^{a_i}$. Throughout this section we fix a flag

$$\mathbb{P}^{a_n} \subseteq \mathbb{P}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{P}^{a_1},$$

together with hyperplanes $H_i \subseteq \mathbb{P}^{a_i}$ for all $i \in [n]$ satisfying the compatibility condition $H_{i+1} = \mathbb{P}^{a_{i+1}} \cap H_i$ for all $i = 1, \dots, n-1$. Observe that the subvarieties $\mathbb{C}^{a_i} \cong \mathbb{P}^{a_i} \setminus H_i$ form a flag

$$\mathbb{C}^{a_n} \subseteq \mathbb{C}^{a_{n-1}} \subseteq \cdots \subseteq \mathbb{C}^{a_1}.$$

For simplicity we will consider $H_1 = V(X_0)$, and we often write H instead of H_1. Let $\mathbf{d} = (a_1, \dots, a_1)$, then the flag induces an embedding $\iota_{\mathbf{a}} : \mathbb{P}^{\mathbf{a}} \hookrightarrow \mathbb{P}^{\mathbf{d}} = \prod_{i=1}^n \mathbb{P}^d$ such that

$$\iota_{\mathbf{a}}(\mathbb{P}^{a_1} \times \cdots \times H_i \times \cdots \times \mathbb{P}^{a_n}) = (\mathbb{P}^{a_1} \times \cdots \times H \times \cdots \times \mathbb{P}^{a_1}) \cap \iota_{\mathbf{a}}(\mathbb{P}^{\mathbf{a}}).$$

Lemma 4.3. *Let $\mathcal{G}_{\mathbb{P}^d, H}$ be the building set on $\mathbb{P}^{\mathbf{d}}$ defining $(\mathbb{P}^d)_H^{[n]}$ as in Equation (4.1), and let*

$$\mathcal{G}_{\mathbf{a}} = \{H_S \cap \mathbb{P}^{\mathbf{a}} \mid H_S \in \mathcal{G}_{\mathbb{P}^d, H}\}$$

be its restriction to $\mathbb{P}^{\mathbf{a}}$. Then $\mathcal{G}_{\mathbf{a}}$ is a building set on $\mathbb{P}^{\mathbf{a}}$.

Proof. The building set $\mathcal{G}_{\mathbb{P}^d, H}$ is closed under intersection, so $\mathcal{G}_{\mathbf{a}}$ is as well. Therefore, it is enough to show that the intersection of its elements is clean.

Since $H_{i+1} = \mathbb{P}^{a_{i+1}} \cap H_i$, we obtain that $H_{i+1} = \mathbb{P}^{a_{i+1}} \cap H$. Therefore, the subvariety $H_S \cap \mathbb{P}^{\mathbf{a}}$ is the product subvariety of $\mathbb{P}^{\mathbf{a}}$ obtained by taking $\mathbb{P}^{a_i}$ in the components labeled by $i \in [n] \setminus S$, and H_i in those labeled by $i \in S$. The fact that the elements of $\mathcal{G}_{\mathbf{a}}$ intersect cleanly is a direct consequence of the fact that $H_S \cap \mathbb{P}^{\mathbf{a}}$ is a multilinear subspace. Indeed, by identifying $H_S \cap \mathbb{P}^{\mathbf{a}}$ with its embedded tangent space one obtains that

$$T_{H_{S_1} \cap \mathbb{P}^{\mathbf{a}}, y} \cap T_{H_{S_2} \cap \mathbb{P}^{\mathbf{a}}, y} = H_{S_1} \cap \mathbb{P}^{\mathbf{a}} \cap H_{S_2} \cap \mathbb{P}^{\mathbf{a}} = H_{S_1} \cap H_{S_2} \cap \mathbb{P}^{\mathbf{a}} = T_{H_{S_1} \cap H_{S_2} \cap \mathbb{P}^{\mathbf{a}}, y}$$

for all $y \in H_{S_1} \cap H_{S_2} \cap \mathbb{P}^{\mathbf{a}}$ and $S_1, S_2 \subseteq [n]$. □

Definition 4.4. The generalized Fulton-MacPherson compactification of configurations of points in the flag $\mathbb{P}^{\mathbf{a}}$ lying outside of the subflag $H_n \subseteq \cdots \subseteq H_1$ is the wonderful compactification $\mathbb{P}_H^{[\mathbf{a}]}$ of $\mathbb{P}^{\mathbf{a}}$ with respect to the building set $\mathcal{G}_{\mathbf{a}}$ defined in Lemma 4.3.

Proposition 4.5. *Given $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{d} = (a_1, \dots, a_1)$. Then, there exists a unique closed immersion which makes the following diagram commute*

$$\begin{array}{ccc} \mathbb{P}_H^{[\mathbf{a}]} & \hookrightarrow & \mathbb{P}_H^{[\mathbf{d}]} \\ \downarrow & & \downarrow \\ \mathbb{P}^{\mathbf{a}} & \xhookrightarrow{\iota_{\mathbf{a}}} & \mathbb{P}^{\mathbf{d}}. \end{array}$$

Proof. Let $X = \mathbb{P}^{\mathbf{d}}$, $Y = \mathbb{P}^{\mathbf{a}}$, and let $\mathcal{G} = \mathcal{G}_{\mathbf{d}}$ and $\mathcal{H} = \mathcal{G}_{\mathbf{a}}$. Lemma 4.3 shows that the intersections with Y of the elements of the building set $\mathcal{G}$ on X are precisely the elements of the building set $\mathcal{H}$ on Y, and therefore the conclusion follows by Lemma 3.5. □

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Finally, we remark that there is a "universal" family $(\mathbb{P}^{[\mathbf{a}]}_H)^+ \to \mathbb{P}^{[\mathbf{a}]}_H$ parametrizing stable degenerations of $\mathbb{P}^{\mathbf{a}}$ with n smooth labeled points away from H. This family is defined to be the pullback of the universal family $(\mathbb{P}^{[\mathbf{d}]}_H)^+ \to \mathbb{P}^{[\mathbf{d}]}_H$ constructed by Kim and Sato, as described in Theorem 4.2.

4.2. The polystellahedral variety as a blow-up of a product of projective spaces

In this section we construct a collection of fans that interpolate between the fan of a given polystellahedral variety and the fan of a product of projective spaces. This interpolation is by explicit star subdivisions, and the cones in the intermediate fans are defined using the notion of *compatible triples*, which we introduce as a refinement of the compatible pairs from [EL24, Definition 2.1]. This provides an explicit description of any given polystellahedral variety as an iterated blow-up of a product of projective spaces in Theorem 4.32. For intuition, we will keep a fixed small caging with cage $\mathbf{a} = (3, 2, 2)$ in the background and refer to it in several examples below.

Example 4.6 (Running example: cage $\mathbf{a} = (3, 2, 2)$). In order to keep track of the combinatorics in this subsection, we fix a small caging $\pi\colon A \to E$ with cage $\mathbf{a} = (3, 2, 2)$, as in Section 3. Thus $E = \{1, 2, 3\}$ and A decomposes as a disjoint union

$$A = A_1 \sqcup A_2 \sqcup A_3, \qquad |A_1| = 3,\ |A_2| = 2,\ |A_3| = 2,$$

where each $A_j = \pi^{-1}(j)$. For concreteness, we label the elements of each fiber by

$$A_1 = \{11, 12, 13\}, \quad A_2 = \{21, 22\}, \quad A_3 = \{31, 32\}.$$

We will repeatedly return to this running example in Examples 4.20, 4.30, and 4.33 in order to illustrate the general constructions for polystellahedral varieties.

Notation 4.7. Let us denote the canonical basis of $\mathbb{R}^A$ by $\{e_i : i \in A\}$ and for each $I \subseteq A$ let $e_I = \sum_{i \in I} e_i$. We let $\{e_i^* : i \in A\}$ denote the associated dual basis of the dual vector space $\left(\mathbb{R}^A\right)^*$. We denote the classes of e_i and e_I in the quotient vector space $\mathbb{R}^A/\langle e_A\rangle$ by $\overline{e}_i$ and $\overline{e}_I$, for any e_i and e_I.

Definition 4.8 [EL24, Definition 2.1]. A *compatible pair* with respect to a surjective function between finite sets $\pi : A \to E$ is a pair $I \leq \mathcal{F}$ consisting of a subset $I \subseteq A$ and a chain

$$\mathcal{F} = \{F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\}$$

of proper subsets of E such that if $\pi^{-1}(S) \subseteq I$ for a subset $S \subseteq E$, then $S \subseteq F_1$.

Remark 4.9. In the language of building sets, Larson and Eur [EL24, Proof of Proposition 2.2] show that the polystellahedron with caging $\pi : A \to E$ is the nestohedron coming from the building set

$$\{\{i\} \,|\, i \in A\} \cup \{\{0\} \cup S \,|\, \emptyset \subseteq S \subseteq E\}.$$

Definition 4.10. A *compatible triple* with respect to a surjective function between finite sets $\pi : A \to E$ is a triple $(I, \mathcal{F}, J)$ consisting of subsets $I \subseteq A$ and $J \subseteq E$, and a chain

$$\mathcal{F} = \{F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} \subseteq E \setminus J\}$$

of proper subsets of a set $F_{k+1} \subseteq E \setminus J$, satisfying the following compatibility condition. For each $S \subseteq E$ such that $\pi^{-1}(S) \subseteq I$: If $\mathcal{F} \neq \emptyset$ then $S \subseteq F_1$ and if $\mathcal{F} = \emptyset$ then $S \cap J = \emptyset$ (see Convention 4.11 for the precise meaning of $\mathcal{F}$ being empty).

Definition 4.10 builds upon the preexisting terminology of compatible pairs. Compatible triples provide the parameters needed to describe all the cones in the explicit blow-up construction of all polystellahedral varieties that we provide in this section.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Convention 4.11 (When is the flag $\mathcal{F}$ considered to be empty). *The conditions in Definition 4.10 are interpreted in analogy to the case of compatible pairs, as we now explain. The sets F_l are allowed to be empty. Just like in the case of compatible pairs, the chain $\mathcal{F}$ is considered to be empty when there are no sets F_l for $1 \leq l \leq k$. Hence, the flag $\mathcal{F}$ is empty and we write $\mathcal{F} = \emptyset$ when there is only F_{k+1}. This convention becomes visible in Definition 4.21 below, and it plays an essential role in the proof of Proposition 4.28.*

Remark 4.12. For any compatible triple $(I, \mathcal{F}, J)$, we have that $S \cap J = \emptyset$ for any $S \subseteq E$ such that $\pi^{-1}(S) \subseteq I$. This holds by definition if $\mathcal{F} = \emptyset$, and it follows from the other assumptions in Definition 4.10 when $\mathcal{F} \neq \emptyset$. Indeed, if $\mathcal{F} \neq \emptyset$ and $\pi^{-1}(S) \subseteq I$, we have $S \cap J \subseteq F_1 \cap J \subseteq F_{k+1} \cap J = \emptyset$, so $S \cap J = \emptyset$.

Definition 4.13. Given a compatible pair $I \leq \mathcal{F}$ as in Definition 4.8, [EL24, Definition 2.1] defines its *associated cone* $\sigma_{I \leq \mathcal{F}}$ as the cone in the space $\mathbb{R}^A$ given by

$$\sigma_{I \leq \mathcal{F}} = \operatorname{cone}(e_i \mid i \in I) + \operatorname{cone}\Big(-e_{A \setminus \pi^{-1}(F_1)}, \ldots, -e_{A \setminus \pi^{-1}(F_k)}\Big).$$

Definition 4.14. Given a compatible triple $(I, \mathcal{F}, J)$ as in Definition 4.10, we define its *associated cone* $C_{I,\mathcal{F},J}$ as the cone in the space $\mathbb{R}^A$ given by

$$C_{I,\mathcal{F},J} = \operatorname{cone}(e_i \mid i \in I) + \operatorname{cone}\Big(-e_{A \setminus \pi^{-1}(F_1)}, \ldots, -e_{A \setminus \pi^{-1}(F_k)}\Big) + \operatorname{cone}(-e_{\pi^{-1}(j)} \mid j \in J).$$

Remark 4.15. The cones $\sigma_{I \leq \mathcal{F}} \subseteq \mathbb{R}^A$ associated to the compatible pairs in [EL24, Definition 2.1] are precisely the cones in Definition 4.14 associated to the compatible triples satisfying $F_{k+1} = E$. Indeed, for any compatible pair $I \leq \mathcal{F}$ we have that $F_{k+1} = E$; there is a unique possible extension to a compatible triple $(I, \mathcal{F}, J)$, namely with $J = \emptyset$; and $\sigma_{I \leq \mathcal{F}} = C_{I,\emptyset,\mathcal{F}}$. Conversely, given any compatible triple $(I, \mathcal{F}, J)$ with $F_{k+1} = E$, then $I \leq \mathcal{F}$ is a compatible pair, $J = \emptyset$, and $C_{I,\mathcal{F},J} = C_{I,\emptyset,\mathcal{F}} = \sigma_{I \leq \mathcal{F}}$.

Lemma 4.16. *Let $(I, \mathcal{F}, J)$ be a compatible triple with respect to $\pi : A \to E$. Then, the vectors*

$$\{e_i \mid i \in I\} \cup \left\{-e_{A \setminus \pi^{-1}(F_1)}, \ldots, -e_{A \setminus \pi^{-1}(F_k)}\right\} \cup \{-e_{\pi^{-1}(j)} \mid j \in J\}$$

are the minimal set of integral generators of the cone $C_{I,\mathcal{F},J}$. Moreover, the cone $C_{I,\mathcal{F},J}$ is smooth.

Proof. Recall the vector space basis $\{e_i^* : i \in A\}$ of $\left(\mathbb{R}^A\right)^*$ dual to the basis $\{e_i : i \in A\}$ of $\mathbb{R}^A$. It is enough to exhibit lattice vectors in $\left(\mathbb{R}^A\right)^*$, u_i for each $i \in I$; v_l for each $1 \leq l \leq k$; and w_j for each $j \in J$, such that for each $i_0 \in I$, $1 \leq l_0 \leq k$, and $j_0 \in J$, we have

$$u_{i_0}(v) = \begin{cases} 1 & \text{if } v = e_{i_0}, \\ 0 & \text{if } v = e_i,\ i \in I \setminus \{i_0\}, \\ 0 & \text{if } v = -e_{A \setminus \pi^{-1}(F_l)},\ 1 \leq l \leq k, \\ 0 & \text{if } v = -e_{\pi^{-1}(j)},\ j \in J. \end{cases} \qquad v_{l_0}(v) = \begin{cases} 1 & \text{if } v = -e_{A \setminus \pi^{-1}(F_{l_0})}, \\ 0 & \text{if } v = -e_{A \setminus \pi^{-1}(F_l)},\ 1 \leq l \leq k,\ l \neq l_0, \\ 0 & \text{if } v = -e_{\pi^{-1}(j)},\ j \in J. \end{cases}$$

$$w_{j_0}(v) = \begin{cases} 1 & \text{if } v = -e_{\pi^{-1}(j_0)},\ j \in J, \\ 0 & \text{if } v = -e_{\pi^{-1}(j)},\ j \in J,\ j \neq j_0. \end{cases} \tag{4.2}$$

Let us first assume $\mathcal{F} \neq \emptyset$. For each $i \in I$ with $\pi(i) \notin F_1$, choose an element $i' \in A \setminus I$ with $\pi(i) = \pi(i')$. For each $1 \leq l \leq k+1$, choose an element $i_l \in A$ such that $\pi(e_{i_l}) \in F_l \setminus F_{l-1}$, where $F_0 := \emptyset$. For each $j \in J$, choose an element $i_j \in A$ such that $\pi(i_j) = j$ (notice we could also assume $i_j \notin I$). Now, for each $i \in I$, each $1 \leq l \leq k$, and each $j \in J$, let us define

$$u_i = \begin{cases} e_i^* & \text{if } \pi(i) \in F_1, \\ e_i^* - e_{i'}^* & \text{if } \pi(i) \notin F_1, \end{cases} \qquad v_l = e_{i_l}^* - e_{i_{l+1}}^*, \qquad w_j = -e_{i_j}^*. \tag{4.3}$$

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

It is straightforward to verify that the vectors in Equation (4.3) satisfy the desired conditions in Equation (4.2). Now, let us consider the remaining case, namely that $\mathcal{F} = \emptyset$. For each $i \in A$ such that $i \in I \cap \pi^{-1}(J)$ we choose $i' \in A$ such that $\pi(i') = \pi(i)$ and $i' \notin I$, and for each $j \in J$, we choose an element $i_j \in A$ such that $\pi(i_j) = j$ (notice we could also assume $i_j \notin I$). For each $i \in I$ and each $j \in J$, let us define

$$u_i = \begin{cases} e_i^* & \text{if } \pi(i) \notin J, \\ e_i^* - e_{i'}^* & \text{if } \pi(i) \in J, \end{cases} \qquad w_j = -e_{i_j}^*. \tag{4.4}$$

It is straightforward to verify that the vectors in Equation (4.4) satisfy the desired conditions in Equation (4.2) relevant to the case $\mathcal{F} = \emptyset$, and this completes the proof. □

Remark 4.17. In view of Lemma 4.16 for each compatible triple $(I, \mathcal{F}, J)$ the faces of the cone $C_{I,\mathcal{F},J}$ are precisely the cones generated by subsets of the minimal set of generators, and hence each such face is itself the cone $C_{I',\mathcal{F}',J'}$ associated to some compatible triple $(I', \mathcal{F}', J')$.

Remark 4.18. The cones that are maximal with respect to containment among the collection of all cones associated to compatible triples have the following two forms depending on whether the flag $\mathcal{F}$ is empty. If the flag $\mathcal{F}$ is empty, there exists a section $\theta : J \to A$ of π over J (i.e., $\pi \circ \theta = \mathrm{Id}_J$), such that $I = A \setminus \theta(J)$. Hence, in this case $C_{I,\mathcal{F},J}$ is determined by the function $\theta : J \to A$ and has the form

$$C_{I,\mathcal{F},J} = C_\theta = \mathrm{cone}(e_i \mid i \in A \setminus \theta(J)) + \mathrm{cone}(-e_{\pi^{-1}(j)} \mid j \in J).$$

If the flag $\mathcal{F}$ is not empty, there exists a section $\theta : (E \setminus F_1) \to A$ of π over $E \setminus F_1$ (i.e., $\pi \circ \theta = \mathrm{Id}_{E \setminus F_1}$), such that $I = A \setminus \theta(E \setminus F_1)$. Moreover, in this case $F_{k+1} = E \setminus J$ and $\mathcal{F}$ is a flag of proper subsets of $E \setminus J$, with F_1 possibly empty, which is full between F_1 and F_{k+1}, that is, $|F_l| = |F_1| + l - 1$ for each $1 \le l \le k+1$. Hence, in this case $C_{I,\mathcal{F},J}$ has the form

$$C_{I,\mathcal{F},J} = \mathrm{cone}(e_i \mid i \in A \setminus \theta(E \setminus F_1)) + \mathrm{cone}\Big(-e_{A\setminus\pi^{-1}(F_1)}, \ldots, -e_{A\setminus\pi^{-1}(F_k)}\Big) + \mathrm{cone}(-e_{\pi^{-1}(j)} \mid j \in J).$$

Remark 4.19. All cones that are maximal with respect to containment among the collection of all compatible triples are themselves maximal cones, that is, they are full-dimensional in $\mathbb{R}^A$. We can see this by verifying the cardinality of the minimal generating set is always $|A|$. We use the notation and description of the maximal cones in Remark 4.18 and we use Lemma 4.16 to obtain the minimal generating sets. When $\mathcal{F}$ is empty we have $\dim C_{I,\mathcal{F},J} = |A| - |\theta(J)| + |J| = |A|$. Similarly, when $\mathcal{F}$ is not empty we have

$$\dim C_{I,\mathcal{F},J} = |A \setminus \theta(E \setminus F_1)| + |k| + |J| = (|A| - |E| + |F_1|) + (|F_{k+1}| - |F_1|) + (|E| - |F_{k+1}|) = |A|.$$

Example 4.20 (Running example: compatible triples and their cones). We continue with the running example from Example 4.6 to provide intuition for 4.10-4.19. Recall that $\pi\colon A \to E$ has cage $\mathbf{a} = (3, 2, 2)$, $E = \{1, 2, 3\}$, and $A_1 = \{11, 12, 13\}$, $A_2 = \{21, 22\}$, $A_3 = \{31, 32\}$.

Let us compute sample cones $C_{I,\mathcal{F},J}$ corresponding to different choices of compatible triples $(I, \mathcal{F}, J)$. By Definition 4.10, there are two possible behaviors for compatible triples, depending on whether the flag $\mathcal{F}$ is empty or not.

Case 1: a compatible triple with empty flag. Let $\mathcal{F} = \emptyset$ and choose any $J \subseteq E$, for example, $J = \{2, 3\}$. Then $I \subseteq A$ is any subset satisfying the compatibility condition: I cannot contain either of the fibers $\pi^{-1}(2)$ or $\pi^{-1}(3)$. Aiming to describe a maximal cone, in view of Remark 4.18, we choose a section $\theta\colon J \to A$ of π over J, say $\theta(2) = 21$ and $\theta(3) = 31$. Let us take

$$I := A \setminus \theta(J) = \{11, 12, 13, 22, 32\}.$$

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Then $(I, \mathcal{F}, J)$ is a compatible triple in the sense of Definition 4.10. Its associated cone is maximal and given by

$$\begin{aligned} C_{I,\mathcal{F},J} &= \operatorname{cone}(e_i \mid i \in I) + \operatorname{cone}(-e_{\pi^{-1}(2)}, -e_{\pi^{-1}(3)}) \\ &= \operatorname{cone}(e_{11}, e_{12}, e_{13}, e_{22}, e_{32}) + \operatorname{cone}(-e_{21} - e_{22}, -e_{31} - e_{32}). \end{aligned}$$

Case 2: a compatible triple with nonempty flag. Let $J = \{3\}$ and consider the flag $\mathcal{F} = \{F_1 \subsetneq F_2 \subseteq E \setminus J\}$ such that

$$F_1 = \{1\} \subsetneq F_2 = \{1, 2\} \subseteq E \setminus J = \{1, 2\}.$$

Define $I := \pi^{-1}(\{1\}) = \{11, 12, 13\}$. Here, any subset $S \subseteq E$ with $\pi^{-1}(S) \subseteq I$ satisfies $S \subseteq \{1\} = F_1$, so $(I, \mathcal{F}, J)$ is a compatible triple. In this case the associated cone is

$$\begin{aligned} C_{I,\mathcal{F},J} &= \operatorname{cone}(e_i \mid i \in \pi^{-1}(\{1\})) + \operatorname{cone}(-e_{A\setminus\pi^{-1}(F_1)}) + \operatorname{cone}(-e_{\pi^{-1}(3)}) \\ &= \operatorname{cone}(e_{11}, e_{12}, e_{13}) + \operatorname{cone}(-e_{21} - e_{22} - e_{31} - e_{32}) + \operatorname{cone}(-e_{31} - e_{32}). \end{aligned}$$

By Lemma 4.16 both of the cones $C_{I,\mathcal{F},J}$ we described are smooth. This small example is a model for the general situation.

Definition 4.21. In the setting of Definition 4.14, for each $0 \leq s \leq |E|$, let

$$\Delta_{\pi,s} = \left\{ C_{I,\mathcal{F},J} \,:\, (I, \mathcal{F}, J) \text{ is a compatible triple with respect to } \pi \text{ and } |F_{k+1}| = s \right\}.$$

Remark 4.22. For a compatible triple $(I, \mathcal{F}, J)$ with $\mathcal{F}$ empty, the condition that $C_{I,\mathcal{F},J}$ is in $\Delta_{\pi,s}$ is equivalent to requiring that $|J| \leq |E| - s$ and that I does not contain $\pi^{-1}(j)$ for any $j \in J$.

Remark 4.23. Each collection $\Delta_{\pi,s}$ is closed under taking faces, by Remark 4.17. We will prove in Proposition 4.28 that they are actually smooth projective fans.

Example 4.24. The collection of cones $\Delta_{\pi,0}$ is the standard fan in $\mathbb{R}^A = \mathbb{R}^{a_1} \times \mathbb{R}^{a_2} \times \cdots \times \mathbb{R}^{a_r}$ defining the product of projective spaces $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_r}$ as a toric variety. To see this, consider a cone $C_{I,\mathcal{F},J} \in \Delta_{\pi,0}$ where $(I, \mathcal{F}, J)$ is a compatible triple. Since $|F_{k+1}| = s = 0$, then the flag $\mathcal{F}$ is empty. Moreover, $I \subseteq A$ and $J \subseteq E$ can be any subsets such that I contains no fibers of π over elements of J, by the compatibility condition. Hence, the cone $C_{I,\mathcal{F},J}$ has the form

$$\begin{aligned} C_{I,\mathcal{F},J} &= \operatorname{cone}(e_i \mid i \in I) + \operatorname{cone}(-e_{\pi^{-1}(j)} \mid j \in J) \\ &= \sum_{j \in E} \operatorname{cone}(e_i \mid i \in I \cap \pi^{-1}(j)) + \sum_{j \in J} \operatorname{cone}(-e_{\pi^{-1}(j)}) = \sum_{j \in E} \sigma_j \end{aligned}$$

where for each $j \in E$,

$$\sigma_j = \begin{cases} \operatorname{cone}(e_i \mid i \in I \cap \pi^{-1}(j)) & \text{if } j \notin J, \\ \operatorname{cone}(e_i \mid i \in I \cap \pi^{-1}(j)) + \operatorname{cone}(-e_{\pi^{-1}(j)}) & \text{if } j \in J. \end{cases}$$

By the compatibility condition, when $j \in J$ we have that $I \cap \pi^{-1}(j) \subsetneq \pi^{-1}(j)$. Therefore, for each $j \in E$ the possible cones σ_j are precisely all the cones generated by proper subsets of the set of vectors $\{e_i : i \in \pi^{-1}(j)\} \cup \{-e_{\pi^{-1}(j)}\}$. Since for each $j \in E$ the cones in $\mathbb{R}^{a_j}$ generated by all proper subsets of set of vectors $\{e_i : i \in \pi^{-1}(j)\} \cup \{-e_{\pi^{-1}(j)}\}$ form the standard fan of $\mathbb{P}^{a_j}$, we see that $\Delta_{\pi,0}$ is the fan of $\mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \cdots \times \mathbb{P}^{a_r}$ as we claimed.

Example 4.25. The collection of cones $\Delta_{\pi,|E|}$ is the fan of the polystellahedral variety PS_π. Indeed, the compatible triples $(I, \mathcal{F}, J)$ such that $C_{I,\mathcal{F},J} \in \Delta_{\pi,|E|}$ are those with $F_{k+1} = E$. Therefore, given

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

one such compatible triple the set J is empty;

$$\mathcal{F} = \{F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E\}$$

is a possibly empty flag of proper subsets of $F_{k+1} = E$; and $I \subseteq A$ is any subset such that if $\pi^{-1}(j) \subseteq I$ for some $j \in E$, then $j \in F_1$. Hence, $I \trianglelefteq \mathcal{F}$ is a compatible pair as in Definition 4.8. Moreover, all compatible pairs arise in this way from a unique compatible triple with $F_{k+1} = E$, since for each compatible pair $I \trianglelefteq \mathcal{F}$, we have that $(I, \mathcal{F}, \emptyset)$ is a compatible triple with $F_{k+1} = E$. It is clear that $C_{I,\mathcal{F},\emptyset} = \sigma_{I \trianglelefteq \mathcal{F}}$ for any compatible triple $(I, \mathcal{F}, J)$ with $J = \emptyset$. Therefore, $\Delta_{\pi,|E|}$ is precisely the fan of the polystellahedral variety PS_π presented in [EL24, Section 2.1 and Proposition 2.3].

Notation 4.26. Given $\pi : A \to E$, for each $J \subseteq E$ we define the cone C_J as the cone associated to the compatible triple $(\emptyset, \emptyset, J)$, that is,

$$C_J := C_{\emptyset,\emptyset,J} = \operatorname{cone}(-e_{\pi^{-1}(j)} \mid j \in J).$$

Remark 4.27. By Remark 4.22, for each $J \subseteq E$ the cone $C_J = C_{\emptyset,\emptyset,J}$ is in $\Delta_{\pi,s}$ for each $s \leq |E| - |J|$.

The next proposition shows that each $\Delta_{\pi,s}$ is a smooth projective fan and it is obtained from the previous one by an iterated star subdivision along specific cones C_J.

Proposition 4.28. *For each $0 \leq s \leq |E|$, the collection of cones $\Delta_{\pi,s}$ forms a smooth projective fan. Moreover, for each $0 \leq s \leq |E| - 1$, the fan $\Delta_{\pi,s+1}$ is the iterated star subdivision of the fan $\Delta_{\pi,s}$ along the cones $C_J = \operatorname{cone}(e_j \mid j \in \pi^{-1}(J))$, for all $J \subseteq E$ with $|J| = |E| - s$, in any order.*

In order to prove Proposition 4.28, in the following lemma we study when a cone C_{J_0} can be contained in the cone $C_{I,\mathcal{F},J}$ associated to a compatible triple $(I, \mathcal{F}, J)$ because the iterated star subdivisions depend on these containments.

Lemma 4.29. *Let $0 \leq s \leq |E| - 1$, let $(I, \mathcal{F}, J)$ be a compatible triple such that $C_{I,\mathcal{F},J}$ is in $\Delta_{\pi,s}$, and let $J_0 \subseteq E$ be any subset. If $C_{J_0} \subseteq C_{I,\mathcal{F},J}$, then $J_0 \subseteq J$. In particular,*

1. *If $|J| < |E| - s$, then $C_{I,\mathcal{F},J}$ contains no cone C_{J_0} such that $|J_0| = |E| - s$.*
2. *If $|J| = |E| - s$, then $C_{I,\mathcal{F},J}$ contains exactly one cone C_{J_0} such that $|J_0| = |E| - s$, namely C_J.*

Example 4.30 (Running example: Illustrating Lemma 4.29). Consider our running example with $E = \{1, 2, 3\}$, and let us take $s = 1$. The second part of Lemma 4.29 says that the cones $C_{I,\mathcal{F},J}$ with $|J| = |E| - s = 2$, which are all in $\Delta_{\pi,1}$ (by Remark 4.27), each contains exactly one of the cones C_{12}, C_{13}, C_{23}, namely C_J. For example, consider the cone

$$C_{I,\mathcal{F},J} = \operatorname{cone}(e_{11}, e_{12}, e_{13}, e_{22}, e_{23}) + \operatorname{cone}(-e_{21} - e_{22}, -e_{31} - e_{32})$$

from Example 4.20, corresponding to $I = \{11, 12, 13, 22, 32\}$, $\mathcal{F} = \emptyset$, and $J = \{2, 3\}$. Then, this cone clearly contains the cone C_{23}, given by

$$C_{23} = \operatorname{cone}(-e_{\pi^{-1}(2)}, -e_{\pi^{-1}(3)}) = \operatorname{cone}(-e_{21} - e_{22}, -e_{31} - e_{32}).$$

In contrast, it can be seen directly that $C_{I,\mathcal{F},J}$ does not contain either C_{12} or C_{13}. On the other hand, taking $s = 0$, the first part of the lemma applied to the same cone $C_{I,\mathcal{F},J}$, which is in $\Delta_{\pi,0}$ (by Remark 4.27), says $C_{I,\mathcal{F},J}$ does not contain $C_{\{1,2,3\}}$, which can be seen directly.

The uniqueness of C_J inside each $C_{I,\mathcal{F},J}$ is what allows the iterated star subdivisions in Proposition 4.28 below to be performed in any order, since it guarantees each star subdivision in the process is independent of the others.

Proof of Lemma 4.29. Let us assume that $C_{J_0} \subseteq C_{I,\mathcal{F},J}$ and let us show that $J_0 \subseteq J$. We have that $C_{J_0} = \operatorname{cone}(-e_{\pi^{-1}(j)} \mid j \in J_0) \subseteq C_{I,\mathcal{F},J}$, and then $-e_{\pi^{-1}(j)} \in C_{I,\mathcal{F},J}$ for each $j \in J_0$. We will now show that $-e_{\pi^{-1}(j)} \notin C_{I,\mathcal{F},J}$ for each $j \in E \setminus J$, from which $J_0 \subseteq J$ clearly follows. Let us then fix $j \in E \setminus J$.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

If the flag $\mathcal{F}$ is empty, we define $I' = I \cup \pi^{-1}(j)$. We have that $(I', \mathcal{F}, J)$ is a compatible triple and $C_{J_0} \subseteq C_{I,\mathcal{F},J} \subseteq C_{I',\mathcal{F},J}$. We see that $e_{\pi^{-1}(j)} \in \operatorname{cone}(e_i \,|\, i \in I') \subseteq C_{I',\mathcal{F},J}$. But then $-e_{\pi^{-1}(j)} \notin C_{I',\mathcal{F},J}$ because the cone $C_{I',\mathcal{F},J}$ is pointed, since it is smooth. Hence, $-e_{\pi^{-1}(j)} \notin C_{I,\mathcal{F},J}$, as desired.

We can now suppose that the flag $\mathcal{F} = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} = E \setminus J\}$ is not empty. We consider the cases $j \in F_1$ and $j \notin F_1$ separately.

If $j \in F_1$ we proceed as before. That is, we define $I' = I \cup \pi^{-1}(j)$ and then $(I', \mathcal{F}, J)$ is a compatible triple such that $C_{J_0} \subseteq C_{I,\mathcal{F},J} \subseteq C_{I',\mathcal{F},J}$. Then, $-e_{\pi^{-1}(j)} \notin C_{I',\mathcal{F},J}$ because $C_{I',\mathcal{F},J}$ is pointed, and then $-e_{\pi^{-1}(j)} \notin C_{I,\mathcal{F},J}$, as desired.

If $j \notin F_1$, then $j \in E \setminus (F_1 \cup J)$. By the compatibility condition, for each $j' \in E \setminus F_1$ the fiber $\pi^{-1}(j')$ is not contained in I. Then, for each $j' \in E \setminus F_1$ we can choose $\theta(j') \in A$ such that $\theta(j') \in \pi^{-1}(j')$ and $\theta(j') \notin I$. We consider the basis $\{e_i^* \,|\, i \in A\}$ of $(\mathbb{R}^A)^*$ which is dual to the basis $\{e_i \,|\, i \in A\}$ of $\mathbb{R}^A$. We notice that J is not empty since $|J| = |E| - s \neq 0$ and that $J \subseteq E \setminus F_1$. Then we can define the nonzero functional $\psi := -\sum_{j' \in J} e^*_{\theta(j')} \in (\mathbb{R}^A)^*$. For each $j' \in J$ we have $\pi^{-1}(j') \subseteq \pi^{-1}(J) \subseteq A \setminus \pi^{-1}(F_l)$ for each $1 \leq l \leq k$, since $J \subseteq E \setminus F_{k+1} \subseteq E \setminus F_l$ for each $1 \leq l \leq k$. By construction, the values of ψ on the minimal integral generating set of the cone $C_{I,\mathcal{F},J}$ satisfy $\psi(e_i) = 0$ for all $i \in I$, $\psi(-e_{A \setminus \pi^{-1}(F_l)}) > 0$ for all $1 \leq l \leq k$, and $\psi(-e_{\pi^{-1}(j')}) > 0$ for all $j' \in J$. Then ψ induces a supporting hyperplane of the cone $C_{I,\mathcal{F},J}$ and it defines the face $C_{I,\mathcal{F},J} \cap \{u \in \mathbb{R}^A \,|\, \psi(u) = 0\} = \operatorname{cone}(e_i \,|\, i \in I)$ of $C_{I,\mathcal{F},J}$. Since $j \notin J$, we see that $\psi(-e_{\pi^{-1}(j)}) = 0$. Then, if we assume by contradiction that $-e_{\pi^{-1}(j)} \in C_{I,\mathcal{F},J}$, we would deduce that $-e_{\pi^{-1}(j)} \in \operatorname{cone}(e_i \,|\, i \in I)$. But this is a contradiction because the functional $\gamma = e^*_{\theta(j)}$ takes the value zero on all elements in $\{e_i \,|\, i \in I\}$ and the negative value -1 on $-e_{\pi^{-1}(j)}$. This completes the proof that $C_{J_0} \subseteq C_{I,\mathcal{F},J}$ implies $J_0 \subseteq J$, and from this (1) follows since $J_0 \subseteq J$ is not possible in that case and (2) follows since only $J_0 = J$ is possible in that case. □

The first paragraph of the proof of Proposition 4.28 explains the proof's strategy. For a concrete illustration of the geometric meaning of this lemma and its key role in the proof of the main result of this section, Theorem 4.32, all in the context of the running example of cage $\mathbf{a} = (3, 2, 2)$, we refer the reader to Example 4.33 below.

Proof of Proposition 4.28. The collection of cones $\Delta_{\pi,0}$ forms a smooth projective fan by Example 4.24. Let $0 \leq s \leq |E| - 1$ and assume that $\Delta_{\pi,s}$ is a smooth projective fan. Let Σ_{s+1} be the smooth projective fan obtained by the iterated star subdivision of the fan $\Delta_{\pi,s}$ along the cones C_J, for all $J \subseteq E$ with $|J| = |E| - s$, in any order. Since there is no cone in $\Delta_{\pi,s}$ containing two distinct cones C_J by Lemma 4.29, then the resulting fan Σ_{s+1} is independent of the order of the star subdivisions along the C_J. Geometrically these star subdivisions correspond to blow-ups along disjoint loci. The claims in the statement all follow if we show that $\Sigma_{s+1} = \Delta_{\pi,s+1}$. Since the collections of cones Σ_{s+1} and $\Delta_{\pi,s+1}$ are both closed under taking faces, it suffices to show that any cone that is maximal within either of the collections Σ_{s+1} or $\Delta_{\pi,s+1}$ also belongs to the other collection.

Let τ be a maximal cone in Σ_{s+1}. Hence, there exists a maximal cone τ_0 in $\Delta_{\pi,s}$ such that τ is one of the cones obtained from τ_0 by the star subdivisions along all cones C_J, for all $J \subseteq E$ with $|J| = |E| - s$. By Lemma 4.29 we know that τ_0 is subdivided by at most one of these star subdivisions. Since $\tau_0 \in \Delta_{\pi,s}$, there exists a compatible triple $(I_0, \mathcal{F}_0, J_0)$ such that $\tau_0 = C_{I_0,\mathcal{F}_0,J_0}$ and $|F_{k=1}| = s$, where $\mathcal{F}_0$ has the form $\mathcal{F}_0 = \{F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} \subseteq E \setminus J_0\}$ (here $\mathcal{F}_0$ may be empty, which by our conventions means $k = 0$ and $\mathcal{F}_0 = \{F_{k+1} \subseteq E \setminus J_0\}$). Whether or not the flag $\mathcal{F}_0$ of proper subsets of F_{k+1} is empty, we know that $J_0 \subseteq E \setminus F_{k+1}$, so $|J_0| \leq |F_{k+1}| = |E| - s$. We consider the cases $|J_0| < |E| - s$ and $|J_0| = |E| - s$.

If $|J_0| < |E| - s$ then the cone $\tau_0 = C_{I_0,\mathcal{F}_0,J_0}$ does not contain any of the cones C_J with $|J_0| = |E| - s$, and therefore $\tau = \tau_0$. We must then show that $\tau = \tau_0$ is in $\Delta_{\pi,s+1}$. Since $J_0 \subsetneq E \setminus F_{k+1}$ we can choose and element $j_0 \in E \setminus F_{k+1}$ such that $j_0 \notin J_0$. We define a new flag $\mathcal{F}_0'$ by $\mathcal{F}_0' = \left\{F_1' \subsetneq F_2' \subsetneq \cdots \subsetneq F_k' \subsetneq F_{k+1}' = E \setminus J_0\right\}$ where $F_l' := F_l'$ for each $1 \leq l \leq k$ and $F_{k+1}' := F_{k+1} \cup \{j_0\}$. It follows that $(I_0, \mathcal{F}_0', J_0)$ is a compatible triple and since $|F_{k+1}'| = |F_{k+1}| + 1 = s+1$, then $C_{I_0,\mathcal{F}_0',J_0} \in \Delta_{\pi,s+1}$. By construction $C_{I_0,\mathcal{F}_0',J_0} = C_{I_0,\mathcal{F}_0,J_0}$, and hence $\tau = \tau_0 = C_{I_0,\mathcal{F}_0',J_0} \in \Delta_{\pi,s+1}$, as desired.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

If $|J_0| = |E| - s$, then $J_0 = E \setminus F_{k+1}$, and by Lemma 4.29(2) the cone $\tau_0 = C_{I_0,\mathcal{F}_0,J_0}$ contains exactly one of the cones C_J with $|J| = |E| - s$, namely for $J = J_0$. We deduce that τ is one of the $|J_0|$ maximal cones that arise when we perform the star subdivision of the cone $C_{I_0,\mathcal{F}_0,J_0}$ along C_{J_0}. The minimal integral generators of $C_{I_0,\mathcal{F}_0,J_0}$ are $\{e_i : i \in I_0\} \cup \{-e_{A\setminus\pi^{-1}(F_1)}, \dots, -e_{A\setminus\pi^{-1}(F_k)}\} \cup \{-e_{\pi^{-1}(j)} : j \in J_0\}$ and C_{F_1} is minimally generated by $\{-e_{\pi^{-1}(j)} : j \in J_0\}$. Then there exists $j_0 \in J_0$ such that τ is generated by the set of vectors obtained by taking the minimal generators of $C_{I_0,\mathcal{F}_0,J_0}$, and removing the vector $-e_{\pi^{-1}(j_0)}$ and including the vector

$$\sum_{j\in J_0} -e_{\pi^{-1}(j)} = \sum_{j\in E\setminus F_{k+1}} -e_{\pi^{-1}(j)} = -\sum_{i\in\pi^{-1}(E\setminus F_{k+1})} e_i = -\sum_{i\in A\setminus\pi^{-1}(F_{k+1})} e_i = -e_{A\setminus\pi^{-1}(F_{k+1})}.$$

Hence τ is generated by $\{e_i : i \in I_0\} \cup \{-e_{A\setminus\pi^{-1}(F_1)}, \dots, -e_{A\setminus\pi^{-1}(F_{k+1})}\} \cup \{-e_{\pi^{-1}(j)} : j \in J_0 \setminus \{j_0\}\}$.

Let us define $I_1 := I_0$, $J_1 := J_0 \setminus \{j_0\}$, and $\mathcal{F}_1 := \{F'_1 \subsetneq F'_2 \subsetneq \cdots \subsetneq F'_{k+1} \subsetneq F'_{k+2} \subseteq E \setminus J_1\}$ where $F'_l := F_l$ for each $1 \le l \le k+1$ and $F'_{k+2} = F_{k+1} \cup \{j_0\}$. Let us show that $(I_1, \mathcal{F}_1, J_1)$ is a compatible triple. Since $(I_1, \mathcal{F}_1, J_1)$ has the form required in Definition 4.10, we must simply verify the compatibility condition, and we do this considering the cases $k = 0$ and $k > 0$ separately. Note that the flag $\mathcal{F}_1$ is not empty, so we only need to show that any fiber of π contained in $I_1 = I_0$ is contained in $F'_1 = F_1$. If $k = 0$, since $(I_0, \mathcal{F}_0, J_0)$ is a compatible triple then any fiber of π contained in $I_1 = I_0$ is contained in $E \setminus J_0 = F_{k+1} = F_1 = F'_1$, as needed. If $k > 0$, since $I_1 = I_0$ and $F'_1 = F_1$, the compatibility condition for $(I_1, \mathcal{F}_1, J_1)$ is the same as that for $(I_0, \mathcal{F}_0, J_0)$, which holds since the latter is a compatible triple. Therefore, $(I_1, \mathcal{F}_1, J_1)$ is a compatible triple. The minimal integral generators set of the cone $C_{I_1,\mathcal{F}_1,J_1}$ are exactly the generators of τ that we described before, and therefore $C_{I_1,\mathcal{F}_1,J_1} = \tau$. Since $|F'_{k+2}| = |F_{k+1}| + 1 = s + 1$, then $\tau = C_{I_1,\mathcal{F}_1,J_1} \in \Delta_{\pi,s+1}$, as desired.

Now, let τ be a maximal cone in $\Delta_{\pi,s+1}$. Then there exists a compatible triple $(I, \mathcal{F}, J)$ of the form $I \subseteq A$, $\mathcal{F} = \{F_1 \subsetneq F_2 \subsetneq F_3 \subsetneq \cdots \subsetneq F_k \subsetneq F_{k+1} \subseteq E \setminus J\}$ for some $k \ge 0$, such that $|F_{k+1}| = s + 1$ and

$$\tau = C_{I,\mathcal{F},J} = \operatorname{cone}(e_i \mid i \in I) + \operatorname{cone}\left(-e_{A\setminus\pi^{-1}(F_1)}, \dots, -e_{A\setminus\pi^{-1}(F_k)}\right) + \operatorname{cone}(-e_{\pi^{-1}(j)} \mid j \in J).$$

We show that τ is in Σ_{s+1} considering the cases $k = 0$ and $k > 0$.

In the case $k = 0$, we have $|F_{k+1}| = |F_1| = s + 1 > 0$, then we can choose $j_0 \in F_{k+1} = F_1$. We define $I_1 := I$, $J_1 := J$, and $\mathcal{F}_1 := \{F'_1 \subseteq E \setminus J_0\}$ where $F'_1 := F_1 \setminus \{j_0\}$. Let us show that $(I_1, \mathcal{F}_1, J_1)$ is a compatible triple. We have that $(I_1, \mathcal{F}_1, J_1)$ has the form in Definition 4.10 so it is enough to verify the compatibility condition in the case that the flag is empty. Since $I_1 := I$, $J_1 := J$ the compatibility condition for $(I_1, \mathcal{F}_1, J_1)$ states the same as that for $(I, \mathcal{F}, J)$, which holds as the latter is a compatible triple. Then, $(I_1, \mathcal{F}_1, J_1)$ is a compatible triple as well. Since $|F'_{k+1}| = |F_{k+1}| - 1 = (s + 1) - 1 = s$, then $C_{I_1,\mathcal{F}_1,J_1}$ is a cone in $\Delta_{\pi,s}$. From the definition of the cone associated to a compatible triple, we have $C_{I_1,\mathcal{F}_1,J_1} = C_{I,\mathcal{F},J}$. We have that $|J_1| = |J| \le |E| - |F_{k+1}| = |E| - (s + 1) < |E| - s$. Then, by Lemma 4.29(1) the cone $C_{I_1,\mathcal{F}_1,J_1}$ does not contain any of the cones C_{J_0} with $J_0 \subseteq E$ such that $|J_0| = |E| - s$. Therefore the cone $C_{I_1,\mathcal{F}_1,J_1}$ in $\Delta_{\pi,s}$ is not modified by the star subdivisions that produce Σ_{s+1} out of $\Delta_{\pi,s}$. Hence, $C_{I,\mathcal{F},J} = C_{I_1,\mathcal{F}_1,J_1}$ is in Σ_{s+1}, as desired.

Let us assume now that $k > 0$. Since $\tau = C_{I,\mathcal{F},J}$ is maximal in $\Delta_{\pi,s+1}$ we deduce that $J = E \setminus F_{k+1}$; that there is a function $\theta : E \setminus F_1 \to A$ such that $\pi \circ \theta$ is the identity over $E \setminus F_1$ and $I = A \setminus \theta(E \setminus F_1)$; and that $\mathcal{F}$ is such that $|F_l| = |F_1| + (l - 1)$ for each $1 \le l \le k + 1$. Indeed, if one of these conditions did not hold, one could use it to enlarge I, $\mathcal{F}$ or J and obtain a new compatible triple that yields a cone in $\Delta_{\pi,s+1}$ strictly containing $C_{I,\mathcal{F},J}$, but this is not possible by the maximality assumption. We have that $|F_{k+1} \setminus F_k| = 1$, and we denote by $j_0 \in E$ the unique element such that $F_{k+1} \setminus F_k = \{j_0\}$. Notice that $j_0 \notin F_1$ and $j_0 \notin J$.

Let us define $I_1 := I$, $J_1 := J \cup \{j_0\}$, and $\mathcal{F}_1 := \{F'_1 \subsetneq F'_2 \subsetneq \cdots \subsetneq F'_k \subseteq E \setminus J_1\}$ where $F'_l := F_l$ for each $1 \le l \le k$. Let us show that $(I_1, \mathcal{F}_1, J_1)$ is a compatible triple, considering the cases $k = 1$ and $k > 1$ separately. If $k = 1$, then $\mathcal{F}_1 := \{F'_1 \subseteq E \setminus J_1\}$ and we must show that for each $j \in J_1$ the fiber $\pi^{-1}(j)$ is not contained in $I_1 = I$. But this holds by the compatibility condition of the compatible triple

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

$(I, \mathcal{F}, J)$ since $J_1 = J \cup \{j_0\} \subseteq E \setminus F_1$. If $k > 1$, we must show that for each $j \in E \setminus F_1'$ the fiber $\pi^{-1}(j)$ is not contained in $I_1 = I$. But this holds by the compatibility condition of the compatible triple $(I, \mathcal{F}, J)$ since $F_1' = F_1$ and $I_1 = I$. Therefore $(I_1, \mathcal{F}_1, J_1)$ is a compatible triple. Since $|F_k'| = |F_k| = |F_{k+1}| - 1 = s$, then the cone $C_{I_1,\mathcal{F}_1,J_1}$ is in $\Delta_{\pi,s}$.

We have that $|J_1| = |J| + 1 = |E| - |F_{k+1}| + 1 = |E| - (s+1) + 1 = |E| - s$. Then, by Lemma 4.29(2) the cone $C_{I_1,\mathcal{F}_1,J_1}$ contains exactly one of the cones C_{J_0} with $J_0 \subseteq E$ such that $|J_0| = |E| - s$, namely for $J_0 = J_1 = J \cup \{j_0\}$. Therefore, the effect of the star subdivisions that on $\Delta_{\pi,s}$ that yield Σ_{s+1} on the cone $C_{I_1,\mathcal{F}_1,J_1}$ in $\Delta_{\pi,s}$ is to subdivide it by the star subdivision along C_{J_1}. The minimal integral generators of $C_{I_1,\mathcal{F}_1,J_1}$ are

$$\begin{aligned}&\{e_i : i \in I_1\} \cup \{-e_{A\setminus\pi^{-1}(F_1')}, \dots, -e_{A\setminus\pi^{-1}(F_{k-1}')}\} \cup \{-e_{\pi^{-1}(j)} : j \in J_1\}\\ &= \{e_i : i \in I\} \cup \{-e_{A\setminus\pi^{-1}(F_1)}, \dots, -e_{A\setminus\pi^{-1}(F_{k-1})}\} \cup \{-e_{\pi^{-1}(j)} : j \in J\} \cup \{-e_{\pi^{-1}(j_0)}\}\end{aligned}$$

and C_{J_1} is minimally generated by $\{-e_{\pi^{-1}(j)} : j \in J_1\} = \{-e_{\pi^{-1}(j)} : j \in J\} \cup \{-e_{\pi^{-1}(j_0)}\}$. Therefore, after performing the star subdivisions on $\Delta_{\pi,s}$ that yield Σ_{s+1}, we obtain a cone τ_1 in Σ_{s+1} whose set of generators is obtained by taking the minimal set of integral generators of $C_{I_1,\mathcal{F}_1,J_1}$, and removing the vector $-e_{\pi^{-1}(j_0)}$ and including the vector

$$\begin{aligned}\sum_{j\in J_1} -e_{\pi^{-1}(j)} &= \sum_{j\in J\cup\{j_0\}} -e_{\pi^{-1}(j)} = \sum_{j\in(E\setminus F_{k+1})\cup\{j_0\}} -e_{\pi^{-1}(j)} = \sum_{j\in E\setminus(F_{k+1}\setminus\{j_0\})} -e_{\pi^{-1}(j)}\\ &= \sum_{j\in E\setminus F_k} -e_{\pi^{-1}(j)} = -\sum_{i\in\pi^{-1}(E\setminus F_k)} e_i = -\sum_{i\in A\setminus\pi^{-1}(F_k)} e_i = -e_{A\setminus\pi^{-1}(F_k)}.\end{aligned}$$

Then, the cone τ_1 in Σ_{s+1} is generated by

$$\{e_i : i \in I\} \cup \{-e_{A\setminus\pi^{-1}(F_1)}, \dots, -e_{A\setminus\pi^{-1}(F_k)}\} \cup \{-e_{\pi^{-1}(j)} : j \in J\}.$$

Hence, by the definition of the cone associated to a compatible triple, $C_{I,\mathcal{F},J} = \tau_1$. Therefore, $C_{I,\mathcal{F},J}$ is in Σ_{s+1}, and this completes the proof. □

Remark 4.31 (The cone of a strict transform). Let X be a toric variety with fan Δ, and let Y and Z be torus-invariant subvarieties of X, with corresponding cones σ_Y and σ_Z in Δ. The fan $\widetilde{\Delta}$ of the toric variety $\widetilde{X} = \mathrm{Bl}_Z X$ is given by the star subdivision of Δ along the cone σ_Z. If we assume that σ_Y does not contain σ_Z, then the cone σ_Y is in $\widetilde{\Delta}$ and its corresponding torus-invariant subvariety in $\widetilde{X}$ is the strict transform $\widetilde{Y}$ of Y under the blow-up map $\widetilde{X} \to X$. Indeed, first notice that σ_Y is in $\widetilde{\Delta}$ by the definition of the star subdivision operation and that by assumption Y is not contained in Z. Since σ_Y maps to itself under the induced map of fans $\widetilde{\Delta} \to \Delta$, the torus-invariant subvariety in $\widetilde{X}$ corresponding to σ_Y has the same dimension as Y and maps surjectively onto Y under $\widetilde{X} \to X$. But these properties characterize the strict transform $\widetilde{Y}$ of Y among the subvarieties of $\widetilde{X}$.

Let $\pi : A \to E$ be a surjective function between nonempty finite sets. We now present a construction of the polystellahedral variety PS_π as an explicit blow-up of the product of projective spaces $\prod_{j\in E} \mathbb{P}^{|\pi^{-1}(j)|}$. We may assume without losing generality that $E = \{1, 2, \dots, n\}$, for some $n \in \mathbb{Z}^+$, and we define $a_j := |\pi^{-1}(j)|$ for each $j \in E$. In this way $PS_\pi = PS_{(a_1,\dots,a_n)}$ is the polystellahedral variety with cage $(a_1, \dots, a_n)$ and $\prod_{j\in E} \mathbb{P}^{|\pi^{-1}(j)|} = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$.

Theorem 4.32. *The polystellahedral variety $PS_\pi = PS_{(a_1,\dots,a_n)}$ is equal to the iterated blow-up of the product of projective spaces $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ along the subvarieties $H_J := \cap_{j\in J} H_j$ and their strict transforms for all nonempty $J \subseteq \{1, 2, \dots, n\}$, in any order such that $|J|$ is nondecreasing, where for each $1 \le j \le n$, H_j is the pullback to $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ of a hyperplane in $\mathbb{P}^{a_j}$ under the projection map.*

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Example 4.33 (Running example: Theorem 4.32 for $\mathbf{a} = (3,2,2)$). We specialize Theorem 4.32 to the cage $\mathbf{a} = (3,2,2)$ and illustrate the idea of its proof. The variety

$$X_0 \;:=\; \mathbb{P}^{a_1} \times \mathbb{P}^{a_2} \times \mathbb{P}^{a_3} \;=\; \mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^2,$$

is represented by the fan $\Delta_{\pi,0}$ by Example 4.24. The cones $C_{\{1\}}, C_{\{2\}}, C_{\{3\}}$ are rays in this fan $\Delta_{\pi,0}$ by Remark 4.27, and they correspond to divisors $H_1, H_2, H_3 \subseteq X_0$ arising as the pullback of a hyperplane from each factor. In particular,

$$H_1 \cong \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2, \quad H_2 \cong \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^2, \quad H_3 \cong \mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^1.$$

The intersections $H_J = \bigcap_{j \in J} H_j$ are products where the j-th factor has dimension $a_j - 1$ whenever $j \in J$, and dimension a_j otherwise. For example,

$$H_{\{1,2\}} \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2, \quad H_{\{2,3\}} \cong \mathbb{P}^3 \times \mathbb{P}^1 \times \mathbb{P}^1, \quad H_{\{1,2,3\}} \cong \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1.$$

Theorem 4.32 says that $PS_{(3,2,2)}$ is obtained from X_0 by the following sequence of blow-ups:

1. First blow up $H_{\{1,2,3\}}$, the smallest stratum. This corresponds to the star subdivision of the fan $\Delta_{\pi,0}$ along the cone $C_{\{1,2,3\}}$ (which is in this fan by Remark 4.27), and produces the fan $\Delta_{\pi,1}$ by Proposition 4.28.
2. Next blow up the (strict transforms of the) three varieties $H_{\{1,2\}}, H_{\{1,3\}}, H_{\{2,3\}}$. By Remark 4.31 this corresponds to the star subdivisions along the cones $C_{\{1,2\}}$, $C_{\{1,3\}}$, $C_{\{2,3\}}$, respectively (which are in this fan by Remark 4.27). The resulting fan is $\Delta_{\pi,2}$ by Proposition 4.28.
3. Finally blow up the (strict transforms of the) three divisors $H_{\{1\}}, H_{\{2\}}, H_{\{3\}}$. Which similarly corresponds to the star subdivisions along the cones $C_{\{1\}}, C_{\{2\}}, C_{\{3\}}$, and the resulting fan is $\Delta_{\pi,3}$ by Proposition 4.28, which is the fan of the polystellahedral variety $PS_{(3,2,2)}$ by Example 4.25.

Thus, for this explicit cage, the abstract procedure of iterated star subdivisions in Proposition 4.28 becomes a concrete, three-step sequence of blow-ups of $\mathbb{P}^3 \times \mathbb{P}^2 \times \mathbb{P}^2$ along the torus-invariant loci H_J to produce the polystellahedral variety $PS_{(3,2,2)}$.

Proof of Theorem 4.32. Let $\widetilde{\Sigma}_0$ be the standard fan of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ in $\mathbb{R}^A := \mathbb{R}^{a_1} \times \cdots \times \mathbb{R}^{a_n}$. We can assume that H_j is the pull back to $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ of the torus-invariant divisor in $\mathbb{P}^{a_j}$ associated to the ray spanned by $(-1,\dots,-1) \in \mathbb{R}^{a_i}$, for each $1 \le j \le n$. We define $X_0 = \mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ and for each $1 \le s \le n$ we inductively define the variety X_s to be the toric variety obtained by blowing up X_{s-1} along the strict transforms of the varieties H_J for all $J \subseteq \{1,2,\dots,n\}$ with $|I| = n - s + 1$ in some particular order that we chose and fix (we will see below that the variety X_s is independent of this order). For each $1 \le s \le n$ let $\widetilde{\Sigma}_s$ be the fan of the toric variety X_s.

Since $\Delta_{\pi,0}$ is the fan of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$ by Example 4.24 and $\Delta_{\pi,n}$ is the fan of $PS_\pi = PS_{(a_1,\dots,a_n)}$ by Example 4.25, then the theorem follows if we show inductively that for each $0 \le s \le n$ the fan $\widetilde{\Sigma}_s$ agrees with the fan $\Delta_{\pi,s}$ (see Definition 4.21). Notice that this would also give us that iterated blow-ups to obtain X_s from X_{s-1} can be performed in any order. For $s = 0$, we have that $\widetilde{\Sigma}_0$ coincides with $\Delta_{\pi,0}$ by Example 4.24. Let us assume that $\widetilde{\Sigma}_s$ coincides with $\Delta_{\pi,s}$ for some $0 \le s \le n-1$, and let us show that $\widetilde{\Sigma}_{s+1}$ coincides with $\Delta_{\pi,s+1}$.

For each $J \subseteq \{1,2,\dots,n\}$ with $|J| = n - s$, let σ_J be the cone in $\widetilde{\Delta}_s$ corresponding to the strict transform of the torus-invariant subvariety H_J. Notice that for each $J \subseteq \{1,2,\dots,n\}$ with $|J| = n - s$ the cone σ_J^0 in $\widetilde{\Sigma}_0$ corresponding to H_J does not contain any of the cones along which we performed star subdivisions in the previous steps. Then, then $\sigma_J = \sigma_J^0$ by Remark 4.31. Hence, the fan $\widetilde{\Sigma}_{s+1}$ is obtained as the iterated star subdivision of $\widetilde{\Sigma}_s$ along the all cones $\sigma_J = \sigma_J^0$ for $J \subseteq \{1,2,\dots,n\}$ with $|J| = n - s$, in the order that we fixed before. On the other hand, by Proposition 4.28 the fan $\Delta_{\pi,s+1}$ is the iterated star subdivision of the fan $\Delta_{\pi,s}$ along the cones C_J in Notation 4.26, for all $J \subseteq E = \{1,2,\dots,n\}$ with

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

$|J| = |E| - s = n - s$, in any order. Therefore, to conclude the proof it is enough to show that $\sigma_J = C_J$ for all $J \subseteq E$ with $|J| = |E| - s = n - s$.

Let $J \subseteq E = \{1, 2, \ldots, n\}$ with $|J| = |E| - s = n - s$. Let $A_j := \{1, 2, \ldots, a_i\}$ for each $1 \leq j \leq n$ and let $\{e_{ji} \mid 1 \leq j \leq n, i \in A_j\}$ denote the canonical basis of $\mathbb{R}^A = \mathbb{R}^{a_1} \times \cdots \times \mathbb{R}^{a_n}$. In the present notation, the cone C_J is the cone generated by $\{-\sum_{i \in A_j} e_{ji} \mid j \in J\}$. For each $j \in J$, the cone in $\widetilde{\Sigma}_0$ corresponding to H_j is the ray generated by $\{-\sum_{i \in A_j} e_{ji}\}$ in $\mathbb{R}^A$. Since $H_J := \cap_{j \in J} H_j$, then the cone σ_J is the sum of the cones in $\mathbb{R}^A$ corresponding to the varieties H_j for $j \in J$. Then, H_J is the cone in $\mathbb{R}^A$ generated by $\{-\sum_{i \in A_j} e_{ji} \mid j \in J\}$. Therefore $\sigma_J = C_J$ and the proof is complete. □

4.3. The polypermutohedral variety as a quotient of the polystellahedral variety

In this subsection we prove that the polypermutohedral variety is a geometric quotient of an open subset of the polystellahedral variety in a canonical way.

Lemma 4.34. *Let X be a smooth toric variety with torus T, and let Z be a T-invariant subvariety of X. Let U denote the complement of the union of T-invariant subvarieties of X that do not intersect Z, and let E be the exceptional divisor in the blow-up of X along Z. Then, there exists a canonical geometric quotient:*

$$(U \setminus Z) /\!\!/ \mathbb{G}_m \cong E.$$

Proof. We show that there exists a canonical morphism of toric varieties $\pi : U \to E$ such that the restriction $\pi : (U \setminus Z) \to E$ is a geometric quotient for the action of the pointwise stabilizer $\mathrm{Stab}_T(E) \cong \mathbb{G}_m \subseteq T$ on U. The desired conclusion for a particular fan also yields the analogous conclusion for a subfan, then we may assume that the maximal cones in the fan of X are top dimensional.

Let U_E denote the complement of the union of T-invariant subvarieties of $\mathrm{Bl}_Z X$ that do not intersect E. Then, the blow-up morphism gives the identification $U_E \setminus E = U \setminus Z$. Therefore, we can assume for the rest of the proof that $Z = D$ is a T-invariant prime divisor on X and $E = D$.

Let Σ be the fan of X in $N_{\mathbb{R}} = N \otimes \mathbb{R}$ and $\rho \in \Sigma$ be the ray corresponding to D. Let us recall the structure of D as a toric variety; see [CLS11, Proposition 3.2.7.] for details. Let N_ρ be the sublattice of N spanned by the points in $\rho \cap N$, and let $N(\rho) = N/N_\rho$. For each cone $\sigma \in \Sigma$ containing ρ, let $\overline{\sigma}$ be the image cone in $N(\rho)_{\mathbb{R}}$ under the quotient map $N_{\mathbb{R}} \longrightarrow N(\rho)_{\mathbb{R}}$. We consider the collections of cones

$$\mathrm{Star}_X(\rho) = \{\sigma \subseteq N_{\mathbb{R}} \mid \rho \preceq \sigma \in \Sigma\} \quad \text{and} \quad \mathrm{Star}(\rho) = \{\overline{\sigma} \subseteq N(\rho)_{\mathbb{R}} \mid \rho \preceq \sigma \in \Sigma\}$$

Then, by [CLS11, Proposition 3.2.7.], $\mathrm{Star}(\rho)$ is a fan in $N(\rho)_{\mathbb{R}}$ and D is the toric variety $X_{\mathrm{Star}(\rho)}$. Moreover, the assignment $\sigma \longmapsto \overline{\sigma}$ is a bijection between the cones in $\mathrm{Star}_X(\rho)$ and those in $\mathrm{Star}(\rho)$. We have that

$$\Sigma' := \{\sigma \in \Sigma \mid \exists \tau \in \Sigma \text{ such that } \rho \preceq \tau \text{ and } \sigma \preceq \tau\}$$

is a subfan of Σ and U is the toric variety $X_{\Sigma'}$. Similarly, we have that

$$\Sigma'' := \{\sigma \in \Sigma \mid \rho \not\preceq \sigma \text{ but } \exists \tau \in \Sigma \text{ such that } \rho \preceq \tau \text{ and } \sigma \preceq \tau\}$$

is a subfan of Σ and $U \setminus D$ is the toric variety $X_{\Sigma''}$.

The quotient map $N_{\mathbb{R}} \longrightarrow N(\rho)_{\mathbb{R}}$ induces morphisms of toric varieties $U = X_{\Sigma'} \to D$ and $(U \setminus D) = X_{\Sigma''} \to D$. We claim that these morphisms are respectively a good quotient and a geometric quotient for the action of $\mathbb{G}_m$. These claims can be verified locally on the base.

The assignment $\sigma \mapsto \sigma + \rho$ is a bijection between the cones in the fan Σ'' and the cones in $\mathrm{Star}_X(\rho)$. Hence, the assignment $\sigma \mapsto \overline{(\sigma + \rho)}$ is a bijection between the cones in the fan Σ'' and the cones in $\mathrm{Star}(\rho)$. We also notice that any σ in Σ'' and $\sigma + \rho$ both map onto $\overline{(\sigma + \rho)}$ under the projection $N_{\mathbb{R}} \longrightarrow N(\rho)_{\mathbb{R}}$.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Since the toric variety X is smooth, all toric varieties under consideration are smooth, and hence we can choose coordinates and reduce to the following affine setting. Let τ be a fixed maximal cone in $\mathrm{Star}(\rho)$, which we know has the form $\overline{(\sigma+\rho)}$ for a unique cone σ in Σ''. We can choose affine coordinates such that $U_{\sigma+\rho} = \operatorname{Spec}\mathbb{C}[x_0, x_1, \ldots, x_n]$, $D \cap U_{\sigma+\rho} = Z(x_0) \subseteq U_{\sigma+\rho}$, and $U_\sigma = \operatorname{Spec}\mathbb{C}[x_0, x_1, \ldots, x_n]_{x_0}$. We are reduced to considering the action of $\mathbb{G}_m$ on $\mathbb{C}[x_0, x_1, \ldots, x_n]$ via $t \cdot f(x_0, x_1, \ldots, x_n) = f(tx_0, x_1, \ldots, x_n)$, for any $t \in \mathbb{G}_m$ and $f \in \mathbb{C}[x_0, x_1, \ldots, x_n]$. We must verify that the ring homomorphisms $\mathbb{C}[x_0, x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[x_0, x_1, \ldots, x_n]_{x_0} \to \mathbb{C}[x_1, \ldots, x_n]$ respectively induce a good quotient $U_{\sigma+\rho} \to U_{\overline{\sigma+\rho}}$ and a geometric quotient $U_\sigma \to U_{\overline{\sigma+\rho}}$ for this action, but this clearly holds since $\mathbb{C}[x_0, x_1, \ldots, x_n]^{\mathbb{G}_m} = \mathbb{C}[x_1, \ldots, x_n]$ and $\mathbb{C}[x_0, x_1, \ldots, x_n]_{x_0}^{\mathbb{G}_m} = \mathbb{C}[x_1, \ldots, x_n]$ clearly satisfy the required properties. □

Corollary 4.35. *Let X be a smooth toric variety with torus T, and let D be a T-invariant prime divisor on X. Let U denote the complement of the union of T-invariant subvarieties of X that do not intersect D. Then, there exist a good quotient and a geometric quotient:*

$$U/\mathbb{G}_m \cong D \quad \text{and} \quad (U \setminus D)/\!\!/\mathbb{G}_m \cong D.$$

Proof. We established these quotients within the proof of Lemma 4.34 in the case $Z = D$. □

We achieve the goal of this subsection by relating the polystellahedral and polypermutohedral varieties via a geometric quotient in the following proposition.

Proposition 4.36. *Let $\pi : A \to E$ be a surjective function between finite sets and let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{>0}$ be the associated cage. Then, there exists a nonempty open subset $(PS_{\mathbf{a}})^\circ$ of the polystellahedral variety $PS_{\mathbf{a}}$ with a canonical $\mathbb{G}_m$ action, which admits as a geometric quotient the polypermutohedral variety $PP_{\mathbf{a}}$, that is,*

$$PP_{\mathbf{a}} \cong (PS_{\mathbf{a}})^\circ/\!\!/\mathbb{G}_m.$$

Proof. By [EL24, Section 6], there is an embedding of $PP_{\mathbf{a}}$ as a torus-invariant divisor in $PS_{\mathbf{a}}$. The result follows by applying Corollary 4.35 to this embedding $PP_{\mathbf{a}} \hookrightarrow PS_{\mathbf{a}}$. Explicitly, since the polypermutohedron is the facet of the polystellahedron corresponding to the ray spanned by $e_A = (1, \ldots, 1) \in \mathbb{R}^A$, then $(PS_{\mathbf{a}})^\circ$ is defined by the subfan of the polystellahedral fan consisting of the cones which do not contain e_A but together with e_A span a cone. □

4.4. *Proof of Theorem 1.2*

To conclude this section, we provide a proof of Theorem 1.2. The different parts of these results are spread throughout the previous subsections, so this proof is meant to aid the reader in navigating the previous results.

Proof of Theorem 1.2. The variety $\mathbb{P}^{[\mathbf{a}]}_H$ is defined to be a wonderful compactification with respect to the building set from Lemma 4.3. Since H is part of the toric boundary of $\mathbb{P}^d$, the building set $\mathcal{G}_{\mathbf{a}}$ is contained in the toric boundary of $\mathbb{P}^{\mathbf{a}}$. Therefore, Claim (i) follows from Proposition 2.15. The varieties $\mathbb{P}^{[\mathbf{a}]}_H$ and $PS_{\mathbf{a}} = PS_{(a_1,\ldots,a_n)}$ are both iterated blow-ups of $\mathbb{P}^{a_1} \times \cdots \times \mathbb{P}^{a_n}$. Therefore, Claim (ii) follows by showing that the centers of these blow-ups are the same. However, this is clear from comparing the blow-up centers given by the building set $\mathcal{G}_{\mathbf{a}}$ with the centers described in Theorem 4.32. Lastly, since we have established that $T^{\mathbf{a}}_{LM}$ is isomorphic to the polypermutohedral variety $PP_{\mathbf{a}^-}$ in Theorem 1.1(i) and $\mathbb{P}^{[\mathbf{a}^-]}_H$ is isomorphic to the polystellahedral variety $PS_{\mathbf{a}^-}$ in Theorem 1.2(ii), then Claim (iii) is given by Proposition 4.36. □

Acknowledgments. The authors are grateful for the supportive environment provided by their home institutions, the University of California, Riverside (PG and JLG), and Santa Clara University (JGA). Javier González Anaya is also thankful to the Harvey Mudd College, where part of this work was conducted. The authors are grateful to Matt Larson for pointing out that the moduli spaces studied in [GGAGR25] are polypermutohedral varieties, and other helpful conversations. We also thank the Western Algebraic Geometry Symposium (WAGS) for providing the venue where this connection was originally pointed out to us.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press

Competing interests. The authors have no competing interest to declare.

Funding statement. José González was supported by a grant from the Simons Foundation (Award Number 710443). Patricio Gallardo is partially supported by the National Science Foundation under Grant No. DMS-2316749.

Data availability statement. No data was used in this work.

Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

Author contributions. All authors contributed to the research, writing, and revision of the manuscript, and approved the final version.

References

[CCRL+20] F. Castillo, Y. Cid-Ruiz, B. Li, J. Montaño and N. Zhang, 'When are multidegrees positive?', *Adv. Math.* **374** (2020), 107382.

[CD06] M. Carr and S. L Devadoss, 'Coxeter complexes and graph-associahedra', *Topol. Its Appl.* **153**(12) (2006), 2155–2168.

[CGK09] L. Chen, A. Gibney and D. Krashen, 'Pointed trees of projective spaces', *J. Algebraic Geom.* **18**(3) (2009), 477–509.

[CHL+22] C. Crowley, J. Huh, M. Larson, C. Simpson and B. Wang, 'The Bergman fan of a polymatroid', 2022, arXiv preprint arXiv:2207.08764.

[CLS11] D. A. Cox, J. B. Little and H. K. Schenck, 'Toric Varieties', in *Graduate Studies in Mathematics*, volume 124 American Mathematical Society, Providence, RI, 2011.

[CT15] A. Maria Castravet and J. Tevelev, '$\overline{M}_{0,n}$ is not a Mori dream space', *Duke Math. J.* **164**(8) (2015), 1641–1667.

[Edm70] J. Edmonds, 'Submodular functions, matroids, and certain polyhedra', in *Combinatorial Structures and Their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)* (Gordon and Breach, New York, 1970), 69–87.

[EHL23] C. Eur, J. Huh and M. Larson, 'Stellahedral geometry of matroids', in *Forum of Mathematics, Pi*, volume 11 (Cambridge University Press, 2023), e24.

[EL24] C. Eur and M. Larson, 'Intersection theory of polymatroids', *Int. Math. Res. Not.* **2024**(5) (2024), 4207–4241.

[FM94] W. Fulton and R. MacPherson, 'A compactification of configuration spaces', *Ann. Math.* **139**(1) (1994), 183–225.

[FM05] E. Feichtner and I. Müller, 'On the topology of nested set complexes', *Proc. Amer. Math. Soc.* **133**(4) (2005), 999–1006.

[FY04] E. Maria Feichtner and S. Yuzvinsky, 'Chow rings of toric varieties defined by atomic lattices', *Invent. Math.* **155**(3) (2004), 515–536.

[GGAGR25] P. Gallardo, J. González-Anaya, J. Luis González and E. Routis, 'Higher-dimensional losev–manin spaces and their geometry', *Int. Math. Res. Not.* **2025**(23) (2025).

[GK16] J. Luis González and K. Karu, 'Some non-finitely generated Cox rings', *Compos. Math.* **152**(5) (2016), 984–996.

[GR17] P. Gallardo and E. Routis, 'Wonderful compactifications of the moduli space of points in affine and projective space', *Eur. J. Math.* **3** (2017), 520–564.

[Har13] R. Hartshorne, 'Algebraic Geometry', in *Graduate Texts in Mathematics*, Springer, New York, NY, 1977.

[Has03] B. Hassett, 'Moduli spaces of weighted pointed stable curves', *Adv. Math.* **173**(2) (2003), 316–352.

[HK00] Y. Hu and S. Keel, 'Mori dream spaces and GIT', *Michigan Math. J.* **48**(1) (2000), 331–348.

[HKL18] J. Hausen, S. Keicher and A. Laface, 'On blowing up the weighted projective plane', *Math. Z.* **290** (2018), 1339–1358.

[KKO14] B. Kim, A. Kresch and Y.-G. Oh, 'A compactification of the space of maps from curves', *Trans. Amer. Math. Soc.* **366**(1) (2014), 51–74.

[KS09] B. Kim and F. Sato, 'A generalization of Fulton–MacPherson configuration spaces', *Selecta Math.* **15** (2009), 435–443.

[Li09] L. Li, 'Wonderful compactification of an arrangement of subvarieties', *Michigan Math. J.* **58**(2) (2009), 535–563.

[LM00] A. Losev and Y. Manin, 'New moduli spaces of pointed curves and pencils of flat connections', *Michigan Math. J.* **48**(1) (2000), 443–472.

[Pan95] R. Pandharipande, 'A Geometric Invariant Theory compactification of $\overline{M}_{g,n}$ via the Fulton-MacPherson configuration space', 1995, arXiv preprint alg-geom/9505022.

[Pos09] A. Postnikov, 'Permutohedra, associahedra, and beyond', *Int. Math. Res. Not.* **2009**(6) (2009), 1026–1106.

[PP23] R. Pagaria and G. Marco Pezzoli, 'Hodge theory for polymatroids', *Int. Math. Res. Not.* **2023**(23) (2023), 20118–20168.

[PRW08] A. Postnikov, V. Reiner and L. Williams, 'Faces of generalized permutohedra', *Doc. Math.* **13** (2008), 207–273.

[Rou14] E. Routis, 'Weighted compactifications of configuration spaces and relative stable degenerations', 2014, arXiv preprint arXiv:1411.2955.

[Sch03] A. Schrijver, 'Combinatorial Optimization: Polyhedra and Efficiency', in *Algorithms and Combinatorics*, Springer, Berlin, Heidelberg, 2003.

[Zel06] A. Zelevinsky, 'Nested complexes and their polyhedral realizations', *Pure Appl. Math. Q.* **2**(3) (2006), 655–671.

https://doi.org/10.1017/fms.2026.10219 Published online by Cambridge University Press