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On holomorphic critical quasi-circle maps

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Dedicated to the memory of Michel Herman

Abstract. The so-called Herman–Świątek theorem is generalized to holomorphic self-homeomorphisms of quasi-circles. This result implies an unpublished theorem of Michel Herman: If a Siegel disk or Arnold–Herman ring for a rational map has a boundary component, which is a quasi-circle containing a critical point, then the associated rotation number is Diophantine of exponent 2.

1. *Introduction*

Definition 1.1. A *holomorphic quasi-circle map* on a quasi-circle $\Gamma \subset \mathbb{C}$ is a self-homeomorphism $f : \Gamma \to \Gamma$, which extends to a holomorphic map $f : U \to \mathbb{C}$ defined in an open neighbourhood U of Γ. If, in addition, Γ contains a critical point for the holomorphic map f, then f is a *(holomorphic) critical quasi-circle map*.

The main result in this paper is the following generalization of the so-called Herman–Świątek theorem.

THEOREM 1. *Let $f : \Gamma \to \Gamma$ be a critical quasi-circle map with irrational rotation number $\theta \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$. Then θ has bounded type (or equivalently is Diophantine of exponent 2) if and only if f is quasi-conformally linearizable on Γ, i.e. if and only if there exists a quasi-conformal homeomorphism $\phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ with $\phi(\Gamma) = \mathbb{S}^1$ and such that*

$$\text{for all } z \in \mathbb{S}^1, \quad \phi \circ f \circ \phi^{-1}(z) = R_\theta(z) = e^{i2\pi\theta} z.$$

Theorem 1 contains as a special case the following theorem of Michel Herman, a theorem which remained unpublished until his death.

THEOREM 1.2. (Herman 89) *Let $f : U \to \overline{\mathbb{C}}$ be a holomorphic map with a k-periodic Siegel disk (or Arnold–Herman ring) Δ of rotation number θ. If $\partial\Delta$ (a component of the boundary of Δ) is a quasi-circle, contained in U and containing an f^k-critical point, then θ is of bounded type.*

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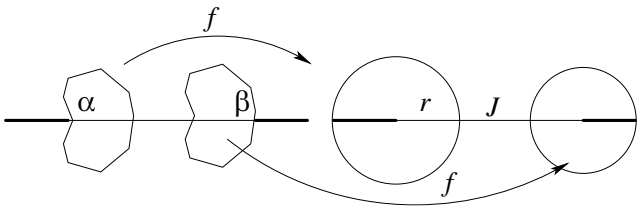


FIGURE 1. A nice diffeomorphism.

The proof of Theorem 1 is based on two main ingredients. One ingredient is the so-called Herman–Świątek theorem [**Her, Pet, Swi2**], which states that Theorem 1 holds in the case where the quasi-circle Γ is a true Euclidean circle. The other and new ingredient is a geometrization of a theorem of Świątek. The original theorem of Świątek states that under certain analytical conditions on a circle map the Świątek cross-ratio distortion inequality holds, even in the presence of a finite number of critical points [**Swi1**] (see also equation (2) later). The inequality is a main ingredient in the proof of the Herman–Świątek theorem. What is proposed here is to replace the analytical conditions on the circle map near the critical points with geometric-analytic conditions, which are more adequate for surgery applications.

In what follows there shall be no distinction between the unit circle $\mathbb{S}^1$ and the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Also a circle map $f : \mathbb{T} \to \mathbb{T}$ and its lifts to $\mathbb{R}$ under the natural projection map will not be distinguished. A map $f : U \to \mathbb{C}$, where $U \subseteq \mathbb{C}$ is some domain, is called quasi-regular, if f belongs to the Sobolev space $H^{1,2}_{\mathrm{loc}}$, i.e. if $f \in L^1_{\mathrm{loc}}$ and the distributional partial derivatives $\partial f/\partial z, \partial f/\overline{\partial z} \in L^2_{\mathrm{loc}}$ and if $|\partial f/\overline{\partial z}| \leq k|\partial f/\partial z|$ for some $0 \leq k < 1$. By the Morrey–Ahlfors–Bers integration theorem for almost complex structures, the map f is quasi-regular if and only if there exists a quasi-conformal homeomorphism $\phi : V \to U$ such that $f \circ \phi : V \to \mathbb{C}$ is holomorphic. A critical point w for a quasi-regular map f is a point where the local degree $\deg(f, w)$ is strictly bigger than one or equivalently, with ϕ as above, a point $w = \phi(c)$, where c is a critical point for $f \circ \phi$.

2. *The main ingredients*

Holomorphic quasi-circle maps can be transformed into special circle homeomorphisms, which are real analytic diffeomorphisms except at a finite set of 'critical points' (see Proposition 2.3). This is the motivation for the following definition of particular types of maps and critical points.

Note that any homeomorphism between bounded intervals $f : J' \to J$ uniquely extends to a homeomorphism between the closed intervals.

Definition 2.1. Let $f :]\alpha, \beta[\to J$, $\alpha < \beta$, be a real analytic diffeomorphism between bounded intervals, say normalized by $f(\alpha) = 0$. The map f *is nice at* α if and only if there exists a biholomorphic extension $f : U \to \mathbb{D}(0, |f(\gamma)|)\backslash[0, -f(\gamma)]$, where $U \subset \mathbb{C}$ satisfies $U \cap \mathbb{R} =]\alpha, \gamma[\subset]\alpha, \beta[$. Niceness of f at β is defined analogously. The map f *is a nice diffeomorphism* if it is nice at both ends. (See also Figure 1.)

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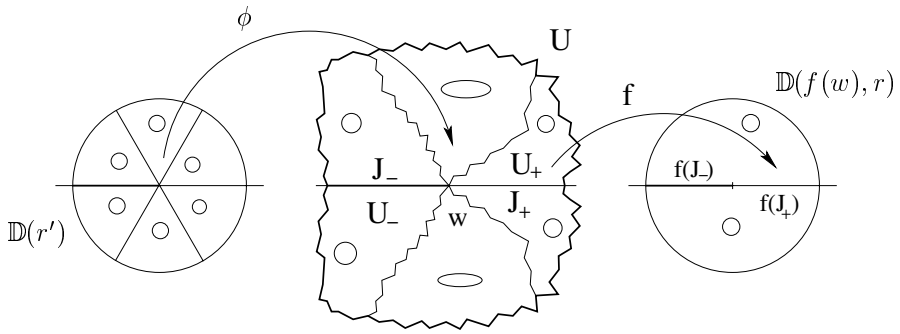


FIGURE 2. A quasi-regularly nice critical point with a degree three-quasi-regular extension. Domains of conformality and quasi-conformality for f are indicated with small circles, respectively ellipses, in U. The small circles in the disk to the left indicate that the composition $f \circ \phi$ is holomorphic.

More generally an interval homeomorphism $f : J' \to J$ is a *nice interval homeomorphism* if there exists a discrete subset $\mathcal{C} \subset J'$ such that the restriction of f to any connected component of $J' \backslash \mathcal{C}$ is a nice diffeomorphism. The points of the minimal such set $\mathcal{C}$ are *the critical points* of f. A critical point w is either a glueing point where f fails to be analytic or a point with $f'(w) = 0$.

Every critical point w is *nice*: there exist domains $U_\pm$ with $\mathbb{R} \cap U_\pm =: J_\pm \subset J'$ and biholomorphic extensions $f_\pm : U_\pm \to \mathbb{D}(f(w), r) \backslash f(J_\mp)$.

A nice critical point is *quasi-regularly nice* if $U_+ \cap U_- = \emptyset$ and if the biholomorphic extensions $f_\pm$ can be interpolated (extended) on a neighbourhood $U \supset U_+ \cup U_-$ of w to a proper quasi-regular map $f : U \to \mathbb{D}(f(w), r)$ of (odd) degree $d \geq 3$.

A nice interval homeomorphism $f : J' \to J$ is a *quasi-regularly nice interval homeomorphism* if and only if all its critical points are quasi-regularly nice.

At a quasi-regularly nice critical point w the local quasi-regular extension of f is not unique, only the germs of the analytic extensions are unique. In fact, the local degree at w of the quasi-regular extensions varies with the extensions. However, only the existence of a quasi-regular extension is used in this paper and hence the non-uniqueness shall not be pursued any further. As the holomorphic extensions naturally are real symmetric we can suppose, replacing one side with the reflection of the other, that quasi-regular extensions are real symmetric. With this assumption there exists a real symmetric neighbourhood U of w and a real symmetric quasi-conformal homeomorphism $\phi : \mathbb{D}(0, r') \to U$, such that $f \circ \phi(z) = z^d$, where d is the local degree at w of the quasi-regular extension and $r' = r^{1/d}$. In terms of ϕ we have $U_\pm = \phi(\Sigma_\pm(r', 1/d))$, where

$$\Sigma_\pm(\rho, \epsilon) = \{\pm s e^{i2\pi\eta} \mid 0 < s < \rho, |\eta| < \epsilon/2\}. \tag{1}$$

Definition 2.2. A circle map $f : \mathbb{T} \to \mathbb{T}$ (orientation-preserving homeomorphism) is *nice/quasi-regularly nice* if and only if its lifts to $\mathbb{R}$ are nice/quasi-regularly nice interval homeomorphisms.

PROPOSITION 2.3. *Let $f : \Gamma \to \Gamma$ be a holomorphic quasi-circle map. Then there exists a quasi-regularly nice circle map $\widehat{f} : \mathbb{S}^1 \to \mathbb{S}^1$ and a quasi-conformal homeomorphism*

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$\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, *with* $\psi(\Gamma) = \mathbb{S}^1$, *such that the diagram*

$$\begin{array}{ccc} \Gamma & \xrightarrow{f} & \Gamma \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathbb{S}^1 & \xrightarrow{\widehat{f}} & \mathbb{S}^1 \end{array}$$

commutes. Moreover, if f *is a critical quasi-circle map, then* $\widehat{f}$ *can be chosen to have at least one quasi-regularly nice critical point on* $\mathbb{S}^1$.

Proof. We shall suppose that f is a critical quasi-circle map, leaving the easier non-critical case to the reader. Let $f : U \to \mathbb{C}$ be a holomorphic extension of f and let $z_0 \in \Gamma$ be a critical point for the extension. Then at least one component V of the two connected components of the complement of Γ intersects $f^{-1}(\Gamma)$ in any neighbourhood of z_0. Let $\psi : \overline{V} \to \overline{\mathbb{D}}$ be a homeomorphically extended Riemann map mapping z_0 to 1. Define $\omega_1 = \psi(U)$ and extend ψ to a quasi-conformal homeomorphism $\psi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ by reflection in Γ and $\mathbb{S}^1$. Define a continuous map, quasi-regular in the interior, $\widehat{f} : \omega_1 \to \overline{\mathbb{C}}$ by $\widehat{f} = \psi \circ f \circ \psi^{-1}$. Extend $\widehat{f}$ by reflection in $\mathbb{S}^1$ to a quasi-regular map defined on $\omega = \omega_1 \cup \tau(\omega_1)$, where $\tau(z) = 1/\overline{z}$ denotes the reflection through $\mathbb{S}^1$. The maps $\widehat{f}$ and ψ are easily seen to fulfil the conclusions of the proposition, with 1 being a quasi-regularly nice critical point. □

A real four-tuple (a, b, c, d) is said to be admissible if either $a < b < c < d$ or $d < c < b < a$. If additionally $a, b, c, d \in \mathbb{T}$ we shall also infer $|d - a| \leq 1$. For such a four-tuple the cross ratio is defined as

$$[a, b, c, d] = \frac{b-a}{c-a}\frac{d-c}{d-b}$$

and, given a homeomorphism $f : \mathbb{R} \to \mathbb{R}$, the cross-ratio distortion is defined as

$$D(a, b, c, d; f) = \frac{[f(a), f(b), f(c), f(d)]}{[a, b, c, d]}.$$

The main new technical result in this paper is the following geometrization of the Świątek cross-ratio distortion inequality.

THEOREM 2.4. *Let* $f : \mathbb{T} \to \mathbb{T}$ *be a quasi-regularly nice circle map. Then:*

(1) *(Geometric Świątek cross-ratio distortion inequality) there exists a constant* $C = C(Df) > 1$ *such that, for all* $N \in \mathbb{N}$ *and for all families of admissible quadruples* $\{(a_i, b_i, c_i, d_i)\}_{i \in I}$ *for which the covering number* $\sup_{z \in \mathbb{T}} \#\{i \in I \mid z \in]a_i, d_i[\}$ *is at most* N,

$$\prod_{i \in I} D(a_i, b_i, c_i, d_i; f) \leq C^N; \tag{2}$$

(2) *for any critical point* $w \in \mathbb{T}$ *there exist* $C', \nu > 1$ *(depending only on* Df*) such that, for all* $x, y \in \mathbb{T}$ *with* $|x - w| \leq |y - w|$,

$$\left|\frac{f(x) - f(w)}{f(y) - f(w)}\right| \leq C' \left|\frac{x - w}{y - w}\right|^{\nu}. \tag{3}$$

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We shall see (Theorem 3.1) that for f as above Df is continuous with value zero at the critical points. Combining this theorem with the previous Proposition 2.3 and the following Theorem 2.5 of Herman, our main Theorem 1 follows as an immediate Corollary.

THEOREM 2.5. (Herman) *Suppose $f : \mathbb{T} \to \mathbb{T}$ is a circle homeomorphism of irrational rotation number θ. If f satisfies the cross-ratio distortion inequality (2) and θ is of bounded type, then f is quasi-symmetrically conjugate to the rigid rotation R_θ. On the other hand, if f is quasi-symmetrically conjugate to R_θ, satisfies (2) and has a critical point w which satisfies (3) for some constants C', $\nu > 1$, then the rotation number θ is of bounded type.*

Proof. See any of the papers [**Her, Pet, Swi2**]. □

The rest of this paper is dedicated to the proof of Theorem 2.4. *Any bounds (bounding constants) that will occur from here on will without explicit mention be functions of Df only.*

3. *Proof of Theorem 2.4*

The proof has the following three main steps.

THEOREM 3.1. *Let $f : J \to J'$ be a quasi-regularly nice interval map and let $w \in J$ be a critical point. Then there exist $\delta > 0$, $C > 1$ and $\nu_+ > \nu_- > 1$ such that*

$$\text{for all } x, 0 < |x| \le \delta, \quad \frac{1}{C} \le \frac{f(w+x) - f(w)}{f(w) - f(w-x)} \le C, \tag{4}$$

$$\nu_- \le \frac{xf'(w+x)}{f(w+x) - f(w)} \le \nu_+. \tag{5}$$

It follows that, for all x, y with $0 < |y| \le \delta$ and $x \in]0, y[$,

$$\left(\frac{x}{y}\right)^{\nu_+} \le \frac{f(w+x) - f(w)}{f(w+y) - f(w)} \le \left(\frac{x}{y}\right)^{\nu_-}. \tag{6}$$

Consequently, f is continuously differentiable (as a real function) at w with derivative 0.

THEOREM 3.2. *Let $f : \mathbb{T} \to \mathbb{T}$ be a quasi-regularly nice circle map. Then there exists $K > 1$ such that, for any admissible quadruple (a, b, c, d),*

$$D(a, b, c, d; f) \le K.$$

THEOREM 3.3. *Let $f : \mathbb{T} \to \mathbb{T}$ be a nice circle map. Then there exists $l_0 > 0$ such that for any admissible quadruple $(a, b, c, d) \in \mathbb{T}$, for which $]a, d[$ does not contain a critical point,*

$$D(a, b, c, d; f) \le \exp(l_0 \cdot |f(d) - f(a)|).$$

Proof of Theorem 2.4. First, item (1) of Theorem 2.4 follows immediately from the combination of the two Theorems 3.2 and 3.3. Simply take $C = K^n \cdot e^{l_0}$, where n is the number of critical points. Secondly item (2) of Theorem 2.4 follows immediately from the two inequalities (4) and (6) in Theorem 3.1, and the 1-periodicity of f. □

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Thus we are left with the task of proving Theorems 3.1, 3.2 and 3.3. Having presented the core theorems, let us comment on their proofs and their relation to, in particular, the work of Świątek. Theorem 3.2 is nearly a corollary of Theorem 3.1. But Theorem 3.3 is completely independent of the other two. In fact, it is more general as it does not suppose that f has a quasi-regular extension. In particular, it also holds for circle maps f which come from completely general holomorphic 'Jordan curve maps' through a uniformization procedure similar to the one in the proof of Proposition 2.3. For a nice, but not quasi-regularly nice, critical point w the domains of univalent extension $U_{\pm}$ on the two sides of w need not be disjoint, thus allowing for glued maps like an affine map on the negative reals and x^2 on the positive reals. Theorem 3.1, on the other hand, relies strongly on the fact that f has a quasi-regular extension: the inequality (4) captures that f is quasi-symmetric across the quasi-regularly nice critical point w. The inequality (5) and the integrated versions (6) hinge on the fact that the sets of univalence $U_{\pm}$ are disjoint quasi-sectors. Theorem 3.2 depends strongly on the quasi-symmetric property (4) and the power-like property (6) of f at the critical points.

As a comparison with Świątek's original work, he asks that Df be bounded in a neighbourhood of each critical point by multiples of a given odd power. Here this condition is replaced by the more general conditions (4) and (5). The part of his proof which corresponds to Theorem 3.3 relies on the following two assumptions: f has negative Schwarzian derivative in punctured neighbourhoods of each critical point and Df has bounded variation on the remaining parts of the circle. Here these hypotheses are replaced by the assumption that f is a nice diffeomorphism in between the critical points. Analyticity is of course a strong condition, but it comes for free in the application Proposition 2.3. Also proving that the Schwarzian derivative is negative near the critical points for the maps obtained by the surgery does not seem possible, presumably it might not be true in general.

4. *Proof of Theorem 3.1*

Define $\mathbb{H}_+ = \{z = x + iy \mid x > 0\}$ and $\mathbb{D}_+ = \mathbb{D} \cap \mathbb{H}_+$. We need the following Julia–Wolf-type lemma.

LEMMA 4.1. *Let $f : \mathbb{D}_+ \to \mathbb{H}_+$ be a holomorphic function. Then on $\mathbb{R}_+$*

$$\limsup_{x \to 0} \frac{x|f'(x)|}{|f(x)|} \leq 1. \tag{7}$$

Proof. Let $\lambda_1(z)|dz|$ and $\lambda_2(z)|dz|$ denote the hyperbolic metrics on $\mathbb{D}_+$ and $\mathbb{H}_+$, respectively. Then $\lambda_2(x + iy) = 1/x$ and, for all $z = x + iy \in \mathbb{D}_+$, $1 \geq \lambda_2(z)/\lambda_1(z) \to 1$ as $z \to 0$. By the Schwarz–Picks lemma for hyperbolic domains,

$$\text{for all } x \in \mathbb{R} \cap \mathbb{D}_+, \quad 1 \geq \frac{\lambda_2(f(x))|f'(x)|}{\lambda_1(x)} \geq \frac{x|f'(x)|}{|f(x)|} \cdot \frac{\lambda_2(x)}{\lambda_1(x)}. \tag{8}$$

Thus (7) follows. □

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Proof of Theorem 3.1. Given $0 < x < y \leq \delta$ the inequality (6) follows by integrating the logarithmic derivative $f'(t+w)/(f(t+w)-f(w))$ from x to y and bounding the integrand using (5).

To prove (4) and (5) note that both inequalities are invariant under both pre and post composition of f with affine maps. Thus we can start out assuming that f is order preserving and $w = f(w) = 0$. Let $f : U \to \mathbb{D}(f(w), r)$ be a degree d quasi-regularly nice extension of f. We can further assume that $r = 1 = r^{1/d} = r'$. Let $\phi : \mathbb{D} \to U$ be the quasi-conformal homeomorphism, such that $f \circ \phi(z) = z^d$, $\phi(0) = 0$ and ϕ maps $]-1, 1[$ increasingly into $\mathbb{R}$. Rescaling f in the domain and range if necessary, we can suppose that $\phi(\overline{\mathbb{D}}) = \overline{U}$ is a quasi-disk for which $\overline{U} \cap \mathbb{R}$ is a closed subinterval of J. Inequality (4) immediately follows, because ϕ is quasi-symmetric on the real axis.

Using our freedom to pre compose f by linear maps, we can assume that $\mathbb{D} \subseteq U \subseteq \mathbb{D}_R$ for some $R > 1$. Recall that the domains of univalence $U_\pm$ for f are given by $U_\pm = \phi(\Sigma_\pm(1, 1/d))$ where $\Sigma_\pm(\cdot,\cdot)$ were defined in (1). We claim that $U_\pm$ are quasi-sectors, i.e. there exist $\nu_+ > \nu_- > 1$ such that

$$\Sigma_\pm(1, 1/\nu_+) \subseteq U_\pm \subseteq \Sigma_\pm(R, 1/\nu_-). \tag{9}$$

In fact, as U is a quasi-disk with $U \cap \mathbb{R}$ connected, we need only show that

$$\pi/\nu'_+ < \liminf_{\partial\Sigma_\pm \ni z \to 0} |\arg(\pm\phi(z))|, \qquad \limsup_{\partial\Sigma_\pm \ni z \to 0} |\arg(\pm\phi(z))| < \pi/\nu'_- \tag{10}$$

for some constants $\nu'_+ > \nu'_- > 1$, which depends only on the degree d and the dilatation of ϕ. Moreover, we need only prove the first of the two inequalities in (10) because U_+ and U_- are disjoint. Finally by symmetry we need only prove the inequality for U_+. By the assumptions on the geometry of U we have a quasi-conformal restriction $\phi : \mathbb{D} \setminus [0, 1[\to \mathbb{D}(R) \setminus [0, R[$. Such a map expands hyperbolic distances by at most a bounded amount depending only on the dilatation of ϕ and the distance. We have $\phi(]-1, 0[) \subset]-R, 0[$ and for $z \in \mathbb{D} \setminus [0, 1[=: B$ with $\arg(z) = \eta$ the hyperbolic distance in B, $\mathrm{d}_B(z,]-1, 0])$, converges to $\log\cot(|\eta|/4)$ as $z \to 0$. Thus the bounds (10) and hence (9) follow.

Suppose $x > 0$ (the case $x < 0$ is similar). Define $g(z) = (f^{-1}(z^2))^{\nu_-/2}$ and $h(z) = f(z^{2/\nu_+})^{1/2}$ on $\mathbb{D}_+$ Applying the Julia–Wolf lemma (7) to g and h we obtain

$$\begin{aligned}\nu_- &\leq \liminf_{x\to 0} \frac{\nu_- g(x)}{x \cdot g'(x)} = \liminf_{x\to 0} \frac{f^{-1}(x^2)}{x^2 \cdot (f^{-1})'(x^2)} \\ &= \liminf_{x\to 0} \frac{f^{-1}(x)}{x \cdot (f^{-1})'(x)} = \liminf_{x\to 0} \frac{x \cdot f'(x)}{f(x)}\end{aligned}$$

and

$$\nu_+ \geq \limsup_{x\to 0} \frac{\nu_+ \cdot x \cdot h'(x)}{h(x)} = \limsup_{x\to 0} \frac{x^{2/\nu_+} \cdot f'(x^{2/\nu_+})}{f(x^{2/\nu_+})} = \limsup_{x\to 0} \frac{x \cdot f'(x)}{f(x)}.$$

Thus increasing ν_+ slightly and decreasing $\nu_- > 1$ slightly, if necessary there exists $\delta > 0$ such that the inequality (5) holds. Finally, (5) and (6) imply that f is continuously differentiable (as a real function) at w with derivative 0. □

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5. *Proof of Theorem 3.2*

The proof uses the same strategy as Świątek in [**Swi1**]. Here the proof relies on Theorem 3.1, however. To facilitate the proof of Theorem 3.2, we introduce Świątek's half cross-ratio distortion:

$$D_h(a,b,c;f)=\frac{f(b)-f(a)}{b-a}\frac{c-a}{f(c)-f(a)},$$

where (a,b,c) is any real triple with $b\in]a,c[\subseteq \mathrm{Domain}(f)$. The half-cross ratio distortion is handy, because it is also invariant under pre and post composition by affine maps and it can be used to compute the cross-ratio distortion:

$$D(a,b,c,d;f)=D_h(a,b,c;f)\cdot D_h(d,c,b;f). \tag{11}$$

PROPOSITION 5.1. *Suppose $f:]w-\delta,w+\delta[\to\mathbb{R}$, $\delta>0$ is differentiable and satisfies (4)–(6) for some constants C,ν^+,ν^-, as in the conclusion of Theorem 3.1. Then there exists a constant $C_1>1$ depending only on C,ν^+,ν^- such that, for all triples (a,b,c) with $b\in]a,c[\subseteq]w-\delta,w+\delta[$,*

$$D_h(a,b,c;f)\le C_1. \tag{12}$$

Proof. We can suppose that $w=f(w)=0$ and we can further normalize so that given a point $x\neq 0$ we have $x=f(x)=-1$. Let $x_-=C^{-1/(\nu^+-1)}$ and $x_+=C^{1/(\nu^--1)}$. If $f(-1)=-1$ then

$$\begin{gathered}\text{for all } x<-1,\quad f(x)<x,\\ \text{for all } x,-1<x<0,\quad f(x)>x,\\ \text{for all } x,0<x<x_-,\quad f(x)<x,\\ \text{for all } x>x_+,\quad f(x)>x.\end{gathered}$$

Moreover,

$$\text{for all } x,-1\le x\le 0,\quad 1\le\frac{f(x)+1}{x+1}\le\nu_+.$$

For instance, if $x>x_+$ then

$$f(x)\ge -C^{-1}f(-x)\ge C^{-1}xx^{\nu^--1}>xC^{-1}x_+^{\nu^--1}=x.$$

By symmetry and using the freedom to normalize, it suffices to consider the following four different cases.

(1) $c<-1=b=f(b)<a\le 0$, where we obtain

$$D_h(a,b,c,f)=\frac{a-c}{f(a)-f(c)}\cdot\frac{f(a)+1}{a+1}\le\frac{f(a)+1}{a+1}\le\nu^+.$$

(2) $-1=a=f(a)<b<c\le 0$, yielding

$$D_h(a,b,c,f)=\frac{c+1}{f(c)+1}\cdot\frac{f(b)+1}{b+1}\le\frac{f(b)+1}{b+1}\le\nu^+.$$

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(3) $c = f(c) = -1 < b \le 0 < a$, where we obtain

$$D = D_h(a, b, c, f) = \frac{a+1}{f(a)+1} \cdot \frac{f(a)-f(b)}{a-b}.$$

If $f(a) \le a$ then $a \le x_+$ and

$$D \le \frac{a+1}{f(a)+1} \le x_+ + 1.$$

If $f(a) \ge a$ then $a \ge x_-$ and

$$D \le \frac{a+1}{a-b} \le \frac{x_-+1}{x_--b} \le \frac{x_-+1}{x_-},$$

where the middle inequality stems from the fact that the Möbius transformation $a \mapsto (a+1)/(a-b)$ is decreasing on the interval $[b, \infty] \supset \mathbb{R}_+$.

(4) $a = f(a) = -1 < b < 0 < c$, where we obtain

$$D = D_h(a, b, c, f) = \frac{c+1}{f(c)+1} \cdot \frac{f(b)+1}{b+1}.$$

If $f(c) \ge c$ then

$$D \le \frac{f(b)+1}{b+1} \le \nu^+.$$

If $f(c) \le c$ then $c \le x_+$ and $D \le \nu^+ \cdot (x_+ + 1)$. This completes the proof. □

PROPOSITION 5.2. *Let f be a C^1 circle homeomorphism with at most finitely many critical points $w_1, \dots, w_n$ each satisfying the inequalities (4)–(6) for some constants $\delta_i > 0, C_i > 1$ and $\nu_i^+ > \nu_i^- > 1$. Then there exists $K > 1$ such that, for any admissible quadruple (a, b, c, d),*

$$D(a, b, c, d; f) \le K.$$

Proof. It suffices to prove that there exists $K_1 > 1$ such that, for every triple (a, b, c) with $b \in]a, c[$,

$$D_h(a, b, c; f) \le K_1,$$

because of equation (11). Write $2\delta_i$ in place of δ_i and let

$$M = \sup\{f'(z) \mid z \in \mathbb{T}\}, \quad m = \inf\left\{ f'(z) \mid z \in \mathbb{T} \Big\backslash \bigcup_{i=1}^{n}]w_i - \delta_i, w_i + \delta_i[\right\}.$$

There are three different cases to be considered separately.

(1) There exists i such that $]a, c[\subset [w_i - 2\delta_i, w_i - 2\delta_i]$. The conclusion then follows from Proposition 5.1.

(2) For every i, $]a, c[\cap [w_i - \delta_i, w_i + \delta_i] = \emptyset$. Then by the mean-value theorem $D_h(a, b, c, f) \le M/m$.

(3) Neither (1) nor (2). Then there exists i such that either $]w - 2\delta_i, w - \delta_i[\subseteq]a, c[$ or $]w + \delta_i, w + 2\delta_i[\subseteq]a, c[$. In either case $D_h(a, b, c; f)$ is bounded by

$$M \cdot \max_{1 \le i \le n} \left\{ \frac{1}{f(w_i - \delta_i) - f(w_i - 2\delta_i)}, \frac{1}{f(w_i + 2\delta_i) - f(w_i + \delta_i)} \right\}. \qquad \square$$

Theorem 3.2 is a corollary of the above Proposition 5.2 and Theorem 3.1.

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Proof of Theorem 3.2. By assumption, f is analytic except possibly at the critical points w_i. By Theorem 3.1 it is continuously differentiable at those points, thus f is C^1. Moreover, also by Theorem 3.1, each of the critical points satisfies the inequalities (4)–(6) for some constants $\delta_i > 0, C_i > 1$ and $\nu_i^+ > \nu_i^- > 1$. Thus the hypotheses of Proposition 5.2 are satisfied. This completes the proof. □

6. *Proof of Theorem 3.3*

Finally, we prove Theorem 3.3. To this end we shall transform the cross-ratio distortion bound into a bound on contraction of hyperbolic length.

For a bounded open interval $J \subset \mathbb{R}$ the doubly slit plane with gap J is defined as $\mathbb{C}_J = (\mathbb{C} \setminus \mathbb{R}) \cup J$. Denote by $\mathrm{d}_J(\cdot,\cdot)$ and $l_J(\cdot)$ the hyperbolic distance and arclength functions in $\mathbb{C}_J$, respectively.

An elementary calculation shows that the cross ratio $[a,b,c,d]$ is related to the hyperbolic distance between b and c in the doubly slit plane $\mathbb{C}_{]a,d[}$ by

$$\mathrm{d}_{]a,d[}(b,c) = -\log[a,b,c,d].$$

Hence the cross-ratio distortion $D(a,b,c,d;f)$ can be calculated as

$$D(a,b,c,d;f) = \exp(\mathrm{d}_{]a,d[}(b,c) - \mathrm{d}_{]f(a),f(d)[}(f(b),f(c))),$$

and the cross-ratio distortion inequality (2) becomes

$$\sum_{i\in I} \mathrm{d}_{]a_i,d_i[}(b_i,c_i) - \mathrm{d}_{]f(a_i),f(d_i)[}(f(b_i),f(c_i)) \le N\log C.$$

In particular, the cross-ratio distortion inequality implies that the map f only shortens the hyperbolic length of $[b,c] \subset \mathbb{C}_{]a,d[}$ by a finite amount. In this perspective the Świątek cross-ratio distortion inequality can be viewed as a real predecessor of the Carleson–Jones–Yoccoz bound for the distortion of proper analytic maps of simply connected domains [**CJY**].

For $R > 0$ and an open interval J, define the slit R-neighbourhood of J as

$$\Omega_R(J) = \{z \in \mathbb{C}_J \mid \mathrm{dist}(z,J) < R\}$$

($\mathrm{dist}(\cdot,\cdot)$ being the standard Euclidean distance). Any nice diffeomorphism $f : J' \to J$ has a biholomorphic extension $f : W \to \Omega_R(J)$ for some $R > 0$ and some domain W with $J' \subset W \subseteq \mathbb{C}_{J'}$, because f is analytic. Define R_f as the supremum over the set of such R. This supremum is clearly achieved for some domain W, that is either R_f is finite and a maximum or infinite and f has a univalent inverse $f^{-1} : \mathbb{C}_J \to \mathbb{C}_{J'}$. Theorem 3.3 is a corollary of the following theorem.

THEOREM 6.1. *There exists a function $L_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any nice diffeomorphism $f : J' \to J$ and any admissible quadruple (a,b,c,d) with $]a,d[\subseteq J'$*

$$D(a,b,c,d;f) \le \exp(l_0 \cdot |f(d) - f(a)|),$$

where $l_0 \cdot R_f = L_0(|J|/R_f)$.

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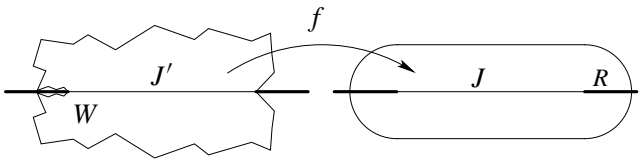


FIGURE 3. Extension of a nice diffeomorphism $f : J' \to J$ as a biholomorphic map onto a slit R-neighbourhood of J.

Proof of Theorem 3.3. Let $w, w' \in \mathbb{T}$ be a pair of neighbouring critical points bounding an open interval $J' =]w, w'[$ and let $J = f(J')$. Then the restriction $f : J' \to J$ is a nice diffeomorphism. Thus Proposition 6.1 proves Theorem 3.3 for J'. As f has finitely many critical points we are done. □

Theorem 6.1 is a simple consequence of the following lemma in hyperbolic geometry.

LEMMA 6.2. *There exists a strictly increasing function $L : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for any $R > 0$ and any bounded non-empty open interval $J =]a, d[\subset \mathbb{R}$ and any pair of points $b, c \in J$,*

$$l_{\Omega_R(J)}([b, c]) \le l_J([b, c]) + L\left(\frac{|J|}{R}\right). \tag{13}$$

Moreover, the function L satisfies

$$L(r) = \mathcal{O}(r) \quad \text{as } r \to 0^+. \tag{14}$$

Proof of Theorem 6.1. Define $L_0(q) = \sup_{0<r\le q} L(q)/q$, which is finite by the bound (14) and increasing by definition. Let $R = R_f$ and let $f : W \to \Omega_R(J)$ be the maximal biholomorphic extension of f with co-domain an R-slit neighbourhood of J. Define $l_0 = L_0(|J|/R)/R$, then for $I \subseteq J$ we have $L(|I|/R) \le l_0 \cdot |I|$. Let (a, b, c, d) be an admissible quadruple with $I' =]a, d[\subseteq J'$. Define $I =]f(a), f(d)[$ and note that $\Omega_R(I) \subseteq \Omega_R(J)$. Define $W_{I'}$ to be the preimage of $\Omega_R(I)$ inside W. Then we have

$$\begin{aligned}
\log(D(a, b, c, d; f)) &= \mathrm{d}_{I'}(b, c) - \mathrm{d}_I(f(b), f(c)) \\
&\le \mathrm{d}_{W_{I'}}(b, c) - \mathrm{d}_I(f(b), f(c)) \\
&= \mathrm{d}_{\Omega_R(I)}(f(b), f(c)) - \mathrm{d}_I(f(b), f(c)) \\
&\le L(|I|/R) \le l_0|I| = l_0 \cdot |f(d) - f(a)|.
\end{aligned}$$ □

For a hyperbolic domain $U \subset \mathbb{C}$ write the infinitesimal hyperbolic metric as $\lambda_U(z)|dz|$. Moreover, let $\mathrm{d}_U(\cdot, \cdot)$ and $l(\cdot)$ denote the hyperbolic distance and arclength, respectively.

Proof of Lemma 6.2. Fix J and R as above and note that $\lambda_{\Omega_R(J)}(z) > \lambda_J(z)$ for any $z \in \Omega_R(J)$. Thus

$$\begin{aligned}
l_{\Omega_J}([b, c]) - l_J([b, c]) &= \int_b^c (\lambda_{\Omega_J}(x) - \lambda_J(x))\, dx \\
&\le \int_a^d (\lambda_{\Omega_J}(x) - \lambda_J(x))\, dx = \int_J (\lambda_{\Omega_J}(x) - \lambda_J(x))\, dx.
\end{aligned}$$

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We need to prove that the latter integral is finite and depends only on the ratio $|J|/R$. Let the open interval J' and $R' > 0$ satisfy $|J'|/R' = |J|/R$ or equivalently $|J'|/|J| = R'/R = \alpha > 0$. Then there exists $\beta \in \mathbb{R}$ such that the affine map $A(z) = \alpha z + \beta$ maps J onto J', $\mathbb{C}_J$ biholomorphically onto $\mathbb{C}_{J'}$ and $\Omega_R(J)$ biholomorphically onto $\Omega_{J'}(R')$. In particular, A acts as a hyperbolic isometry between both pairs of domains. Thus

$$\begin{aligned}\int_{J'} (\lambda_{\Omega_{J'}}(x) - \lambda_{J'}(x))\,dx &= \int_J (\lambda_{\Omega_{J'}}(A(x)) - \lambda_{J'}(A(x))) \cdot A'(x)\,dx \\ &= \int_J (\lambda_{\Omega_J}(x) - \lambda_J(x))\,dx, \end{aligned} \tag{15}$$

which proves that (15) depends on $|J|/R$ only. Denote by $L(|J|/R)$ the common value in (15). By the Schwartz lemma L is increasing given that the integrals defining L are bounded. Thus it suffices to consider the case $J =]-1, 1[$ with $|J| = 2$, which we shall do hereafter. Recall that, if $V \subseteq W$ are hyperbolic domains, then

$$\text{for all } z \in V, \quad 1 \le \lambda_V(z)/\lambda_W(z) \le \coth(\tfrac{1}{2}\,\mathrm{d}_W(z, \partial V)). \tag{16}$$

Suppose $\gamma : [0, l] \to V$ is an arc which is parametrized by hyperbolic W-length and which satisfies

$$\text{for all } t \in [0, l], \quad \mathrm{d}_W(\gamma(t), \partial V) \ge t + k \tag{17}$$

for some $k \ge 0$. Then by integration of (16),

$$\begin{aligned} 0 \le l_V(\gamma) - l_W(\gamma) &= \int_0^l (\lambda_V(\gamma(t))/\lambda_W(\gamma(t)) - 1)\,dt \\ &\le \int_0^l (\coth \tfrac{1}{2}(t + k) - 1)\,dt \\ &\le 2 \log \frac{1 - e^{-(k+l)}}{1 - e^{-k}} \le 2 \log \frac{1}{1 - e^{-k}}. \end{aligned} \tag{18}$$

To prove boundedness of $L(|J|/R) = L(2/R)$ it suffices to prove that, for some $0 \le x < 1$, the two geodesic segments $]-1, -x]$ and $[x, 1[$ satisfy (17) for some $k > 0$ (here $\mathbb{C}_J = W$, $\Omega_R(J) = V$ and $l = \infty$). Moreover, by symmetry it suffices to consider the arc $[x, 1[$ only. Similarly, to prove that $L(r) = \mathcal{O}(r)$ for $r = 2/R$ small, we shall prove that there exists $c > 0$ such that for R big enough $[0, 1[$ satisfies (17) with $k \ge \log(R/c)$. From this (14) follows.

To produce these hyperbolic bounds on distances, it is convenient to change coordinates using $M(z) = (z^2 - 1)/(z^2 + 1)$. This degree-two rational map restricts to an isomorphism $M : \mathbb{H}_+ \to \mathbb{C}_J$, and it maps $\mathbb{R}_+$ increasingly onto J. For $R > 0$ let $\Delta_\pm(R)$ denote the two connected components of the preimage of $\overline{\mathbb{C}} \setminus \overline{\Omega_R(J)}$ centred at $\pm i$. Let $\rho = \rho(R) > 0$ denote the smallest radius such that $\Delta_\pm(R) \subseteq \overline{\mathbb{D}(\pm i, \rho(R))}$. Set $k = 1 + \log \rho$. Then the map $\gamma(t) = e^{t+k} : [0, \infty[\to [e^k, \infty[$ is a parametrization of the latter interval by hyperbolic length in $\mathbb{H}_+$ and it satisfies (17). This proves boundedness of the integral in (15) and hence the existence of the function L. To prove the asymptotic behaviour (14) of L we note that

$$M(z \pm i) = \frac{\pm i}{z} + \frac{1}{2} + \mathcal{O}(|z|) \quad \text{for small } |z|.$$

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It follows that $\rho(R) = \mathcal{O}(1/R)$. As above $\gamma(t) = e^t : [0, \infty] \to [1, \infty]$ is a parametrization of the latter interval by hyperbolic length in $\mathbb{H}_+$ and it satisfies (17) with $k(R) = -\log \rho(R)$. That is, there exists $R_0 > 0$ and $c > 0$ such that, for $R \geq R_0$, $k(R) \geq \log(R/c)$, thus completing the proof. □

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