Ergod. Th. & Dynam. Sys. (2004), **24**, 329–346
DOI: 10.1017/S0143385703000555
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Printed in the United Kingdom

Hyperbolic times: frequency versus integrability

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(*Received* 30 *January* 2003 *and accepted in revised form* 29 *September* 2003)

Abstract. We consider dynamical systems on compact manifolds that are local diffeomorphisms outside an exceptional set (a compact submanifold). We are interested in analyzing the relation between the integrability (with respect to Lebesgue measure) of the first hyperbolic time map and the existence of positive frequency of hyperbolic times. We show that a (strong) integrability of the first hyperbolic time map implies a positive frequency of hyperbolic times. We also present an example of a map with a positive frequency of hyperbolic times at Lebesgue almost every point, but whose first hyperbolic time map is not integrable with respect to the Lebesgue measure.

1. *Introduction*

In the last few decades, many dynamicists have dedicated their attention to the understanding of the dynamical features of systems exhibiting some non-uniformly hyperbolic behavior. In this direction we mention [**7, 9, 12**] for quadratic maps, [**8, 10, 16, 18, 20**] for Hénon-like diffeomorphisms and [**1, 5, 19**] for a generalized higher dimensional version of the quadratic and Hénon-like maps. The dynamics of all these systems are characterized by the existence of regions of the phase space where the system displays some hyperbolicity, together with *critical regions* where some strong non-hyperbolic behavior appears. The strategy for dealing with the loss of hyperbolicity on the quadratic and Hénon-like maps is based on the existence of well-defined *recovering periods* during which the non-hyperbolic effect of the critical region is compensated for.

It is no longer possible to use the recovering period argument in the class of endomorphisms introduced in [**19**] due to the fact that positive iterates of the critical region have unavoidable intersections with this region. Instead, the mechanism that enables the non-uniformly expanding behavior to be obtained in that case is of a statistical type and comes from a delicate analysis of the derivative of the map along full orbits.

A new and powerful tool in this setting has been introduced in [**1**] through the notion of *hyperbolic times*. These have become a very useful ingredient in the study of non-hyperbolic dynamical systems, playing an important role in the proof of several results on

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the existence of absolutely continuous invariant measures and their statistical properties; see [**1–6**]. Ideas of hyperbolic times were implicitly contained in Pesin's theory and in the work of Pliss [**17**] and Mañé [**14**]. Also, hyperbolic times have recently been used to study stochastic flows [**11**].

The applications of hyperbolic times have been twofold: on the one hand, some of these works deal with the integrability with respect to the Lebesgue measure of the first hyperbolic time map, while on the other hand, the usefulness of hyperbolic times appears through their positive frequency along typical orbits.

Our aim in this work is to clarify the relations among the integrability of the first hyperbolic time map, the frequency of hyperbolic times and the existence of absolutely continuous invariant measures.

Statement of results. We start by introducing the most relevant concepts and definitions. Let $f : M \to M$ be a continuous map defined on a compact Riemannian manifold M with the induced distance that we denote by dist, and fix a normalized Riemannian volume form m on M that we call the *Lebesgue measure*.

Throughout this work, we assume that f is a local diffeomorphism on all of M except for an exceptional set $\mathcal{S} \subset M$, where $\mathcal{S}$ is a compact submanifold of M with $\dim(\mathcal{S}) < \dim(M)$ (thus $m(\mathcal{S}) = 0$) satisfying some non-degeneracy conditions. Examples of systems satisfying Definition 1 include one-dimensional quadratic maps and Viana maps [**19**].

Definition 1. We say that $\mathcal{S} \subset M$ is a *non-degenerate exceptional set* for f if the following conditions hold. The first essentially states that f *behaves like a power of the distance* to $\mathcal{S}$: there are constants $B > 1$ and $\beta > 0$ such that for every $x \in M \setminus \mathcal{S}$,

$$\frac{1}{B}\operatorname{dist}(x,\mathcal{S})^{\beta} \le \frac{\|Df(x)v\|}{\|v\|} \le B\operatorname{dist}(x,\mathcal{S})^{-\beta} \quad \text{for all } v \in T_xM. \tag{s$_1$}$$

Moreover, we assume that the functions $\log|\det Df(x)|$ and $\log\|Df(x)^{-1}\|$ are *locally Lipschitz* at points $x \in M \setminus \mathcal{S}$ with Lipschitz constant depending on $\operatorname{dist}(x,\mathcal{S})$: for every $x, y \in M \setminus \mathcal{S}$ with $\operatorname{dist}(x,y) < \operatorname{dist}(x,\mathcal{S})/2$ we have

$$\big|\log\|Df(x)^{-1}\| - \log\|Df(y)^{-1}\|\big| \le \frac{B}{\operatorname{dist}(x,\mathcal{S})^{\beta}}\operatorname{dist}(x,y), \tag{s$_2$}$$

$$\big|\log|\det Df(x)| - \log|\det Df(y)|\big| \le \frac{B}{\operatorname{dist}(x,\mathcal{S})^{\beta}}\operatorname{dist}(x,y). \tag{s$_3$}$$

The set $\mathcal{S}$ may be taken as some set of critical points of f or a set where f fails to be differentiable. The case where $\mathcal{S}$ is equal to the empty set may also be considered. For the following definition it will be useful to introduce $\operatorname{dist}_\delta(x,\mathcal{S})$, the δ-*truncated* distance from x to $\mathcal{S}$, defined as $\operatorname{dist}_\delta(x,\mathcal{S}) = \operatorname{dist}(x,\mathcal{S})$ if $\operatorname{dist}(x,\mathcal{S}) \le \delta$, and $\operatorname{dist}_\delta(x,\mathcal{S}) = 1$ otherwise.

Definition 2. Let $\beta > 0$ be as in Definition 1 and fix $b > 0$ such that $b < \min\{1/2, 1/(4\beta)\}$. Given $0 < \sigma < 1$ and $\delta > 0$, we say that n is a (σ,δ)-*hyperbolic*

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time† for a point $x \in M$ if for all $1 \le k \le n$,

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| \le \sigma^k \quad \text{and} \quad \operatorname{dist}_\delta(f^{n-k}(x), \mathcal{S}) \ge \sigma^{bk}. \tag{1}$$

We say that the *frequency of* (σ, δ)*-hyperbolic times* for $x \in M$ is greater than $\theta > 0$ if, for large n, there are $n_1 < n_2 < \cdots < n_\ell \le n$ which are (σ, δ)-hyperbolic times for x and $\ell \ge \theta n$.

We point out that condition (1) implies the dynamically meaningful property

$$\|(Df^k(f^{n-k}(x)))^{-1}\| \le \sigma^k, \quad \text{for } 1 \le k \le n,$$

which states that at any intermediate moment of time between zero and n we get exponential stretching by iteration under f. The work of Viana [**19**] provides interesting higher dimensional examples of maps with many hyperbolic times for most points. The significance of hyperbolic times may be attested by the following result, the proof of which is essentially contained in [**3**].

THEOREM A. *Let $f: M \to M$ be a C^2 local diffeomorphism outside a non-degenerate exceptional set $\mathcal{S} \subset M$. If there are $0 < \sigma < 1$, $\delta > 0$ and $H \subset M$ with $m(H) > 0$ such that the frequency of (σ, δ)-hyperbolic times is bigger than $\theta > 0$ for every $x \in H$, then f has some absolutely continuous invariant probability measure.*

The existence of (σ, δ)-hyperbolic times for Lebesgue almost all points in M allows us to introduce a map $h: M \to \mathbb{Z}^+$ defined Lebesgue almost everywhere and assigning to each $x \in M$ its first (σ, δ)-hyperbolic time. The integrability properties of this first hyperbolic time map play an important role in the study of some statistical properties of several classes of dynamical systems, such as stochastic stability and the decay of correlations; see [**2, 4, 5, 9**]. The same conclusion of Theorem A can be obtained under the assumption of integrability with respect to Lebesgue measure of the first hyperbolic time map.

THEOREM B. *Let $f: M \to M$ be a C^2 local diffeomorphism outside a non-degenerate exceptional set $\mathcal{S} \subset M$. If for some $0 < \sigma < 1$ and $\delta > 0$, the first (σ, δ)-hyperbolic time map $h: M \to \mathbb{Z}^+$ is Lebesgue integrable, then f has an absolutely continuous invariant probability measure μ.*

Having in mind Theorems A and B, we are naturally interested in understanding the relation between the existence of a positive Lebesgue measure subset of points in M with a positive frequency of (σ, δ)-hyperbolic times and the integrability with respect to the Lebesgue measure of the first (σ, δ)-hyperbolic time map.

THEOREM C. *Let $f: M \to M$ be a C^2 local diffeomorphism outside a non-degenerate critical set $\mathcal{S} \subset M$. If for $0 < \sigma < 1$ and $\delta > 0$ the first (σ, δ)-hyperbolic time map $h: M \to \mathbb{Z}^+$ belongs to $L^p(m)$ for some $p > 4$, then there are $\hat{\sigma} > 0$ and $\theta > 0$ such that Lebesgue almost every $x \in M$ has a frequency of $(\hat{\sigma}, \delta)$-hyperbolic times bigger than θ.*

† In the case $\mathcal{S} = \emptyset$, the definition of (σ, δ)-hyperbolic time reduces to the first condition in (1) and we simply call it σ-hyperbolic time.

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We do not know whether the need for stronger integrability in the above theorem is due to the methods we have used to prove it or whether some kind of stronger integrability is really necessary. It remains an interesting open question to know the smallest value of $p \geq 1$ for which the first condition in Theorem C still implies the desired conclusion. As a by-product of the proof of Theorem C we will see that the answer to this question is optimal when $\mathcal{S} = \emptyset$.

THEOREM D. *Let $f : M \to M$ be a C^2 local diffeomorphism. If for some $\sigma \in (0, 1)$ the first σ-hyperbolic time map is Lebesgue integrable, then there are $\hat{\sigma} > 0$ and $\theta > 0$ such that Lebesgue almost every $x \in M$ has a frequency of $\hat{\sigma}$-hyperbolic times bigger than θ.*

In the opposite direction, we could ask whether a positive frequency of hyperbolic times is enough for assuring the integrability of the first hyperbolic time map. There is no hope of such a result. In §5, we present an example of a map of the circle (with non-empty exceptional set) having a positive frequency of hyperbolic times for Lebesgue almost every point, but whose first hyperbolic time map is not integrable with respect to the Lebesgue measure.

Another interesting open question is whether a C^2 local diffeomorphism (with no exceptional set) on a compact manifold, admitting a positive frequency of hyperbolic times for Lebesgue almost every point, necessarily has a first hyperbolic time map that is Lebesgue integrable.

Hyperbolic times appear naturally when f is assumed to be *non-uniformly expanding* in some set $H \subset M$: there is some $c > 0$ such that for every $x \in H$, we have

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| < -c; \tag{2}$$

and points in H satisfy some *slow recurrence to the exceptional set*: given any $\varepsilon > 0$ there exists $\delta > 0$ such that for every $x \in H$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} -\log \operatorname{dist}_\delta(f^j(x), \mathcal{S}) \leq \varepsilon. \tag{3}$$

The following result has been proved in [**3**] (see Theorem C and Lemma 5.4 therein) and is used in the proof of our results.

THEOREM E. *Let $f : M \to M$ be a C^2 local diffeomorphism outside a non-degenerate exceptional set $\mathcal{S} \subset M$. If there is some set $H \subset M$ with $m(H) > 0$ such that (2) and (3) hold for all $x \in H$, then there are $0 < \sigma < 1$, $\delta > 0$ and $\theta > 0$ such that the frequency of (σ, δ)-hyperbolic times for the points in H is bigger than θ.*

This work is organized as follows. In §2 we establish the basic properties of hyperbolic times and sketch the proof of Theorem A, using some of the results in [**3**]. In §3 we prove Theorem B and in §4 we prove Theorems C and D. In §5 we present an example of a map of the circle which is a C^2 local diffeomorphism everywhere but in an exceptional set with two points, which has a positive frequency of hyperbolic times and whose first hyperbolic time map is non-integrable with respect to the Lebesgue measure.

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2. *Positive frequency of hyperbolic times*

One of the main features of hyperbolic times is that the corresponding iterates locally behave as those of an expanding map, namely with uniform expansion and uniformly bounded distortion. This is precisely stated in the following result.

LEMMA 2.1. *There are $\delta_1 > 0$ and $C_1 > 0$ such that if n is a (σ, δ)-hyperbolic time for x, then there is a neighborhood V_x of x in M for which:*

(1) *f^n maps V_x diffeomorphically onto the ball of radius δ_1 around $f^n(x)$;*

(2) *for $1 \le k < n$ and $y, z \in V_x$, $\operatorname{dist}(f^{n-k}(y), f^{n-k}(z)) \le \sigma^{k/2} \operatorname{dist}(f^n(y), f^n(z))$;*

(3) *$f^n|V_x$ has distortion bounded by C_1: if $y, z \in V_x$, then*

$$\frac{1}{C_1} \le \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \le C_1.$$

For a proof, see [**3**, Lemma 5.2 and Corollary 5.3].

Let us now give a brief idea on the way we obtain Theorem A from the results in [**3**].

Assume that there are $0 < \sigma < 1$, $0 < \delta$ and $H \subset M$ with $m(H) > 0$ such that for every $x \in H$, the frequency of (σ, δ)-hyperbolic times is bigger than $\theta > 0$. Given an integer $n \ge 1$ we define

$$H_n = \{x \in H : n \text{ is a } (\sigma, \delta)\text{-hyperbolic time for } x\}.$$

PROPOSITION 2.2. *Take δ_1 as in Lemma 2.1. Then there exists a constant $\tau > 0$ such that for any n, there exists a finite subset $\widehat{H}_n$ of H_n for which the balls of radius $\delta_1/4$ around the points $x \in f^n(\widehat{H}_n)$ are pairwise disjoint and their union Δ_n satisfies*

$$f_*^n m(\Delta_n \cap H) \ge f_*^n m(\Delta_n \cap f^n(H_n)) \ge \tau m(H_n).$$

For a proof, see [**3**, Proposition 3.3].

From Proposition 2.2, we may find for each $j \ge 1$ a finite set of points $x_1^j, \dots, x_N^j$ (in principle, with N depending on j) admitting j as a (σ, δ)-hyperbolic time, such that:

(1) $V_{x_1^j}, \dots, V_{x_N^j}$ are pairwise disjoint;

(2) the Lebesgue measure of $W_j = V_{x_1^j} \cup \cdots \cup V_{x_N^j}$ is larger than the Lebesgue measure of H_j, up to a uniform multiplicative constant $\tau > 0$.

We let $(\mu_n)_n$ be the sequence of the averages of the positive iterates of Lebesgue measure on M,

$$\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j m,$$

and let ν_n be the part of μ_n carried on disks of radius δ_1 around points $f^j(x_k^j)$ such that $1 \le j \le n$ is a (σ, δ)-hyperbolic time for x,

$$\nu_n = \frac{1}{n} \sum_{j=0}^{n-1} f_*^j (m \mid W_j).$$

By Proposition 2.2 we have

$$\nu_n(H) \ge \frac{\tau}{n} \sum_{i=0}^{n-1} m(H_i).$$

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So, it suffices to prove that this last expression is larger than some positive constant, for n large. Let ξ_n be the measure in $\{1, \dots, n\}$ defined by $\xi_n(B) = \#B/n$, for each subset B. Then, using Fubini's theorem,

$$\frac{1}{n}\sum_{i=0}^{n-1} m(H_n) = \int \left(\int \chi(x,i)\, dm(x) \right) d\xi_n(i)$$
$$= \int \left(\int \chi(x,i)\, d\xi_n(i) \right) dm(x),$$

where $\chi(x,i) = 1$ if $x \in H_i$ and $\chi(x,i) = 0$ otherwise. Now, since we are assuming a positive frequency of hyperbolic times for points in H, this means that the integral with respect to $d\xi_n$ is larger than $\theta > 0$ for large n. So, the expression on the right-hand side is bounded from below by $\theta m(M)$. This implies that each ν_n has total mass uniformly bounded away from zero. Moreover, as a consequence of the bounded distortion given by Lemma 2.1, every $f_*^j(m \mid W_j)$ is absolutely continuous with respect to Lebesgue measure, with density uniformly bounded from above and so the same is true for every ν_n.

Since we are working with a continuous map in the compact space M, we know that sequences of probability measures in M have weak* accumulation points. Take $n_k \to \infty$ such that both μ_{n_k} and ν_{n_k} converge in the weak* sense to measures μ and ν, respectively. Then μ is an invariant probability measure, $\mu = \nu + \eta$ for some measure η, ν is absolutely continuous with respect to Lebesgue measure and $\nu(H) > 0$. Now, if $\eta = \eta_{\mathrm{ac}} + \eta_{\mathrm{s}}$ denotes the Lebesgue decomposition of η (as the sum of an absolutely continuous and a completely singular measure, with respect to Lebesgue measure), then $\mu_{\mathrm{ac}} = \nu + \eta_{\mathrm{ac}}$ gives the absolutely continuous component in the corresponding decomposition of μ. By the uniqueness of the Lebesgue decomposition and the fact that the push-forward under f preserves the class of absolutely continuous measures, we may conclude that μ_{ac} is an invariant measure. Clearly, $\mu_{\mathrm{ac}}(H) \geq \nu(H) > 0$. Normalizing μ_{ac} we obtain an absolutely continuous f-invariant probability measure.

The next lemma is useful in §5.

LEMMA 2.3. *Take δ_1 as in Lemma 2.1. If $H \subset M$ is a positively invariant set with $m(H) > 0$ for which* (2) *and* (3) *hold, then there exists some disk Δ with radius $\delta_1/4$ such that $m(\Delta \setminus H) = 0$.*

For a proof, see [**3**, Lemma 5.6].

Using this lemma, it is shown in [**3**, §5] that f has a finite number of absolutely continuous invariant probability measures which are ergodic.

3. *Integrability of first hyperbolic time map*

Here we prove Theorem B. As in §2 the strategy is to consider $(\mu_n)_n$ the sequence of averages of forward iterates of Lebesgue measure on M:

$$\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} f_*^j m.$$

Since we are dealing with a continuous map of a compact manifold, we know that the sequence $(\mu_n)_n$ has accumulation points—which belong to the space of f-invariant

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probability measures. Now the idea is to show that such accumulation points are absolutely continuous with respect to the Lebesgue measure.

PROPOSITION 3.1. *There is a constant $C_2 > 0$ (depending on δ_1 and C_1 from Lemma 2.1) such that for every $n \geq 0$,*

$$\frac{d}{dm} f_*^n(m \mid H_n) \leq C_2.$$

Proof. Take $\delta_1 > 0$ given by Lemma 2.1. It suffices to show that there is some uniform constant $C > 0$ such that if $A \subset M$ is a Borel set with diameter smaller than $\delta_1/2$, then

$$m(f^{-n}(A) \cap H_n) \leq Cm(A).$$

Let A be a Borel set in M with diameter smaller than $\delta_1/2$ and B an open ball of radius $\delta_1/2$ containing A. Taking the connected components of $f^{-n}(B)$, we may write

$$f^{-n}(B) = \bigcup_{k \geq 1} B_k,$$

where $(B_k)_{k \geq 1}$ is a (possibly finite) family of pairwise disjoint open sets in M. Taking into account only those B_k that intersect H_n, we choose, for each $k \geq 1$, a point $x_k \in H_n \cap B_k$. For each $k \geq 1$, let V_{x_k} be the neighborhood of x_k given by Lemma 2.1. Since B is contained in $B(f^n(x_k), \delta_1)$, the ball of radius δ_1 around $f^n(x_k)$, and f^n is a diffeomorphism from V_{x_k} onto $B(f^n(x_k), \delta_1)$, we must have $B_k \subset V_{n_k}$ (recall that by the choice of B_k we have $f^n(B_k) \subset B$). As a consequence of this and Lemma 2.1 we have that $f^n \mid B_k \colon B_k \to B$ is a diffeomorphism with uniformly bounded distortion for all $n \geq 1$ and $k \geq 1$:

$$\frac{1}{C_1} \leq \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq C_1 \quad \text{for all } y, z \in B_k.$$

This finally gives

$$\begin{aligned} m(f^{-n}(A) \cap H_n) &\leq \sum_{k \geq 1} m(f^{-n}(A \cap B) \cap B_k) \\ &\leq \sum_{k \geq 1} C_1 \frac{m(A \cap B)}{m(B)} m(B_k) \\ &\leq C_2 m(A), \end{aligned}$$

for some constant $C_2 > 0$ only depending on $C_1 > 0$ and on the volume of the ball B of radius $\delta_1/2$. □

Defining, for each $n \geq 1$,

$$H_n^* = \{x \in M \colon \ n \text{ is the } \textit{first } (\sigma, \delta)\text{-hyperbolic time for } x\},$$

we immediately have

$$\int_M h \, dm = \sum_{k=1}^{\infty} km(H_k^*). \tag{4}$$

It is useful to define, for each $n, k \geq 1$,

$$R_{n,k} = \{x \in H_n \colon f^n(x) \in H_k^*\}.$$

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Observe that $R_{n,k}$ is precisely the set of points $x \in M$ for which n is a (σ, δ)-hyperbolic time and $n + k$ is the next (σ, δ)-hyperbolic time for x after n. Defining the measures

$$\nu_n = f_*^n(m \mid H_n) \tag{5}$$

and

$$\eta_n = \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} f_*^{n+j}(m \mid R_{n,k}), \tag{6}$$

we may write

$$\mu_n \leq \frac{1}{n} \sum_{j=0}^{n-1} (\nu_j + \eta_j).$$

It follows from Proposition 3.1 that

$$\frac{d\nu_n}{dm} \leq C_2 \tag{7}$$

for every $n \geq 0$, with C_2 not depending on n. Our goal now is to control the densities of the measures η_n.

PROPOSITION 3.2. *Given $\varepsilon > 0$, there is $C_3(\varepsilon) > 0$ such that for every $n \geq 1$ we may bound η_n by the sum of two non-negative measures, $\eta_n \leq \omega + \rho$, with*

$$\frac{d\omega}{dm} \leq C_3(\varepsilon) \quad \textit{and} \quad \rho(M) < \varepsilon.$$

Proof. Let A be some Borel set in M. For each $n \geq 0$ we have

$$\begin{aligned} \eta_n(A) &= \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} m(f^{-n-j}(A) \cap R_{n,k}) \\ &\leq \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} m(f^{-n}(f^{-j}(A) \cap H_k^*) \cap H_n) \\ &\leq \sum_{k=2}^{\infty} \sum_{j=1}^{k-1} C_2 m(f^{-j}(A) \cap H_k^*) \end{aligned}$$

(in this last inequality we have used the bound (7) above). Now let $\varepsilon > 0$ be some fixed small number. By the integrability of h and since (4) holds, we may choose some integer $\ell = \ell(\varepsilon)$ for which

$$\sum_{j=\ell}^{\infty} k m(H_k^*) < \frac{\varepsilon}{C_2}.$$

We take

$$\omega = C_2 \sum_{k=2}^{\ell-1} \sum_{j=1}^{k-1} f_*^j(m \mid H_k^*)$$

and

$$\rho = C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-1} f_*^j(m \mid H_k^*). \tag{8}$$

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This last measure satisfies

$$\rho(M) = C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-1} m(H_k^*) \leq C_2 \sum_{k=\ell}^{\infty} km(H_k^*) < \varepsilon.$$

On the other hand, we have

$$\omega \leq C_2 \sum_{k=2}^{\ell-1} \sum_{j=1}^{k-1} f_*^j m,$$

and this last measure has density bounded by some constant by the non-degeneracy conditions of f, since we are taking a finite number of push-forwards of Lebesgue measure. □

It follows from this last proposition and (7) that weak* accumulation points of $(\mu_n)_n$ cannot have singular parts, thus being absolutely continuous with respect to the Lebesgue measure. Since such weak* accumulation points are invariant with respect to f, we have proved Theorem B.

Remark 3.3. This argument proves that *every* weak* accumulation point of $(\mu_n)_n$ is absolutely continuous with respect to Lebesgue measure, whenever the first hyperbolic time function is integrable. This is not known if we only assume a positive frequency of hyperbolic times.

4. *Strong integrability implies positive frequency*

Assume that $f : M \to M$ is a C^2 local diffeomorphism outside a non-degenerate exceptional set $\mathcal{S} \subset M$. We start the proof of Theorem C by obtaining a simple useful result.

PROPOSITION 4.1. *If $\mathcal{S}$ is a compact submanifold of M with* $\dim(\mathcal{S}) < \dim(M)$*, then the function* $\log \operatorname{dist}(x, \mathcal{S})$ *belongs to* $L^p(m)$ *for every* $1 \leq p < \infty$.

Proof. We may assume without loss of generality that $\mathcal{S}$ is connected. Let $\dim(\mathcal{S}) = k < n = \dim(M)$. We may cover $\mathcal{S}$ with finitely many images of charts $\psi_i(U_i)$ $(i = 1, \ldots, p)$ such that $U_i \subset \mathbb{R}^n$ is a bounded open set and $\psi_i^{-1}(\mathcal{S}) \subset U_i \cap (\mathbb{R}^k \times 0^{n-k})$. Denoting by λ the usual n-dimensional volume on $\mathbb{R}^n$ and by d the standard Euclidean distance on $\mathbb{R}^n$, there are constants $C, K > 0$ such that for all $i = 1, \ldots, p$

$$\frac{1}{C} \leq \frac{d(\psi_i^{-1})_* m}{d\lambda} \leq C,$$

and for all $w, z \in U_i$

$$\frac{1}{K} d(w, z) \leq \operatorname{dist}(\psi_i(w), \psi_i(z)) \leq K\, d(w, z).$$

Hence, for showing that $\log \operatorname{dist}(x, \mathcal{S})$ is integrable with respect to m, it is enough to show that $\log d(x, U \cap (\mathbb{R}^k \times 0^{n-k}))$ is integrable with respect to λ for any open and bounded neighborhood U of the origin in $\mathbb{R}^n$. We may assume without loss of generality that U

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is sufficiently small in order for $U \subset B_k \times B_{n-k}$, where B_k and B_{n-k} are the unit balls around the origin in $\mathbb{R}^k$ and $\mathbb{R}^{n-k}$, respectively. For $z = (z_1, \dots, z_n) \in \mathbb{R}^n$ we have

$$d(z, \mathbb{R}^k \times 0^{n-k}) = (z_{k+1}^2 + \cdots + z_n^2)^{1/2}.$$

Hence, we have for $1 \le p < \infty$,

$$\int_U |\log d(z, \mathbb{R}^k \times 0^{n-k})|^p \, d\lambda \le \frac{1}{2^p} \int_{B_k} \left(\int_{B_{n-k}} |\log(z_{k+1}^2 + \cdots + z_n^2)|^p \, dz_{k+1} \cdots dz_n \right) dz_1 \cdots dz_k.$$

Now it is enough to show that the inner integral in the last expression is finite. Actually, denoting by S_ρ^{n-k-1} the $(n-k-1)$-sphere with radius ρ around the origin in $\mathbb{R}^{n-k}$, dA its area element and a the total area of S_1^{n-k-1}, we have

$$\int_{B_{n-k}} |\log(z_{k+1}^2 + \cdots + z_n^2)|^p \, dz_{k+1} \cdots dz_n = \int_0^1 \left(\int_{S_\rho^{n-k-1}} |2 \log \rho|^p \, dA \right) d\rho = a \int_0^1 \rho^{n-k-1} |\log \rho|^p \, d\rho.$$

Since this last integral is finite, we have completed the proof of the result. □

Assume now that h is integrable with respect to the Lebesgue measure. By Theorem B we know that there exists an absolutely continuous invariant probability measure μ for f.

COROLLARY 4.2. *If the density $d\mu/dm$ belongs to $L^q(m)$ for some $q > 1$, then* $\log \operatorname{dist}(x, \mathcal{S})$ *is μ-integrable.*

Proof. This is an immediate application of Hölder inequality. Actually, since

$$\int \log \operatorname{dist}(x, \mathcal{S}) \, d\mu = \int \log \operatorname{dist}(x, \mathcal{S}) \frac{d\mu}{dm} \, dm,$$

and we have $d\mu/dm$ in $L^q(m)$ for some $q > 1$ and $\log \operatorname{dist}(x, \mathcal{S})$ in $L^p(m)$ for every p, then taking p equal to the conjugate of q (that is $p^{-1} + q^{-1} = 1$) the Hölder inequality gives that the integral above is finite. □

We are also interested in obtaining the same conclusion of the previous corollary under the hypothesis that $h \in L^p(m)$ for some $p > 4$. Observe that the absolutely continuous f-invariant measure μ may be obtained as a weak* accumulation point of the sequence $(\mu_n)_n$ of averages of push-forwards of Lebesgue measure. As shown in §3, we may write

$$\mu_n \le \frac{1}{n} \sum_{j=0}^{n-1} (\nu_j + \eta_j),$$

where ν_j and η_j are given by (5) and (6).

LEMMA 4.3. *If the first (σ, δ)-hyperbolic time map $h : M \to \mathbb{Z}^+$ belongs to $L^p(m)$ for some $p > 4$, then* $\log \operatorname{dist}(x, \mathcal{S})$ *is μ-integrable.*

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Proof. We take any $\varepsilon > 0$ and use Proposition 3.2 to ensure the existence of two non-negative measures ω and ρ bounding η_n, where ω has density bounded by some constant and ρ has total mass bounded by ε. Recall that ρ was defined in (8) by

$$\rho = C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-1} f_*^j(m \mid H_k^*),$$

where ℓ is some large integer.

Let us now compute the weight ρ gives to some special family of neighborhoods of $\mathcal{S}$. For $i \geq 1$, let $d_i = \sigma^{bi}$ where $0 < \sigma < 1$ comes from the definition of (σ, δ)-hyperbolic time. Define for $i \geq 1$

$$B_i = \{x \in M : \operatorname{dist}(x, \mathcal{S}) \leq d_i\}.$$

If n is a (σ, δ)-hyperbolic time for $x \in M$, then $f^j(x) \in M \setminus B_i$ for all $j \in \{n-i, \ldots, n-1\}$. This implies that

$$\begin{aligned}
\rho(B_i) &= C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-1} m(H_k^* \cap f^{-j}(B_i)) \\
&= C_2 \sum_{k=\ell}^{\infty} \sum_{j=1}^{k-i} m(H_k^* \cap f^{-j}(B_i)) \\
&\leq C_2 \sum_{k=\max\{\ell,i\}}^{\infty} \sum_{j=1}^{k-i} m(H_k^* \cap f^{-j}(B_i)) \\
&\leq C_2 \sum_{k=i}^{\infty} k m(H_k^*),
\end{aligned}$$

for all $i \geq 1$. Now by Propositions 3.1 and 3.2 we know that

$$\mu_n \leq \frac{1}{n} \sum_{j=0}^{n-1} \nu_j + \omega + \rho \leq \nu + \rho$$

where ν is a measure with uniformly bounded density. Hence any weak* accumulation point μ of the sequence $(\mu_n)_n$ is bounded by $\nu + \rho$. Since we are assuming that $\mathcal{S}$ is a submanifold of M, then $\log \operatorname{dist}(x, \mathcal{S})$ is integrable with respect to ν by Proposition 4.1. On the other hand,

$$\int_M -\log \operatorname{dist}_\delta(x, \mathcal{S})\, d\rho \leq \sum_{i=1}^{\infty} -\rho(B_i) \log d_{i+1} \leq -b \log \sigma \sum_{i=1}^{\infty} (i+1) \sum_{k=i}^{\infty} k m(H_k^*).$$

We have $h \in L^p(m)$ by assumption, which is equivalent to $\sum_{k \geq 1} k^p m(H_k^*) < \infty$. This implies that there is some constant $C > 0$ such that $m(H_k^*) \leq C k^{-p}$ for all $k \geq 1$. Thus we have for $i \geq 2$

$$\sum_{k=i}^{\infty} k m(H_k^*) \leq \sum_{k=i}^{\infty} \frac{C}{k^{p-1}} \leq \int_{i-1}^{\infty} \frac{C}{x^{p-1}}\, dx = \frac{C}{(p-2)(i-1)^{p-2}},$$

and so

$$\sum_{i=2}^{\infty} (i+1) \sum_{k=i}^{\infty} k m(H_k^*) \leq \frac{C}{p-2} \sum_{i=2}^{\infty} \frac{i+1}{(i-1)^{p-2}}.$$

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This last quantity is finite whenever $p > 4$. Hence $\log \operatorname{dist}(x, \mathcal{S})$ is integrable with respect to μ for all $p > 4$. □

As a consequence of the last results and the assumption that f behaves like a power of the distance near the exceptional set $\mathcal{S}$, we obtain the following result.

COROLLARY 4.4. *If the first (σ, δ)-hyperbolic time map h belongs to $L^p(m)$ for some $p > 4$, then $\log \|Df(x)^{-1}\|$ is μ-integrable.*

Proof. It is an easy consequence of condition (s_1) in the definition of the non-degenerate exceptional set that for some $\zeta > \beta$ we have

$$|\log \|Df(x)^{-1}\|| \leq \zeta |\log \operatorname{dist}(x, \mathcal{S})|$$

for all x in a small open neighborhood V of $\mathcal{S}$. Hence

$$\int_V |\log \|Df(x)^{-1}\|| \, d\mu \leq \zeta \int_V -\log \operatorname{dist}(x, \mathcal{S}) \, d\mu < \infty,$$

and since $\log \|Df(x)^{-1}\|$ is bounded on the compact set $M \setminus V$, this function is necessarily integrable with respect to μ on M. □

Now we are ready to conclude the proof of Theorems C and D. Assuming that the first (σ, δ)-hyperbolic time map h belongs to $L^p(m)$ for some $p > 4$, it follows from Lemma 4.3 and Corollary 4.4 that both:

(1) $\log \operatorname{dist}(x, \mathcal{S})$ is integrable with respect to μ; and

(2) $\log \|Df(x)^{-1}\|$ is integrable with respect to μ.

Observe that by the definition of (σ, δ)-hyperbolic time, if n is a (σ, δ)-hyperbolic time for x and if k is a (σ, δ)-hyperbolic time for $f^n(x)$, then $n + k$ is a (σ, δ)-hyperbolic time for x. Moreover, since h is well defined Lebesgue almost everywhere and f preserves sets of Lebesgue measure zero, then Lebesgue almost all points must have infinitely many hyperbolic times. Thus we have

$$\liminf_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq \log \sigma < 0 \tag{9}$$

for Lebesgue almost every $x \in M$ and, hence, for μ almost every $x \in M$. The μ-integrability of $\log \|Df(x)^{-1}\|$ and Birkhoff's ergodic theorem then ensure that

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \|Df(f^j(x))^{-1}\| \leq \log \sigma < 0 \tag{10}$$

for μ almost every $x \in M$.

Remark 4.5. If $\mathcal{S}$ is equal to the empty set, then $\log \|Df(x)^{-1}\|$ is immediately integrable with respect to μ because it is a bounded function. Hence (10) is enough for obtaining Theorem D by applying Theorem E.

The strong integrability condition on h ensures that there exists an absolutely continuous invariant measure μ and that $\log \operatorname{dist}(x, \mathcal{S})$ is μ-integrable after Lemma 4.4. Then we are in the setting of the following result.

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LEMMA 4.6. *If μ is an f-invariant probability measure and* $\log \operatorname{dist}(x, \mathcal{S})$ *is μ-integrable, then for every $0 < \eta < 1$ there is a set R with $\mu(R) > 1 - \eta$ such that points in R have slow recurrence to $\mathcal{S}$.*

Proof. We start by fixing a small $\eta > 0$ and choosing $\alpha > 0$ such that

$$\prod_{n\geq 1}(1 - e^{-\alpha n}) \geq 1 - \eta.$$

The integrability of $\log \operatorname{dist}(x, \mathcal{S})$ with respect to μ and the definition of the δ-truncated distance $\operatorname{dist}_\delta$ ensure that for each $k \in \mathbb{N}$ we may find $\delta_k > 0$ for which

$$\int_M -\log \operatorname{dist}_{\delta_k}(x, \mathcal{S})\, d\mu \leq \frac{1}{k2^{k+1}}.$$

We define for each $k \in \mathbb{N}$

$$\varphi_k(x) = \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_{\delta_k}(f^j(x), \mathcal{S}).$$

This φ_k is well-defined μ almost everywhere in M by Birkhoff's ergodic theorem. Moreover

$$\int_M \varphi_k\, d\mu = \int_M -\log \operatorname{dist}_{\delta_k}(x, \mathcal{S})\, d\mu \leq \frac{1}{k2^{k+1}}.$$

Let

$$E_k = \left\{x \in M : \varphi_k(x) > \frac{1}{k}\right\}.$$

Since $\varphi_k \geq 0$ we have

$$\frac{\mu(E_k)}{k} \leq \int_{E_k} \varphi_k\, d\mu \leq \int_M \varphi_k\, d\mu \leq \frac{1}{k2^{k+1}},$$

which implies that $\mu(E_k) \leq 2^{-(k+1)}$. Hence we may find a big enough $k_1 \in \mathbb{N}$ such that $\mu(M \setminus E_{k_1}) \geq 1 - e^{-\alpha} > 0$. This is the first step in the following construction by induction on n. Assuming that we have chosen $k_1 < k_2 < \cdots < k_n$ satisfying

$$\mu(M \setminus (E_{k_1} \cup \cdots \cup E_{k_j})) \geq (1 - e^{-\alpha j})\mu(M \setminus (E_{k_1} \cup \cdots \cup E_{k_{j-1}})) > 0$$

for all $j = 2, \ldots, n$, then we may find a big enough $k_{n+1} > k_n$ such that

$$\mu(M \setminus (E_{k_1} \cup \cdots \cup E_{k_n} \cup E_{k_{n+1}})) \geq (1 - e^{-\alpha(n+1)})\mu(M \setminus (E_{k_1} \cup \cdots \cup E_{k_n})) > 0.$$

Now, taking $R = M \setminus \bigcup_{k\geq 1} E_{k_n}$ we have

$$\mu(R) \geq \prod_{n\geq 1}(1 - e^{-\alpha n}) \geq 1 - \eta.$$

Let us now show that points in R have slow approximation to $\mathcal{S}$. Given $\varepsilon > 0$ we choose $n \in \mathbb{N}$ for which $\varepsilon > 1/k_n$. If $x \in R$ then, in particular, $x \notin E_{k_n}$ and this implies

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} -\log \operatorname{dist}_{\delta_{k_n}}(f^j(x), \mathcal{S}) = \varphi_{k_n}(x) \leq \frac{1}{k_n} < \varepsilon.$$

This concludes the proof of the result. □

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Since μ is absolutely continuous with respect to m, we deduce from (10) and Lemma 4.6 that there is a set with positive Lebesgue measure on which f is non-uniformly expanding and whose points have slow recurrence to $\mathcal{S}$. Actually these conditions must hold for Lebesgue almost every $x \in M$. Indeed, let H be the set of points for which both (2) and (3) hold, and take $B = M \setminus H$. Observe that B is invariant by f and $h \in L^p(m \mid B)$ with $p > 4$. If $m(B) > 0$, then by the arguments above we would prove the existence of some $A \subset B$ with $m(A) > 0$ where f is non-uniformly expanding and points have slow recurrence to $\mathcal{S}$. This would naturally give a contradiction.

Thus we have proved that f is non-uniformly expanding and points have slow recurrence to $\mathcal{S}$ Lebesgue almost everywhere. Applying Theorem E we prove Theorem C.

Remark 4.7. The hypothesis of h belonging to $L^p(m)$ for some $p > 4$ can be replaced by $d\mu/dm \in L^q(m)$ for some $q > 1$. In fact, the integrability of h has only been used to prove that $\log \operatorname{dist}(x, \mathcal{S})$ is integrable with respect to μ—which implies that $\log \|Df(x)^{-1}\|$ is also integrable with respect to μ by Remark 4.4. As stated in Corollary 4.2 this is a consequence of $d\mu/dm \in L^q(m)$ for some $q > 1$.

5. *An example with non-integrable first hyperbolic time map*

In this section we exhibit a map of the circle, differentiable everywhere except at a single point, having a positive frequency of hyperbolic times at Lebesgue almost every point, but whose first hyperbolic time map is not integrable with respect to the Lebesgue measure. This example is an adaptation of the 'intermittent' Manneville map into a local homeomorphism of the circle; see [**15**]. Consider $I = [-1, 1]$ and the map $\hat{f} : I \to I$ (see Figure 1) given by

$$x \mapsto \begin{cases} 2\sqrt{x} - 1 & \text{if } x \geq 0, \\ 1 - 2\sqrt{|x|} & \text{otherwise.} \end{cases}$$

This map induces a continuous local homeomorphism $f : S^1 \to S^1$ through the identification $S^1 = I/\sim$, where $-1 \sim 1$, not differentiable at the point 0.

Topological mixing. We show that given any open interval $J \subset S^1$ there is some $N \in \mathbb{N}$ such that $f^N(J) = S^1$. Note that f has two inverse branches $g_1 : (-1, 1) \to (0, 1)$ and $g_2 : (-1, 1) \to (-1, 0)$, given by

$$g_1(x) = \left(\frac{1+x}{2}\right)^2 \quad \text{and} \quad g_2(x) = -\left(\frac{1-x}{2}\right)^2.$$

Let $X = \{g_1^n(0), g_2^n(0) : n \geq 0\}$ be a set of points in the pre-orbit of $1 \in S^1$ and let $\emptyset \neq J \subseteq S^1$ be an open interval. Observe that if $X \cap J \neq \emptyset$, then there is $n \geq 1$ such that $1 \in f^n(J)$, thus the interval $f^n(J)$ would contain a neighborhood of 1 in S^1. We easily see that this implies $f^{n+k}(J) = S^1$ for some $k \in \mathbb{N}$. Hence, to prove topological mixing for f it is enough to show that given any open interval $J \subset S^1$ there is $j \in \mathbb{N}$ such that $f^j(J) \cap X \neq \emptyset$.

Let us take an interval $J \subset S^1$ such that $J \cap X = \emptyset$. Hence $0 \notin J$. Assume for definiteness that $J \subset (-1, 0)$. Thus there is an $n \in \mathbb{N}$ such that $f^n|_J$ is a diffeomorphism

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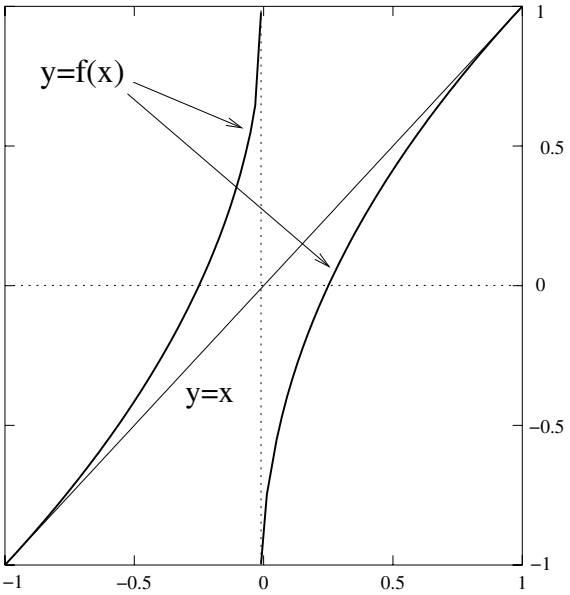


FIGURE 1. A map with a non-integrable first hyperbolic time map.

and $f^n(J) \subset (0,1)$. It is clear that there is $\sigma > 1$ independent of n such that $m(f^{n+1}(J)) \geq \sigma m(J)$. Now let $J_1 = f^{n+1}(J) \subseteq (0,1)$. If $J_1 \cap X \neq \emptyset$, then we are done. Otherwise, by the symmetry of f, we repeat the argument obtaining an iterate $J_2 \subset (-1,0)$ of J with $m(J_2) \geq \sigma^2 m(J)$. Since $(\sigma^k)_{k\geq 1}$ is unbounded, after a finite number of iterates the image of J will eventually hit X.

Invariance of Lebesgue measure. Now we show that Lebesgue measure is invariant by f. Note that $f'(x) = |x|^{-1/2}$ for $x \in S^1 \setminus \{0\}$. It is straightforward to check that for all $x \in S^1 \setminus \{0\}$,

$$\frac{1}{f'(g_1(x))} + \frac{1}{f'(g_2(x))} = \sqrt{\left(\frac{1+x}{2}\right)^2} + \sqrt{\left(\frac{1-x}{2}\right)^2} = 1,$$

where $g_1 : (-1,1) \to (0,1)$ and $g_2 : (-1,1) \to (-1,0)$ are the inverse branches of f. Hence, the transfer operator

$$T_f : L^1(m) \to L^1(m)$$

given by

$$T_f(\varphi)(x) = \sum_{f(z)=x} \frac{\varphi(z)}{|f'(z)|} = \frac{\varphi(g_1(x))}{|f'(g_1(x))|} + \frac{\varphi(g_2(x))}{|f'(g_2(x))|}$$

fixes the constant densities. This means that Lebesgue measure is f-invariant.

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Positive frequency of hyperbolic times. Observe that $\log|(f'(x))^{-1}| = \log\sqrt{|x|}$ is Lebesgue integrable on S^1 and that

$$\int_{S^1} \log|(f'(x))^{-1}|\,dm = \int_{-1}^{1} \tfrac{1}{2}\log|x|(\tfrac{1}{2}\,dx) = \tfrac{1}{2}\int_0^1 \log x\,dx = -\tfrac{1}{2}$$

(recall that we are taking the Lebesgue measure m normalized on S^1). Thus, the invariance of Lebesgue measure and Birkhoff's ergodic theorem ensure that

$$G = \left\{x \in S^1 : \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log|(f'(f^j(x)))^{-1}| \le -\frac{1}{10}\right\}$$

has positive Lebesgue measure. On the other hand, taking $\mathcal{S} = \{0, \pm 1\}$ (the set of points where f fails to be a C^2 local diffeomorphism), we have

$$\int_{S^1} -\log \operatorname{dist}(x, \mathcal{S})\,dm(x) = 2\int_0^{1/2} -\log x\,dx, \tag{11}$$

and so $\log \operatorname{dist}(x, \mathcal{S})$ is integrable with respect to m on S^1. Since m is f-invariant, Lemma 4.6 ensures that there exists a set of points R with slow recurrence to $\mathcal{S}$ whose complement has arbitrarily small Lebesgue measure. We obtain a positively invariant subset $H = G \cap R$ with $m(H) > 0$ whose points satisfy conditions (2) and (3). Then, by Lemma 2.3 we have that there is an interval $J \subset H$, up to a null Lebesgue measure subset. Due to the topological mixing property and the regularity of f (preserves null Lebesgue measure sets) this implies that conditions (2) and (3) hold Lebesgue almost everywhere on S^1. Hence, using Theorem E we conclude that f has a positive frequency of hyperbolic times on Lebesgue almost every point.

Non-integrability of the first hyperbolic time map. We now show that given $0 < \sigma < 1$ and $\delta > 0$ the map $h : S^1 \to \mathbb{Z}_+$, assigning to each $x \in S^1$ the first (σ, δ)-hyperbolic time of x, cannot be Lebesgue integrable in S^1. We observe that for n being a (σ, δ)-hyperbolic time for x it must satisfy

$$|f'(f^{n-1}(x))| \ge \sigma^{-1} > 1.$$

Hence the first (σ, δ)-hyperbolic time for a given $x \in S^1 \setminus \mathcal{S}$ is at least the number of iterates x needs to hit a fixed neighborhood of 0.

If we consider the inverse branch g_1 of f and iterate a point $x_1 \in (0, 1)$ under g_1, we obtain a sequence $(x_n)_{n\ge1}$ in $(0, 1)$ satisfying

$$x_{n+1} = \frac{(1 + x_n)^2}{4}, \quad n \ge 1. \tag{12}$$

According to the observation above, we must have

$$\int_{S^1} h\,dm \gtrsim \sum_{n\ge1} n(x_{n+1} - x_n).$$

The non-integrability of h is then a consequence of the following result.

LEMMA 5.1. $\sum_{n\ge1} n(x_{n+1} - x_n) = +\infty$.

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Proof. We first prove (by induction) that

$$0 \le x_n \le 1 - \frac{1}{2n} \quad \text{for every } n \ge 1. \tag{13}$$

This obviously holds for $n = 1$ since we have chosen $x_1 \in (0, \frac{1}{2})$. Assuming that (13) holds for $n \ge 1$, we then have

$$\begin{aligned} 0 \le x_{n+1} &= \frac{(1+x_n)^2}{4} \\ &\le \frac{(2-1/(2n))^2}{4} \\ &= 1 - \frac{1}{2n} + \frac{1}{16n^2} \\ &= 1 - \frac{1}{2n+2}\left(\frac{n+1}{n} - \frac{n+1}{8n^2}\right). \end{aligned}$$

It is enough to observe that

$$\frac{n+1}{n} - \frac{n+1}{8n^2} = \frac{8n^2+7n-1}{8n^2} > 1, \quad \text{for all } n \ge 1.$$

Using the recurrence relation (12), a simple calculation now shows that

$$x_{n+1} - x_n = \frac{(1-x_n)^2}{4},$$

which together with (13) leads to

$$x_{n+1} - x_n \ge \frac{1}{16n^2}.$$

This is enough for concluding the proof of the result. □

Acknowledgements. We are indebted to Henk Bruin, who suggested to us the example of §5 in conversations during the workshop *Concepts and Techniques in Smooth Ergodic Theory* held at the Imperial College, London, July 2001. We also thank Xavier Bressaud and Sandro Vaienti for helpful conversations at the Institut de Mathématiques de Luminy, Marseille, June 2002.

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