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SET MAPPINGS WITH FREE SETS WHICH ARE ARITHMETIC PROGRESSIONS

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Abstract. If $3 \leqslant n < \omega$ and V is a vector space over $\mathbb{Q}$ with $|V| \leqslant \aleph_{n-2}$, then there is a well ordering of V such that every vector is the last element of only finitely many length-n arithmetic progressions (n-APs). This implies that there is a set mapping $f : V \to [V]^{<\omega}$ with no free set which is an n-AP. If, however, $|V| \geqslant \aleph_{n-1}$, then for every set mapping $f : V \to [V]^{<\omega}$ there is a free set which is an n-AP.

Our starting point is Rado's theorem which says that each vector space over $\mathbb{Q}$ can be colored with countably many colors, so that no arithmetic progression of length 3 (in short, 3-AP) is monocolored (unpublished, but see [1]).

There is a general machinery to obtain similar coloring theorems, and in some cases it even gives the existence of a well ordering such that each point is the last element of only finitely many of the configurations in question. In one example, Erdős and Hajnal proved that $\mathbb{R}^2$ can be colored with countably many colors so the no two points in rational distance get the same color, and their argument gives a well ordering $<$ of $\mathbb{R}^2$ such that for each $x \in \mathbb{R}^2$ the set

$$\{y < x : d(x, y) \in \mathbb{Q}\}$$

is finite (see [2]). The latter implies the former: given a well ordering $<$ as above, the coloring can be constructed by transfinite recursion.

The proof of Rado's theorem, however, goes differently: one fixes a basis $\{b_\alpha : \alpha < \kappa\}$ of V and if

$$x = \sum_{i=0}^{n-1} \lambda_i b_{\alpha_i}$$

for some n non-zero rational numbers $\lambda_0, \dots, \lambda_{n-1}$ such that $\alpha_0 < \dots < \alpha_{n-1}$, then color x with (some code of) the string $\langle \lambda_0, \dots, \lambda_{n-1} \rangle$ (vector 0 obtains a separate color). An easy calculation shows that there is no (non-trivial) 3-AP with all terms of the same color.

In the paper we use the following notation. If S is a set, κ a cardinal, we define $[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$, $[S]^{\geqslant \kappa} = \{x \subseteq S : |x| \geqslant \kappa\}$, $[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$. In particular, $[S]^{<\omega}$ is the set of finite subsets of S.

A set mapping is any function $f : S \to \mathcal{P}(S)$ for some set S. A subset $A \subseteq S$ is free for f, if $x \notin f(y)$ holds for $x, y \in A$, $x \neq y$.

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If $f : S \to [S]^{<\omega}$ is a set mapping, then by Fodor's theorem [5], S is the union of countably many free sets, i.e. $S = S_0 \cup S_1 \cup \cdots$ such that if $x \neq y \in S_i$, then $y \notin f(x)$.

We show that if $|V| \leqslant \aleph_1$ then there is a well order of V such that each vector is the last element of only finitely many 3-APs. In fact, there is a set mapping $f : V \to [V]^{<\omega}$ such that each set that is free for f is linearly independent. This implies, via Fodor's theorem that V is the union of countably many linearly independent sets, a result of Erdős and Kakutani (cf. [3]). On the other hand, if $|V| > \aleph_1$ then even the following holds: for every set mapping $f : V \to [V]^{<\omega}$ there is a free set which is a 3-AP.

Using this we obtain yet another equivalent form of the Continuum Hypothesis (see [9]): there is a set mapping $f : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ with no free set which is a 3-AP.

We also give the general forms of the these theorems: if $3 \leqslant n < \omega$ and V is a vector space over $\mathbb{Q}$ with $|V| \leqslant \aleph_{n-2}$, then there is a well ordering of V such that each vector is the last element of finitely many n-APs, and consequently there is a set mapping $f : V \to [V]^{<\omega}$ with no free n-AP. If, however, $|V| \geqslant \aleph_{n-1}$ then every set mapping $f : V \to [V]^{<\omega}$ contains a free set which is an n-AP.

As an example similar to Corollary 5 we also show that the Continuum Hypothesis is equivalent to the existence of a set mapping $f : \mathbb{R}^2 \to [\mathbb{R}^2]^{<\omega}$ with no free set which consists of the four nodes of a rectangle (Theorem 6).

One might notice that Theorem 4 is somewhat similar to the famous Kuratowski–Sierpiński theorem on set mappings [7, 8] and the proof also bears some resemblance. There is, however, an essential difference; the Kuratowski–Sierpiński theorem concerns set mappings of the form $f : [S]^n \to [S]^{<\omega}$ for some set S with $n > 1$. In our case, no such thing can be done, because if V is a vector space, then there is no free 3-AP for the set mapping $f : [V]^2 \to [V]^1$ where

$$f(\{x, y\}) = \left\{\frac{x+y}{2}\right\}.$$

Our results can be generalized to higher cardinals, as well. One has to replace "finitely many" with "of size $< \mu$" for some infinite cardinal μ and $\aleph_n$ with the nth successor of μ, μ^{+n}. The proofs are identical.

The paper [6] also gives somewhat similar results: the authors investigate Ramsey type theorems on $\mathbb{R}$ where the monocolored set has to satisfy certain arithmetical properties.

THEOREM 1. *If V is a vector space over $\mathbb{Q}$, $|V| = \aleph_1$, then there is a set mapping $f : V \to [V]^{<\omega}$ such that if $\{a_0, \ldots, a_{m-1}\}$ is a linearly dependent system of non-zero vectors, then $a_i \in f(a_j)$ for some $i \neq j$.*

Proof. Let $V = \bigcup\{V_\alpha : \alpha < \omega_1\}$ be an increasing decomposition of V into countable subspaces, which is continuous, i.e. $V_\alpha = \bigcup\{V_\beta : \beta < \alpha\}$ for α limit, and $V_0 = \{0\}$. Enumerate $V_{\alpha+1} - V_\alpha$ as $\{v_n^\alpha : n < \omega\}$. Define f by $f(v_{n+1}^\alpha) = \{v_0^\alpha, \ldots, v_n^\alpha\}$.

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In order to show that f is as claimed, let $A = \{a_0, \ldots, a_m\}$ be a linearly dependent set of non-zero vectors. We can as well assume that A is a minimal linearly dependent system. Let α be minimal such that $A \subseteq V_{\alpha+1}$, say $a_m \in V_{\alpha+1} - V_\alpha$. As A is a minimal linearly dependent system, a_m is a linear combination of $a_0, \ldots, a_{m-1}$. We cannot have that $a_0, \ldots, a_{m-1} \in V_\alpha$ as then a_m would also be in V_α, a contradiction. We must have, therefore, another element of A in $V_{\alpha+1} - V_\alpha$, say $v_i^\alpha, v_j^\alpha \in A$ for some $i \neq j$. But then, if $i < j$, we have $v_i^\alpha \in f(v_j^\alpha)$, and we are finished. □

If V is a vector space over $\mathbb{Q}$, let $\mathcal{H}_k$ be the set of k-element arithmetic progressions in V and

$$\mathcal{H}_{s,k} = \bigcup\{[A]^{\geqslant s} : A \in \mathcal{H}_k\}.$$

THEOREM 2. *If $s < \omega$, V is a vector space over $\mathbb{Q}$, $X \subseteq V$, $|X| \leqslant \aleph_s$, then X has a well ordering such that for each $x \in X$, x is the last element of finitely many $A \in \mathcal{H}_{s+2,k}$.*

Proof. The proof is by induction on s. If $s = 0$, any well ordering of X into type ω is as required.

Assume that we have the result for s, and $|X| = \aleph_{s+1}$. By replacing both X and V with the subspace generated by X, we can assume that $X = V$.

Decompose V as the increasing, continuous union $V = \bigcup\{V_\alpha : \alpha < \omega_{s+1}\}$ such that each V_α is a subspace, $V_0 = \{0\}$, and $|V_\alpha| \leqslant \aleph_s$.

By the inductive hypothesis, there is a well ordering $<_\alpha$ of $V_{\alpha+1} - V_\alpha$ such that for each $x \in V_{\alpha+1} - V_\alpha$ there are only finitely many $B \in \mathcal{H}_{s,k}$, $B \subseteq V_{\alpha+1} - V_\alpha$, whose last element is x. Let V be the sum of the well orders $<_\alpha$, that is, if $x \in V_{\alpha+1} - V_\alpha$, $y \in V_{\beta+1} - V_\beta$, then we let $x < y$ if either $\alpha < \beta$ or else $\alpha = \beta$ and $x <_{\alpha+1} y$. Clearly, $<$ is a well ordering of V.

In order to show that $<$ is as required, assume that $x \in V$ is the last element of $B \in \mathcal{H}_{s+3,k}$. Clearly, x is not in $V_0 = \{0\}$. There is a unique $\alpha < \kappa$ such that $x \in V_{\alpha+1} - V_\alpha$. Necessarily $|B \cap V_\alpha| \leqslant 1$, as otherwise $B \subseteq V_\alpha$ (for V_α is a subspace). We then have $B' = B - V_\alpha \in \mathcal{H}_{s+2,k}$. By the inductive hypothesis, x is the last element of finitely many sets B'. For a fixed B', as $|B'| \geqslant 2$ and it is a subset of an arithmetic progression of length k, there are finitely many $B \in \mathcal{H}_{s+2,k}$ with $B \supseteq B'$, which gives the statement. □

COROLLARY 3. *If V is a vector space over $\mathbb{Q}$, $|V| = \aleph_n$, then there is a well ordering such that each element is the last member of only finitely many arithmetic progressions of length $n + 1$. Consequently, there is a set mapping $f : V \to [V]^{<\omega}$ with no free arithmetic progression of length $n + 1$.*

Proof. The first statement is Theorem 2 with $s = n-1, k = n$. Given the well ordering, we set $f(x)$ as the union of all elements of all arithmetic progressions whose last element is x. □

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THEOREM 4. *If V is a vector space over $\mathbb{Q}$ with $|V| = \aleph_{n-1}$, $f : V \to [V]^{<\omega}$ is a set mapping, then there is a free arithmetic progression of length n.*

Proof. Let $A_j \subseteq V$ be arbitrary with $|A_j| = \aleph_{n-1-j}$ $(j < n)$.

We first choose $a_0 \in A_0$ in such a way that for every sequence

$$\langle x_1, \dots, x_{n-1}\rangle \in A_1 \times \cdots \times A_{n-1}$$

we have

$$ia_0 + \sum_{j=1}^{n-1}(i-j)x_j \notin f\left(-\sum_{j=1}^{n-1} jx_j\right) \quad (1 \leqslant i \leqslant n-1).$$

This is possible, as this requirement for each sequence $\langle x_1, \dots, x_{n-1}\rangle$ excludes finitely many values for a_0, there are $\aleph_{n-2}$ such sequences and $|A_0| = \aleph_{n-1}$.

Assume now that we have selected $a_0, \dots, a_{r-1}$.

Pick $a_r \in A_r$ in such a way that for every sequence

$$\langle x_{r+1}, \dots, x_{n-1}\rangle \in A_{r+1} \times \cdots \times A_{n-1}$$

if

$$b_r = \sum_{j<r}(r-j)a_j + \sum_{j>r}(r-j)x_j$$

then

$$\sum_{j<r}(i-j)a_j + (i-r)a_r + \sum_{j>r}(i-j)x_j \notin f(b_r) \quad (i \neq r).$$

If $r = n-1$ we also require

$$a_0 + \cdots + a_{n-1} \neq 0.$$

This is possible, as there are $\aleph_{n-r-2}$ such sequences $\langle x_{r+1}, \dots, x_{n-1}\rangle$, each excludes finitely many values of a_r and $|A_r| = \aleph_{n-r-1}$. If $r = n-1$, then there are $|A_{n-1}| = \aleph_0$ possible values for a_{n-1} and finitely many of them are excluded.

Finally, when we have selected $a_0, \dots, a_{n-1}$, set

$$v_i = \sum_{j=0}^{n-1}(i-j)a_j.$$

Then $v_0, v_1, \dots, v_{n-1}$ form an arithmetic progression of length n, with difference

$$a_0 + \cdots + a_{n-1} \neq 0,$$

and $v_i \notin f(v_r)$ as the condition

$$v_i = \sum_{j=0}^{n-1}(i-j)a_j \notin f\left(\sum_{j=0}^{n-1}(r-j)a_j\right) = f(v_r)$$

was satisfied when a_r was chosen. □

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COROLLARY 5. *The following are equivalent:*

(a) *the Continuum Hypothesis;*
(b) *there is a well ordering* $<$ *of* $\mathbb{R}$ *such that each real is the maximal element of finitely many 3-APs under* $<$*;*
(c) *there is a set mapping* $f : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ *with no free 3-AP;*
(d) *there is a set mapping* $f : \mathbb{R} \to [\mathbb{R}]^{<\omega}$ *with no linearly dependent free set.*

Proof. The implications (a)→(b)→(c) are Corollary 3.
The implication (c)→(a) is Theorem 4 for $n = 3$.
The implication (a)→(d) is Theorem 1.
The implication (d)→(c) is obvious. □

THEOREM 6. *The following are equivalent:*

(a) *the Continuum Hypothesis;*
(b) *there is a set mapping* $f : \mathbb{R}^2 \to [\mathbb{R}^2]^{<\omega}$ *with no free rectangle.*

Proof. (a)→(b) By the Continuum Hypothesis, $|\mathbb{R}^2| = \aleph_1$, and so $\mathbb{R}^2$ can be written as the increasing, continuous union of countable sets $\mathbb{R}^2 = \bigcup\{V_\alpha : \alpha < \omega_1\}$ so that $V_0 = \{0\}$ and if $a, b, c \in V_\alpha$ are distinct then $a+b-c \in V_\alpha$. Enumerate $V_{\alpha+1} - V_\alpha$ as $V_{\alpha+1} - V_\alpha = \{x_n^\alpha : n < \omega\}$. Define the set mapping $f : \mathbb{R}^2 \to [\mathbb{R}^2]^{<\omega}$ by $f(x_{n+1}^\alpha) = \{x_0^\alpha, \dots, x_n^\alpha\}$. Assume now that $R \subseteq \mathbb{R}^2$ is a rectangle. Let $\alpha < \omega_1$ be minimal such that $R \subseteq V_{\alpha+1}$. We cannot have $|R \cap V_\alpha| = 3$, because in that case if $R = \{a, b, c, d\}$, $R \cap V_\alpha = \{a, b, c\}$, then $d = a + b - c \notin V_\alpha$, contradicting the closure property of V_α. We have, therefore, $|R \cap (V_{\alpha+1} - V_\alpha)| \geqslant 2$, say, $x_i^\alpha, x_j^\alpha \in V_{\alpha+1} - V_\alpha$. But then, if $i < j$, then $x_i^\alpha \in f(x_j^\alpha)$, so R is not free for f.

(b)→(a) Assume that the Continuum Hypothesis fails, yet there is a set mapping $f : \mathbb{R}^2 \to [\mathbb{R}^2]^{<\omega}$ with no free rectangle. By Fodor's theorem, $\mathbb{R}^2$ can be decomposed into the union of countably many free sets: $\mathbb{R}^2 = A_0 \cup A_1 \cup \cdots$. By a result of Erdős and Komjáth (see [4]), if the Continuum Hypothesis fails and $\mathbb{R}^2$ is decomposed into countably many parts, then one of them contains a rectangle: $R \subseteq A_i$. As A_i is free for f, so is R. □

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