Two unusual optimisation problems

Problem 1

Figure 1 shows two symmetrically stacked cylinders with radii r_1, r_2 and heights h_1, h_2 where the only restriction is that $r_1 > r_2$. The problem is to find the cone of minimum volume that contains the two cylinders.

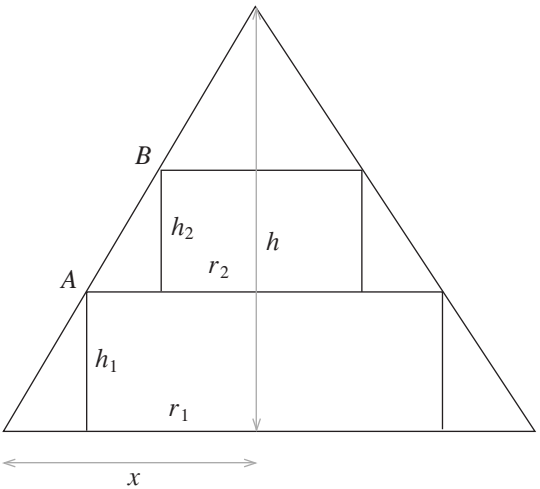


FIGURE 1

By shrinking the containing cone, it is clear that the one with minimum volume must touch the cylinders at A or B (or both).

If it touches at A then, from similar triangles, we have

$$\frac{h}{x} = \frac{h_1}{x - r_1}. \tag{1}$$

If it touches at B, we have

$$\frac{h}{x} = \frac{h_1 + h_2}{x - r_2}. \tag{2}$$

If it touches at both A and B, we have

$$\frac{h_1}{x - r_1} = \frac{h_1 + h_2}{x - r_2} \quad \text{or} \quad x = x_0 = r_1 + \frac{h_1}{h_2}(r_1 - r_2). \tag{3}$$

For $r_1 < x \leqslant x_0$, the minimising cone touches at A and, from (1), its volume is $V_A(x) = \frac{1}{3}\pi x^2\left(\frac{xh_1}{x - r_1}\right) = \frac{\pi}{3} \cdot \frac{h_1 x^3}{x - r_1}$.

For $x_0 \leqslant x < \infty$, the minimising cone touches at B and, from (2), its volume is $V_B(x) = \frac{\pi}{3} \cdot \frac{(h_1 + h_2)x^3}{x - r_2}$.

(This split of cases may be visualised by imagining x steadily increasing

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from $x = r_1$ with the generator of the cone rocking from touching at A to touching at B as x passes through $x = x_0$.)

The graph of $V_A(x)$ for $x > r_1$ is U-shaped with a vertical asymptote at $x = r_1$ and (from differentiation) a minimum at $x = \frac{3}{2}r_1$. The graph of $V_B(x)$ for $x \geqslant x_0$ is U-shaped, but without an asymptote and with a potential minimum at $x = \frac{3}{2}r_2$. The only intersection point of $V_A(x)$ and $V_B(x)$ for $x > r_1$ occurs at $x = x_0$.

Finding the global minimum in this problem thus requires us to minimise the function

$$V(x) = \begin{cases} V_A(x) & r_1 < x \leqslant x_0 \\ V_B(x) & x_0 \leqslant x < \infty \end{cases}$$

and the answer depends on the relative positions of $\frac{3}{2}r_2$, x_0, $\frac{3}{2}r_1$.

There are three cases:

- $x_0 < \frac{3}{2}r_2$:

  The minimum of $V(x)$ occurs at the minimum of $V_B(x)$ when $x = \frac{3}{2}r_2$ with $V\left(\frac{3}{2}r_2\right) = V_B\left(\frac{3}{2}r_2\right) = \frac{9}{4}\pi(h_1 + h_2)r_2^2$.

- $\frac{3}{2}r_2 < x_0 < \frac{3}{2}r_1$;

  Here, the minimum of $V(x)$ occurs at $x = x_0$ with
  $V(x_0) = \dfrac{\pi h_2}{3(r_1 - r_2)}\left[r_1 + \dfrac{h_1}{h_2}(r_1 - r_2)\right]^3$, using (3).

- $\frac{3}{2}r_1 < x_0$:

  The minimum of $V(x)$ occurs at the minimum of $V_A(x)$ when $x = \frac{3}{2}r_1$ with $V\left(\frac{3}{2}r_1\right) = V_A\left(\frac{3}{2}r_1\right) = \frac{9}{4}\pi h_1 r_1^2$.

From a teaching point of view, Problem 1 has several attractive features. It provides a natural example of a function with a split definition (why are students so suspicious of these?). A common initial reaction to this problem is that the minimising cone has to touch at both A and B, but this is not the full story since there are two other cases to consider. And, when the cone touching at both A and B is the minimising one, the minimum for $V(x)$ corresponds to a point at which $V(x)$ is not differentiable because $V_A'(x_0^-) < 0 < V_B'(x_0^+)$.

Problem 2

This problem arose from a strange query in class: when calculating the area of a triangle using the formula $\frac{1}{2}bc\sin A$, does it makes that much difference whether your calculator is in degrees or radians mode? This motivates Problem 2: investigate the function

$$f(x) = \sin x\,(degrees) - \sin x\,(radians)$$

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or $f(x) = \sin\frac{\pi x}{180} - \sin x$ for x in radians.

We will start with domain $\mathbb{R}$ for $f(x)$ and then comment briefly on the domain $0 \leqslant x \leqslant 180$ relevant to the original query. Investigating $f(x)$ using graphing software shows a slowly oscillating $\sin\frac{\pi x}{180}$ component with the $\sin x$ term providing added "noise" and suggests the following questions.

- Where are the zeros of $f(x)$?

  These occur when $\sin\frac{\pi x}{180} = \sin x$, so that $x = n\pi + (-1)^n\frac{\pi x}{180}$ for $n \in \mathbb{Z}$ or

$$x = \frac{n\pi}{1 - (-1)^n\frac{\pi}{180}}. \tag{4}$$

- Is $f(x)$ periodic?

  Suppose that there is a fixed $p > 0$ with $f(x + p) = f(x)$ for all x. Then $f'(x)$ and $f''(x)$ are also periodic with period p. From $f(p) = f(0)$ and $f''(p) = f''(0)$ we obtain $\sin\frac{\pi p}{180} = \sin p$ and $\frac{\pi^2}{180^2}\sin\frac{\pi p}{180} = \sin p$, so that $\sin\frac{\pi p}{180} = 0 = \sin p$. There are thus integers m, n with $\frac{\pi p}{180} = m\pi$ and $p = n\pi$ which gives the contradiction that $\pi = \frac{180m}{n}$ is rational.

- What are the (global) maximum and minimum values of $f(x)$?

  Clearly $-2 \leqslant f(x) \leqslant 2$, but are these bounds attained? The graphical evidence here can provoke fierce debate!

  First, these bounds are not attained. For example, $f(x) = 2$ can only occur when $\sin\frac{\pi x}{180} = 1$ and $\sin x = -1$. But then there are integers m, n with $\frac{n\pi}{180} = (4m + 1)\frac{\pi}{2}$ and $x = (4n - 1)\frac{\pi}{2}$ so, as above, $\pi = \frac{180(4m+1)}{4n-1}$ would be rational.

  But (and this explains the fierce debate) these bounds can be approached arbitrarily closely. For completeness, we sketch the proof of this which uses Kronecker's theorem that $\{e^{(\alpha n+\beta)i} : n \in \mathbb{Z}\}$ is dense in the set of complex numbers with modulus 1 if, and only if, $\frac{\alpha}{\pi}$ is irrational. (Trying to plot the set of such points using a graphing package makes this theorem intuitively plausible.) Applying this theorem with $\alpha = 8$ and $\beta = 2$, we see that there are integers n for which $\sin(8n + 2) \approx -1$ (as close as we like). Let $x = 45(8n + 2)$. Then $\sin\frac{\pi x}{180} = \sin(4n + 1)\frac{\pi}{2} = 1$ and $\sin x = \sin 45(8n + 2)$. But there is a polynomial p of degree 45 for which $\sin 45\theta = p(\sin\theta)$ and $p(-1) = -1$. Thus $\sin 45(8n + 2) = p(\sin(8n + 2)) \approx -1$ (as close as we like) since p is continuous. Overall then $f(45(8n + 2)) \approx 2$ (as close as we like) and a similar proof shows that the lower bound of -2 can also be approached arbitrarily closely.

  In our original context, the domain for $f(x)$ is $0 \leqslant x \leqslant 180$. Equation (4) shows that the zeros in this range satisfy

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$0 \leqslant \frac{n\pi}{1-(-1)^n\frac{\pi}{180}} \leqslant 180$, or $n \leqslant \frac{180-(-1)^n\pi}{\pi}$, an unusual inequality which shows that there are 57 zeros. Since $0 \leqslant \sin\frac{\pi x}{180} \leqslant 1$ and $-1 \leqslant \sin x \leqslant 1$, we have $-1 \leqslant f(x) \leqslant 2$ for $0 \leqslant x \leqslant 180$ but, with just a finite number of local maxima and minima to examine, we can readily check that the lowest minimum occurs at (1.553, −0.973) and the highest maximum occurs at (92.676, 1.999).

From a teaching point of view, this is a challenging example with several unusual features. It provides an interesting example of a non-periodic function, a situation where general solutions of trigonometrical equations are really needed, an application of the irrationality of π, and a glimpse ahead to Kronecker's theorem and ideas of equidistribution.

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