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Jets of differential equations having integrable extensions

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Abstract. We characterize the set of n-jets admitting an extension which is a germ of a differential equation with an analytic first integral, and compute its codimension in the n-jet space. Some applications in the case of the centre-focus problem are given.

1. *Introduction and statement of the results*

An n-jet of a differential equation at a singular point of resonance saddle type is, roughly speaking, *a class* of differential equations of type

$$\begin{aligned} \dot{x} &= -qx + P_n(x, y) + \cdots \\ \dot{y} &= py + Q_n(x, y) + \cdots \end{aligned} \tag{1}$$

where p, q are relatively prime positive integers, P_n, Q_n are polynomials in x, y of degree n, without linear terms, and the dots stand for terms of higher order than n.

In this paper we study the following problem: to find necessary and sufficient conditions for an n-jet of the above specified type to have an integrable extension, i.e. there exists a germ of a differential equation in the class of the jet which has an analytic first integral. This problem is solved (Lemma 4.4) and the solution is used in Theorem 4.2 to prove that the problem of existence of a holomorphic first integral is algebraically solvable in the class of germs of resonant saddles *with fixed ratio* p/q. As a corollary we re-obtain a well-known result of algebraic solvability of the centre-focus problem in the class of monodromic singular germs with nonvanishing 1-jet. As a by-product of that result we prove the functional independence of the *focal values* of the singularity (see the remark at the end of §4). The results contained in this paper were announced in [**10**], where some examples and applications are also given.

2. *Holonomy of jets*

In this section we define the holonomy of n-jets of 1-forms in $\mathbb{C}^2$ as an n-jet of a conformal map containing the common geometric features of all the extensions of a given n-jet of a differential equation. $\mathcal{G}$ and $\mathcal{O}$ are the rings of analytic germs at the origin

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of 1-forms in $\mathbb{R}^2$ and $\mathbb{C}^2$. The canonical projections $j^n : \mathcal{G} \to \mathcal{G}^n$ and $j^n : \mathcal{O} \to \mathcal{O}^n$ define the n-jet spaces $\mathcal{G}^n, \mathcal{O}^n$: we consider only singular germs at the origin, hence $\dim \mathcal{G}^n = \dim \mathcal{O}^n = n(n+1)$. The jets are usually denoted with a 'hat': if $\omega \in \mathcal{G}$, $j^n(\omega) = \hat{\omega} \in \mathcal{G}^n$. An *extension* ω of a given jet $\hat{\omega}$ is a germ satisfying

$$\omega \in [\hat{\omega}] = (j^n)^{-1}(\hat{\omega}).$$

To simplify the notation and the definitions, and having in mind their applications, in this article we define the holonomy for n-jets of resonance saddles only. An n-jet $\hat{\omega}$ of a (p,q)-*resonant saddle*, where p and q are relatively prime positive integers, is an n-jet that can be written in normalizing coordinates as

$$\hat{\omega} = py(1 + \hat{g}(x^p y^q))\,dx + qx(1 + \hat{f}(x^p y^q))\,dy, \tag{2}$$

where $\hat{f}$, $\hat{g}$ are $i(n)$-jets such that

$$\hat{f}(z) = \sum_{i=1}^{i(n)} a_i z^i, \quad \hat{g}(z) = \sum_{i=1}^{i(n)} b_i z^i, \tag{3}$$

where from the condition $i(p+q)+1 \le n$ it follows that

$$i(n) = [(n-1)/(p+q)] = \text{integer part of } \frac{n-1}{p+q}. \tag{4}$$

If $i(n) = 0$, we put $\hat{f} = \hat{g} = 0$. The set of germs of (p,q)-resonant saddles is denoted by $W(p,q)$: the projection $W^n(p,q) = j^n(W(p,q))$ is a semi-algebraic subset of $\mathcal{G}^n$. It is well-known that a germ of a (p,q)-resonance saddle ω has a prenormal form

$$\omega = py(1 + g(x^p y^q))\,dx + qx(1 + f(x^p y^q))\,dy, \tag{5}$$

where $f, g \in \mathbb{R}[[z]]$ if $\omega \in \mathcal{G}$, and $f, g \in \mathbb{C}[[z]]$ if $\omega \in \mathcal{O}$. From elementary normal form theory we get

$$j^{n-1}(f) = \hat{f}, \quad j^{n-1}(g) = \hat{g}. \tag{6}$$

This remark is of basic importance, for it gives a well-defined meaning to the quotient equation

$$\frac{dy}{dx} = -\frac{p}{q}\frac{y}{x}\frac{1+\hat{g}}{1+\hat{f}}, \tag{7}$$

henceforth permitting a good definition of the holonomy of jets. Let us develop this definition. We consider the quotient equation corresponding to (5) and we develop it formally as

$$\frac{dy}{dx} = \sum_{j=1}^{\infty} w_j(x) y^j. \tag{8}$$

From (6) we have the following.

PROPOSITION 2.1. *The functions $w_1, \ldots, w_n$ in (8) satisfy:*

(i) *$w_1, \ldots, w_n$ are polynomials in $1/x$ and in the coefficients of $\hat{f}$ and $\hat{g}$: in symbols*

$$w_1, \ldots, w_n \in \mathbb{C}\left[\frac{1}{x}; \hat{f}, \hat{g}\right];$$

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(ii) *referring to (7),*

$$\frac{dy}{dx} = -\frac{p}{q}\frac{y}{x}\frac{1+\hat{g}}{1+\hat{f}} = w_1(x)y + \cdots + w_n(x)y^n;$$

(iii) $w_1, \ldots, w_n$ *depend only on* $\hat{\omega}$*, i.e. they are independent of the extension of* $\hat{\omega}$*.*

By the above proposition the following variational system of n differential equations associated to $\hat{\omega} \in \mathcal{O}^n$ is well-defined, i.e. independent of the choice of any extension of the jet,

$$\begin{aligned} \dot{\phi}_1 &= w_1\phi_1, \\ \dot{\phi}_2 &= w_1\phi_2 + w_2\phi_1^2, \\ &\vdots \\ \dot{\phi}_n &= \sum_{j=1}^{n} w_j P_{jn}, \end{aligned} \tag{9}$$

where the m_is are positive integers and

$$P_{jn} = \sum_{\substack{m_1+\cdots+m_j=n \\ m_1,\ldots,m_j\in\{1,\ldots,n-j+1\}}} \phi_{m_1}\cdots\phi_{m_j} \tag{10}$$

with the initial conditions

$$\phi_1(x_0) = 1, \quad \phi_2(x_0) = \cdots = \phi_n(x_0) = 0. \tag{11}$$

We denote the corresponding flow as

$$x \mapsto (\phi_1(x; x_0, \gamma), \ldots, \phi_n(x; x_0, \gamma)), \tag{12}$$

where we stressed the dependence on the path $\gamma \in \mathbb{C}_x$, with endpoints x_0 and x.

Definition 2.2. The holonomy map of n-jets of resonance saddles, with respect to the q-marked separatix, is defined as the map

$$h : W^n(p, q) \subset \mathcal{G}^n \mapsto \mathrm{Diff}^n(\mathbb{C}, 0),$$

where

$$h_{\hat{\omega}} = h(\hat{\omega}) = h_1 y + \cdots + h_n y^n$$

with

$$h_k = \phi_k(x_0; x_0, \gamma), \tag{13}$$

where γ is a loop of index one around $x = 0$ in $\mathbb{C}_x$, based at x_0.

From Proposition 2.1 we have the following fundamental property,

$$j^n(h_\omega) = h_{j^n(\omega)}, \tag{14}$$

where h_ω is the holonomy of the germ ω with respect to the q-marked separatrix. Finally, let us remark that $h_{\hat{\omega}}$ is an algebraic function of the coefficients of $\hat{\omega}$. In fact, the computations in (13) involve only the calculus of residues and of the monodromy factors arising from the analytic continuations of exponential-like functions.

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3. *Characterizing jets of differential equations through their holonomies*

A basic result in the local theory of elementary singularities of holomorphic differential equations in $\mathbb{C}^2$ is the correspondence between the orbital equivalence of germs of foliations and the conjugacy of the relative holonomies (cf. [**5, 4**]). In this section we prove an analogous result for n-jets of resonance saddles. The proof of Lemma 3.3 involves an analytic version of a well-known geometric method of suspension of a conjugation between holonomies to give an equivalence of the relative foliations. In Appendix 2 this argument is used to give a modified proof of that result for germs. Let $\mathrm{Diff}(\mathbb{C}^k, 0)$ be the group of germs of analytic diffeomorphisms of $\mathbb{C}^k$ fixing the origin. The corresponding n-jet space is denoted $\mathrm{Diff}^n(\mathbb{C}^k, 0)$. The actions of $\mathrm{Diff}^{n+1}(\mathbb{C}^2, 0)$ on $\mathcal{O}^n$ and of $\mathrm{Diff}^n(\mathbb{C}, 0)$ on itself lead to the following definitions.

Definition 3.1. Two n-jets $\hat{\omega}, \hat{\omega}' \in \mathcal{O}^n$ are *orbitally n-equivalent*, denoted $\hat{\omega} \overset{n}{\sim} \hat{\omega}'$, if there exist $\Phi \in \mathrm{Diff}^{n+1}(\mathbb{C}^2, 0)$ and $\hat{u}$, an n-jet at the origin of $\mathbb{C}^2$ of a germ of an analytic function, $\hat{u}(0) \neq 0$, such that

$$\hat{\Phi}^\star \hat{\omega} = \hat{u}\hat{\omega}'.$$

Definition 3.2. Two n-jets $\hat{f}, \hat{g} \in \mathrm{Diff}^n(\mathbb{C}, 0)$ are *n-conjugated*, denoted $\hat{f} \overset{n}{\sim} \hat{g}$, if there exists $\phi \in \mathrm{Diff}^n(\mathbb{C}, 0)$ such that

$$\hat{\phi}^{-1} \circ \hat{f} \circ \hat{\phi} = \hat{g}.$$

Remark. The relations between jets in the above definitions have a simple analytic interpretation. Let Φ, u, ω, ω' be extensions of $\hat{\Phi}, \hat{u}, \hat{\omega}, \hat{\omega}'$ and let ϕ, f, g be extensions of $\hat{\phi}, \hat{f}, \hat{g}$. Then the given definitions are respectively equivalent to

$$\Phi^\star \omega - u\omega = \mathcal{O}(r^{n+1})$$

and

$$\phi^{-1} \circ f \circ \phi(z) - g(z) = \mathcal{O}(z^{n+1}),$$

where r is any distance from the origin in $\mathbb{C}^2$, and z is any variable parametrizing a neighbourhood of the origin in $\mathbb{C}$. The main result of this section is as follows.

LEMMA 3.3. *Let $\hat{\omega}, \hat{\omega}' \in W^n(p, q)$, and let $h_{\hat{\omega}}, h_{\hat{\omega}'}$ be their holonomies with respect to the q-marked separatrix. Then $\hat{\omega} \overset{n}{\sim} \hat{\omega}'$ if and only if $h_{\hat{\omega}} \overset{n}{\sim} h_{\hat{\omega}'}$.*

Proof. The proof is similar, and obviously simpler, to that of the analogous result for germs, cf. Appendix 2. One implication being obvious, we assume that $h_{\hat{\omega}} \overset{n}{\sim} h_{\hat{\omega}'}$ and we choose local coordinates such that both $\hat{\omega}, \hat{\omega}'$ have the form

$$\frac{p}{q} y(1 + A(x, y))\, dx + x\, dy,$$

where $A(0, 0) = 0$. The conjugation of the holonomies implies that there exists $\hat{\phi} \in \mathrm{Diff}^n(\mathbb{C}, 0)$ satisfying

$$\hat{\phi} = h_{\hat{\omega}} \circ \hat{\phi} \circ {h_{\hat{\omega}'}}^{-1}. \tag{15}$$

The holonomies are computed with respect to the transversal section $S_1 = \{(1, y) \mid |y| < r\}$. We look for a $(n+1)$-jet $\hat{\Phi} \in \mathrm{Diff}^{n+1}(\mathbb{C}^2, 0)$ which is a suspension of $\hat{\phi}$, i.e. such that $\hat{\Phi}|_{S_1} = \hat{\phi}$, and satisfies

$$\hat{\Phi}^\star \hat{\omega} \wedge \hat{\omega}' = 0 \tag{16}$$

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or the equivalent equality for the extensions

$$\Phi\omega \wedge \omega' = \mathcal{O}(r^{n+1}).$$

The particular choice of coordinates permits us to look for $\hat{\Phi}$ of the type

$$\hat{\Phi}(x, y) = (x, \hat{\xi}(x, y)). \tag{17}$$

Therefore, the problem of suspension of $\hat{\phi}$ turns out to be linked to the n-jet version of an initial-boundary problem for a semi-linear partial differential equation:

$$\begin{aligned} \frac{\partial\hat{\xi}(x, y)}{\partial x}(x, y) + \frac{\partial\hat{\xi}(x, y)}{\partial y}\hat{f}^{\hat{\omega}'}(x, y) &= \hat{f}^{\hat{\omega}}(x, y), \\ \hat{\xi}(1, y) &= \hat{\phi}(y), \\ \hat{\xi}_\gamma(1, y) &= \hat{\phi}(y), \end{aligned} \tag{18}$$

where the definition of γ is given below and $\hat{\xi}_\gamma(1, y)$ is an n-jet solution of the above equation computed along the path γ (see Appendix 2 for more details). Moreover,

$$(x, y) \in D_2 \times D_r - \{x = 0\}, \quad D_2 = \{x \mid |x| < 2\}, D_r = \{y \mid |y| < r\}$$

and

$$\hat{f}^{\hat{\omega}} = -\frac{\hat{\omega}_1}{\hat{\omega}_2}, \quad \hat{f}^{\hat{\omega}'} = -\frac{\hat{\omega}'_1}{\hat{\omega}'_2},$$
$$\hat{\omega} = \hat{\omega}_1\, dx + \hat{\omega}_2\, dy, \quad \hat{\omega}' = \hat{\omega}'_1\, dx + \hat{\omega}'_2\, dy$$

with

$$\hat{\omega}_2 = \hat{\omega}'_2 = x.$$

Let ϕ, ω, ω' be any extensions of $\hat{\phi}$, $\hat{\omega}$, $\hat{\omega}'$ and let us define

$$\xi(x, y) = g_\omega(x, 1, \phi({g_\omega}^{-1}(x, 1, y))), \tag{19}$$

where g_ω and g'_ω are the flows of the equations

$$\frac{dy}{dx} = f^\omega(x, y), \quad \frac{dy}{dx} = f^{\omega'}(x, y).$$

In Appendix 2 we prove that $\xi(x, y)$ defined in (19) solves the initial-boundary problem for germs analogous to (18). Hence, $\hat{\xi} = j^n(\xi)$ satisfies (18), at least in $D_2 \times D_r - (\{x = 0\} \cup S_1)$. The bound on the size of r is obtained in Appendix 2, where we prove that such a bound depends only on the first jet of ω. To prove that $\hat{\xi}$ satisfies (18) in $D_2 \times D_r - \{x = 0\}$ we use the hypothesis on conjugation of the holonomies. Let $t \mapsto e^{2\pi i t}$, $0 \le t \le 1$, be a parametrization of the loop γ which defines the holonomies $h_{\hat{\omega}}$, $h_{\hat{\omega}'}$. Then, from the initial conditions in (18) we get

$$\hat{\xi}(1, y) = \hat{\phi}.$$

Therefore, from (19) we obtain

$$\hat{\xi}(1, y) = h_{\hat{\omega}} \circ \hat{\phi} \circ h_{\hat{\omega}'}(y)$$

and from (15) it follows that $\hat{\xi}$ is the solution of (18) in $D_2 \times D_r - \{x = 0\}$. The extension of this n-jet solution to $\{x = 0\}$ is obvious. □

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4. *Characterization of jets admitting an extension with an analytic first integral*

This section is devoted to the proof of the algebraic solvability of the problem of existence of a holomorphic first integral in the class $W(p, q)$. As a consequence, we obtain a proof of a well-known result about the algebraic solvability of the problem of distinguishing the centre from the focus dynamics in the case of germs of singular vector fields in the plane with non-zero 1-jet. Moreover, we will prove the functional independence of the focal values obtained through the Poincaré–Lyapunov algorithm. Let $W \subset \mathcal{G}$ be such that $W^n = j^n(W) \subset \mathcal{G}^n$, where W^n is a semialgebraic subset in $\mathcal{G}^n$. Let $\mathcal{I} \subset \mathcal{G}$ be the set of germs having analytic first integrals. For every positive integer n we have the natural splitting

$$M_+{}^n = \{\hat{\omega} \in W^n \mid \forall \omega \in [\hat{\omega}], \omega \in \mathcal{I}\},$$

$$M_-{}^n = \{\hat{\omega} \in W^n \mid \forall \omega \in [\hat{\omega}], \omega \notin \mathcal{I}\},$$

$$M^n = \{\hat{\omega} \in W^n \mid \exists \omega, \omega' \in [\hat{\omega}], \omega \in \mathcal{I}, \omega' \notin \mathcal{I}\}.$$

Definition 4.1. The problem of existence of an analytic first integral is *algebrically solvable* in $W \subset \mathcal{G}$ if the following two conditions are satisfied:

(i) $M_+{}^n$, $M_-{}^n$, M^n are semialgebraic subsets;

(ii) $\operatorname{codim}_{\mathcal{G}^n} M^n \mapsto \infty$ as $n \mapsto \infty$.

It is known [**9**] that the problem of existence of an analytic first integral is not algebrically solvable if $W = \mathcal{G}$. On the other hand, our main result is as follows.

THEOREM 4.2. *Let p, q be fixed, relatively prime positive integers. The problem of existence of an analytic first integral is algebrically solvable in $W(p, q)$. More precisely, referring $M_+{}^n$, $M_-{}^n$, M^n to $W^n = j^n(W(p, q))$:*

(i) *$M_+{}^n = \emptyset$ for every integer;*

(ii) *M^n is algebraic;*

(iii) *$\operatorname{codim}_{\mathcal{G}^n} M^n = [(n-1)/(p+q)]$ = integer part of $(n-1)/(p+q)$.*

We will give the proof of the theorem after Lemma 4.4. First, we want to discuss a theorem which easily follows from it, when $p = q = 1$.

THEOREM 4.3. *The problem of distinguishing between the centre and the focus dynamics is algebrically solvable in the class of germs in $\mathcal{G}$ having nonzero 1-jet.*

Remark. As we have stated, the last theorem is well-known: usually it is quoted (cf., for instance, [**1**, p. 89 of the English translation]) as a straightforward consequence of a celebrated theorem by Poincaré [**8**] and Lyapunov [**3**] and of a result by Moussu concerning nilpotent centres [**6**]. This argument needs a deeper understanding: let us justify this comment. The 1-jet of a monodromic nondegenerate singularity can be written as

$$j^1\omega = x\,dx + y\,dy.$$

In these coordinates the Poincaré–Lyapunov algorithm permits us to distinguish centres from foci: $\omega(x, y)$ will have a centre if and only if there exists an analytic germ

$$F(x, y) = \tfrac{1}{2}(x^2 + y^2) + \cdots$$

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such that, if $v(x, y)$ is the dual vector field of $\omega(x, y)$, it satisfies

$$L_v F = \sum_{i=1}^{\infty} V_i (x^2 + y^2)^{i+1} \equiv 0. \tag{20}$$

The V_is are the so-called *focal values* and it is easy to prove that if $\hat{\omega} = j^n(\omega)$ then

$$V_i = V_i[\hat{\omega}], \quad i = 1, 2, \ldots, i(n), \quad i(n) = \left[\frac{n-1}{2}\right] \tag{21}$$

or, more explicitly, the first $i(n)$ focal values are algebraic functions of the n-jet of ω. Equation (20) holds true if and only if $V_1 = V_2 = \cdots = 0$. What is usually given as a proof of Theorem 4.3 is the (implicit) claim that

$$(V_1[\hat{\omega}] = \cdots = V_{i(n)}[\hat{\omega}] = 0) = M^n. \tag{22}$$

This claim is actually true, as we will show at the end of this section, but it deserves a proof. It is obviously true that if ${M_+}^n$, M^n refer to $W(1, 1)$ in Definition 4.1, then

$${M_+}^n \cup M^n \subset (V_1 = \cdots = V_{i(n)} = 0).$$

On the other hand, we need to prove that if

$$\hat{\omega} \in (V_1 = V_2 = \cdots = V_{i(n)} = 0)$$

then there exists $\omega \in [\hat{\omega}]$ admitting an analytic first integral. The main tool used to obtain this result is the following lemma, which is an n-jet version of Theorem B_0 in [5]: it is a characterization of those n-jets in $j^n(W(p, q))$ admitting an extension with a first integral.

LEMMA 4.4. *Let $\hat{\omega} \in j^n(W(p, q))$ and let $h_{\hat{\omega}}$ be its holonomy with respect to the q-marked separatrix. The three following propositions are equivalent:*

(i) *$\hat{\omega}$ is n-linearizable (i.e. $\hat{\omega} \overset{n}{\sim} py\,dx + qx\,dy$);*

(ii) *$\hat{\omega}$ has an extension having a first integral;*

(iii) *if $h^{(q)}$ denotes the qth-superpositionary power of a jet, we have*

$$h_{\hat{\omega}}{}^{(q)} = \hat{1}_{\mathbb{C}}.$$

Remark. If

$$L_n = \{\hat{\omega} \in j^n(W(p, q)) \mid \hat{\omega} \overset{n}{\sim} qx\,dy + py\,dx\},$$
$$P_n = \{\hat{\omega} \in j^n(W(p, q)) \mid h_{\hat{\omega}}{}^{(q)} = \hat{1}_{\mathbb{C}}\}$$

the lemma can be stated as

$${M_+}^n \cup M^n = L_n = P_n. \tag{23}$$

Proof. First step: (i) is equivalent to (iii). From Lemma 3.3, $\hat{\omega}$ is n-linearizable if and only if $h_{\hat{\omega}} \overset{n}{\sim} \hat{R}_q$, $\hat{R}_q = e^{2\pi i p/q} z \in \mathcal{O}^n$. Moreover, from Lemma A.1 in Appendix 1 $h_{\hat{\omega}} \overset{n}{\sim} \hat{R}_q$ if and only if $h_{\hat{\omega}}{}^{(q)} = \hat{1}_{\mathbb{C}}$.

Second step: (ii) implies (iii). Let $\omega \in [\hat{\omega}]$, ω admitting a first integral. Then $h_{\omega}^{(q)} = 1_{\mathbb{C}}$, and from (14)

$$\hat{1}_{\mathbb{C}} = j^n(h_{\omega}^{(q)}) = h_{\hat{\omega}}{}^{(q)}.$$

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Third step: (iii) implies (ii). Let $\hat{\omega} \in P_n$. From Lemma A.1 and from Lemma 3.3 we get that if

$$\omega' = qx\,dy + py\,dx$$

then

$$\hat{\omega} \overset{n}{\sim} \hat{\omega}',$$

where $\hat{\omega}' = j^n(\omega')$. From Definition 3.1 there exists $\hat{\Phi}$, $\hat{u}$ such that

$$\hat{\Phi}^*\hat{\omega} = \hat{u}\hat{\omega}'.$$

Let $\hat{\Psi} = \hat{\Phi}^{-1}$: then $\hat{\omega} = \hat{\Psi}^*(\hat{u}\hat{\omega}')$. Let Ψ, u be any extensions of $\hat{\Psi}$, $\hat{u}$. Therefore, $\Psi^*(u\omega')$ has a first integral and the third step of the proof follows from the equality

$$j^n(\Psi^*(u\omega')) = \hat{\Psi}^*(\hat{u}\hat{\omega}') = \hat{\omega}. \qquad \square$$

The next step in proving Theorem 4.2 is the following lemma.

LEMMA 4.5. *L_n is an algebraic subset of $\mathcal{G}^n$, with* $\operatorname{codim}_{\mathcal{G}^n} L_n = [(n-1)/(p+q)]$.

Proof. Let $\tilde{\mathcal{G}}^n$ be the quotient of $\mathcal{G}^n$ with respect to the action of $GL(2,\mathbb{R}) \times \mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$. Hence,

$$\tilde{L}_n = \{\hat{\omega} \in L_n \mid \hat{\omega} \in \tilde{\mathcal{G}}^n\} = \{\hat{\omega} \in L_n \mid j^1(\hat{\omega}) = py\,dx + qx\,dy\}.$$

We will prove that $\tilde{\mathcal{G}}^n$ is algebraic and satisfies

$$\operatorname{codim}_{\tilde{\mathcal{G}}^n} \tilde{L}_n = \left[\frac{n-1}{p+q}\right] \tag{24}$$

and this will end the proof, for

$$\operatorname{codim}_{\tilde{\mathcal{G}}^n} \tilde{L}_n = \operatorname{codim}_{\mathcal{G}^n} L_n. \qquad \square$$

We use the following classical result of normal form theory (cf. [**2**]).

LEMMA 4.6. *Let $\hat{\omega} \in \tilde{L}_n$, and let*

$$\begin{aligned} \dot{x} &= -qx(1 + \hat{f}(x^p y^q)), \\ \dot{y} &= py(1 + \hat{g}(x^p y^q)). \end{aligned} \tag{25}$$

be its (pre)normal form. Then $\hat{\omega}$ is n-linearizable if and only if $\hat{f} = \hat{g}$.

We need to recall briefly the elements of normal form theory, cf. [**5**]. Let $\hat{\omega} \in \tilde{L}_n$: in coordinates (u_1, u_2) the n-jet of the differential equation associated to $\hat{\omega}$ is (in this proof for notational convenience we drop the 'hat' superscript for jets)

$$\dot{u_j} = \lambda_j u_j + \phi_j, \tag{26}$$

where $j = 1, 2$, $\lambda_1 = -q$, $\lambda_2 = p$ and

$$\phi_j = \sum_{2 \le |Q| \le n} \phi_{jQ} u^Q \tag{27}$$

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with $Q = (q_1, q_2)$, $|Q| = q_1 + q_2$, $u^Q = u_1{}^{q_1} u_2{}^{q_2}$ and $\phi_{jQ} \in \mathbb{C}$. Let N be a normalizing $(n+1)$-jet tangent to the identity

$$N(x_1, x_2) = (x_1 + \xi_1(x_1, x_2), x_2 + \xi_2(x_1, x_2)), \quad \xi_j(x_1, x_2) = \sum_{2 \le |Q| \le n+1} \xi_{jQ} x^Q. \tag{28}$$

N transforms (26) in

$$\dot{x}_j = \lambda_j x_j + \psi_j, \tag{29}$$

where

$$\psi_j(x_1, x_2) = \sum_{2 \le |Q| \le n} \psi_{jQ} x^Q \tag{30}$$

with

$$\psi_{jQ} = \begin{cases} 0, & \text{if } \lambda_1 q_1 + \lambda_2 q_2 \neq \lambda_j,\ j = 1, 2 \\ \phi_{jQ} + B_{jQ}, & \text{if } \lambda_1 q_1 + \lambda_2 q_2 = \lambda_j,\ j \in \{1, 2\} \end{cases} \tag{31}$$

and

$$B_{jQ} \in \mathbb{Q}[\phi_{js};\ j = 1, 2,\ 2 \le |s| < |Q|]. \tag{32}$$

Therefore, from Lemma 4.6 and (29)–(31) we get that $\hat{\omega} \in \tilde{L}_n$ if and only if its normal form verifies

$$F_i = \psi_{1(ip+1,iq)} + \frac{q}{p} \psi_{2(ip,iq+1)} = 0, \tag{33}$$

where $i = 1, 2, \ldots, i(n) = [(n-1)/(p+q)]$. From (33) and (31), (32) it follows that $\tilde{L}_n$ is an algebraic subset of $\tilde{\mathcal{G}}^n$. Let us now compute the codimension of $\tilde{L}_n$ in $\tilde{\mathcal{G}}^n$. By using (32) in (33) we obtain

$$F_i = \phi_{1(ip+1,iq)} + \frac{q}{p} \phi_{2(ip,iq+1)} + \left(B_{1(ip+1,iq)} + \frac{q}{p} B_{2(ip,iq+1)} \right) = 0,$$

where $i = 1, 2, \ldots, i(n)$.

Therefore,

$$\frac{\partial F_j}{\partial \phi_{1(kp+1,kq)}} = \begin{cases} 0, & \text{if } j > k \\ 1, & \text{if } j = k \\ \circ, & \text{if } j < k \end{cases}$$

($\circ$ denotes a term of no importance for us). Finally,

$$\frac{\partial(F_1, \ldots, F_{i(n)})}{\partial(\phi_{1(p+1,q)}, \ldots, \phi_{1(i(n)p+1,i(n)q+1)})} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and Lemma 4.5 follows. □

We come back now to the proof of Theorem 4.2.

Proof of Theorem 4.2. We have already proved statements (ii) and (iii) of the theorem: therefore, we just need to prove that $M_+{}^n = \emptyset$ for every $n > 1$. This conclusion is a consequence of the following simple argument. Let us suppose, by contradiction, that there exists an n_0-jet, called $\hat{\omega}_0$, such that

$$\hat{\omega}_0 \in M_+^{n_0}.$$

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For every $m > n_0$ we define

$$I^m(\hat{\omega}_0) = \{\hat{\omega} \in \mathcal{G}^m \mid j^n(\hat{\omega}) = \hat{\omega}_0\}.$$

Clearly $I^m(\hat{\omega}_0)$ is an affine subspace of $\mathcal{G}^m$:

$$I^m(\hat{\omega}_0) = \hat{\omega}' + \{\hat{\omega} \in \mathcal{G}^m \mid j^n(\hat{\omega}) = 0\}$$

with $j^n(\hat{\omega}') = \hat{\omega}_0$. Therefore,

$$\dim I^m(\hat{\omega}_0) = m(m+1) - n_0(n_0+1). \tag{34}$$

Moreover, it is also clear that

$$I^m(\hat{\omega}_0) \subset {M_+}^m \subset {M_+}^m \cup M^m. \tag{35}$$

As a consequence of (34) and (35) and from the computation of the codimension of $L_m = {M_+}^m \cup M^m$ we have, for every $m > n_0$,

$$m(m+1) - n_0(n_0+1) \leq m(m+1) - \left[\frac{m-1}{p+q}\right]$$

and we get the contradiction. □

Remark. We come back now to the comments we made about Theorem 4.3. Let $\hat{\omega} \in \mathcal{G}^n \cap j^n(W(1,1))$. From Lemma 4.4 and the Poincaré–Lyapunov centre theorem, we get that $\hat{\omega}$ is n-linearizable if and only if it has an extension with real dynamics of centre type. By the standard arguments of normal form theory for equations exhibiting a 1:1 resonance we know that $\hat{\omega}$ is n-linearizable if and only if

$$V_1[\hat{\omega}] = \cdots = V_{i(n)}[\hat{\omega}] = 0, \quad i(n) = \left[\frac{n-1}{2}\right],$$

where the V_is are the focal values. Then arguing exactly as in Lemma 4.5 we can conclude that

$$(V_1 = \cdots = V_{i(n)} = 0) = M^n$$

and, moreover, the focal values $V_1, \ldots, V_{i(n)}$ are functionally independent algebraic functions in $\mathcal{G}^n$.

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Appendix 1. Characterization of formal periodic diffeomorphisms and extension of periodic jets

The results contained in this appendix are due to Moussu [**7**]: we thank him for permission to report them here. We recall that an n-jet of a conformal map $\hat{h} \in \mathrm{Diff}^n(\mathbb{C}, 0)$ is q-periodic, $q > 1$, if q is the smallest integer such that $\hat{h}^{(q)} = \hat{1}_{\mathbb{C}}$.

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LEMMA A.1. *Let $\hat{R}_q$ be the n-jet of the rotation germ $z \mapsto e^{2\pi ip/q}z$. Then:*

(i) *$\hat{h} \in \mathrm{Diff}^n(\mathbb{C}, 0)$ is q-periodic if and only if $\hat{h} \overset{n}{\sim} \hat{R}_q$;*

(ii) *every q-periodic $\hat{h} \in \mathrm{Diff}^n(\mathbb{C}, 0)$ has a formal q-periodic extension, i.e. a formal diffeomorphism h such that $h^{(q)} = 1_{\mathbb{C}}$.*

Proof. During this section we will always drop the 'hat' convention on the jets notation. We begin by proving (i): one implication being trivial we only prove that if $\hat{h}^{(q)} = \hat{1}_{\mathbb{C}}$ then $\hat{h} \overset{n}{\sim} \hat{R}_q$. The proof is by induction on the order n of the jets: the case $n = 1$ is trivial†. Therefore, we suppose that there exists an $(n-1)$-jet Φ_{n-1} such that

$$\Phi_{n-1}{}^{-1} \circ h \circ \Phi_{n-1}(x) = h_1 x + \mathcal{O}(x^n),$$

where

$$\Phi_{n-1}(x) = \phi_1 x + \cdots + \phi_{n-1}x^{n-1}.$$

We need to prove the existence of an n-jet Φ_n of the form

$$\Phi_n = \phi_1 x + \cdots + \phi_{n-1}x^{n-1} + \phi_n x^n$$

such that

$$\Phi_n{}^{-1} \circ h \circ \Phi_n(x) = h_1 x + \mathcal{O}(x^{n+1}).$$

By the inductive hypothesis we get

$$\Phi_n{}^{-1} \circ h \circ \Phi_n(x) = h_1 x + \alpha_n x^n + \mathcal{O}(x^{n+1})$$

and therefore from

$$h \circ \Phi_n(x) = \Phi_n(h_1 x + \alpha_n x^n) + \mathcal{O}(x^{n+1})$$

the inductive hypothesis becomes

$$\begin{aligned} h_1\phi_2 + h_2{\phi_1}^2 &= \phi_2 {h_1}^2, \\ &\vdots \\ \sum_{j=1}^{n-1} h_j P_{j,n-1} &= \phi_{n-1}{h_1}^{n-1}, \end{aligned} \tag{36}$$

where

$$P_{j,n-1} = \sum_{i_1+\cdots+i_j=n-1} \phi_{i_1} \cdots \phi_{i_j},$$

while the recursive statement becomes the fundamental equation

$$\phi_n({h_1}^n - h_1) = h_2 P_{2n} + \cdots + h_n {\phi_1}^n. \tag{37}$$

We distinguish two cases.

† Of course h_1 is a root of unity, and we always suppose that $h_1 \neq 1$.

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Case 1. $n-1 \not\equiv 0 \bmod q$. In this case equation (37) has a unique solution given by

$$\phi_n = \frac{h_2 P_{2n} + \cdots + h_n {\phi_1}^n}{{h_1}^n - h_1}. \tag{38}$$

Case 2. $n-1 \equiv 0 \bmod q$. Equation (37) becomes

$$h_2 P_{2n} + \cdots + h_n {\phi_1}^n = 0 \tag{39}$$

which does not involve ϕ_n. We are going to prove that if $\phi_1, \ldots, \phi_{n-1}$ satisfy (36) then (39) is always satisfied. In proving this we use the following lemma.

LEMMA A.2. *Let*

$$A(x) = {\Phi_{n-1}}^{-1} \circ h \circ \Phi_{n-1}(x) = h_1 x + \alpha_n x^n + \mathcal{O}(x^{n+1}).$$

We have

(i) $A^{(j)}(x) = {h_1}^j(x) + \alpha_n \sigma_j(n) x^n + \mathcal{O}(x^{n+1})$, *where*

$$\begin{aligned} &\sigma_0(n) = 0, \\ &\sigma_1(n) = 1, \\ &\sigma_j(n) = {h_1}^{(j-1)n} + {h_1}^n \sigma_{j-1}(n), \quad j > 1; \end{aligned} \tag{40}$$

(ii) $\sigma_q(n) = 0$ *if and only if* $n-1 \not\equiv 0 \bmod q$.

We will give the proof of this lemma at the end of this section. Now we use it to prove Lemma A.1. From the hypothesis $h^{(q)}(x) = x + \mathcal{O}(x^{n+1})$ we get

$$A^{(q)}(x) = {\Phi_{n-1}}^{-1} \circ h^{(q)} \circ \Phi_{n-1}(x) = x + \mathcal{O}(x^{n+1})$$

and from Lemma A.2, statement (i), we obtain

$$x + \mathcal{O}(x^{n+1}) = x + \alpha_n \sigma_q(n) x^n + \mathcal{O}(x_{n+1}).$$

Hence

$$\alpha_n \sigma_q(n) = 0.$$

Then from Lemma A.2, statement (ii), it follows that $\alpha_n = 0$, and statement (i) of the lemma is proved. We now prove statement (ii): the proof is again by induction. We suppose that

$$h_1 x + \cdots + h_{n-1} x^{n-1}$$

is the expression in coordinates of a q-periodic $(n-1)$-jet and we will prove that it admits an extension

$$h^*(x) = h_1 x + \cdots + h_{n-1} x^{n-1} + h_n x^n$$

which is a q-periodic n-jet. From (i), to prove (ii) turns out to be equivalent to proving the existence of an element $\Phi_n \in \mathrm{Diff}^n(\mathbb{C}^2, 0)$ such that

$${\Phi_n}^{-1} \circ h^* \circ \Phi_n(x) = h_1 x + \mathcal{O}(x^{n+1}). \tag{41}$$

The inductive hypothesis is obviously

$${\Phi_{n-1}}^{-1} \circ h^* \circ \Phi_{n-1}(x) = h_1 x + \mathcal{O}(x^n),$$

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where

$$\Phi_n = \Phi_{n-1} \bmod (x^n)$$

and in coordinates $\Phi_n = \phi_1 x + \cdots + \phi_n x^n$. Therefore, ϕ_n and h_n are linked by (37) and (38). We consider again two different cases.

Case 1. $n - 1 \not\equiv 0 \bmod q$, $n > 1$. In this case from (37)

$$h_n = \frac{\phi_n}{{\phi_1}^n}({h_1}^n - h_1) - (h_2 P_{2,n} + \cdots + h_{n-1} P_{n-1,n})$$

and the induction is proved. Let us explicitly note that in this case the choice of h_n is free.

Case 2. $n - 1 \equiv 0 \bmod q$. In this case from (39) it follows that h_n does not depend on the choice of ϕ_n, and is uniquely determined by

$$h_n = -\frac{(h_2 P_{2,n} + \cdots + h_{n-1} P_{n-1,n})}{{\phi_1}^n}.$$

The proof of Lemma A.1 is therefore concluded. □

We now come back to prove Lemma A.2.

Proof of Lemma A.2. From the definition of $A(x)$ we get $\sigma_1(1) = 1$. The proof of statement (i) is by induction, assuming that

$$A^{(j)}(x) = {h_1}^j x + \alpha_n \sigma_j(n) x^n + \mathcal{O}(x^{n+1}),$$

where $\sigma_j(n) = h_1 \sigma_{j-1}(n) + {h_1}^{(j-1)n}$. A straightforward computation leads to

$$A^{(j+1)}(x) = {h_1}^{j+1} x + \alpha_n (h_1 \sigma_j(n) + {h_1}^{jn}) x^n + \mathcal{O}(x^{n+1}).$$

Hence (i) follows. To prove statement (ii) we observe that an easy computation leads to

$$\sigma_q(n) = 0 \text{ if and only if } 1 + {h_1}^{n-1} + \cdots + {h_1}^{(n-1)(q-1)} = 0.$$

If $n - 1 \not\equiv 0 \bmod q$ we have

$$1 + {h_1}^{n-1} + \cdots + {h_1}^{(n-1)(q-1)} = \frac{1 - ({h_1}^{(n-1)q})}{1 - ({h_1}^{n-1})} = 0$$

and therefore in this case $\sigma_q(n) = 0$. On the other hand, if $n - 1 \equiv 0 \bmod q$ we have

$$1 + {h_1}^{n-1} + \cdots + {h_1}^{(n-1)(q-1)} = q$$

and the proof of Lemma A.2 is complete. □

Appendix 2. An analytic proof of the classification theorem for germs of resonance saddles

This appendix contains a proof of a well-known result concerning the holomorphic equivalence between two singular germs of resonance saddle-type differential equations having conjugated holonomies. The scheme of the proof remains the same as in [**5**], but here we translate into an analytic argument the geometric method of suspending the conjugation of the holonomies to give the orbital equivalence. Our motivation for this

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modification of the proof in [**5**] comes from the necessity of applying an analogous result in Lemma 3.3 to jets. Moreover, we also give here some estimates on the domain of definition of the orbital equivalence transformations that have been used in Lemma 3.3: we prove that these bounds depend on the first jet of the germ, hence permitting the application of the same arguments to jets of 1-forms in Lemma 3.3.

THEOREM B.1. *Let ω, $\omega' \in W(p, q)$ and let h_ω, $h_{\omega'}$ be their holonomies with respect to the q-marked separatrix. Then ω, ω' are orbitally holomorphically equivalent if and only if h_ω and $h_{\omega'}$ are conjugated.*

Proof. One implication in the statement is obvious. Therefore, we only prove that the conjugation of the holonomies implies the orbital equivalence. We can restrict our argument to the case when both ω, ω' have normal forms of type

$$\frac{p}{q} y(1 + A(x, y))\, dx + x\, dy,$$

where $A(0, 0) = 0$. Moreover, we suppose that $h_\omega, h_{\omega'}$ are relative to the separatrix $\{y = 0\}$ and to a loop γ, with $x_0 = 1 \in \gamma$ and $t \mapsto x(t) = e^{2\pi i t}$, $0 \leq t \leq 1$, a parametrization of γ. We look for an orbital equivalence in the form of a *fibration*

$$\Phi^* = (x, \xi(x, y)) \tag{42}$$

satisfying

$$\Phi^* \omega \wedge \omega' = 0 \tag{43}$$

in a neighbourhood of the origin in $\mathbb{C}^2$. To simplify notation we put

$$\begin{aligned} \omega &= \omega_1\, dx + \omega_2\, dy, \\ \omega' &= \omega'_1\, dx + \omega'_2\, dy, \end{aligned} \tag{44}$$

where $\omega_2 \equiv \omega'_2 \equiv x$, and

$$\begin{aligned} f^{\omega}(x, y) &= -\frac{\omega_1}{\omega_2}, \\ f^{\omega'}(x, y) &= -\frac{\omega'_1}{\omega'_2}. \end{aligned} \tag{45}$$

A standard application of Riemann's extension theorem reduces the proof to the following statement: there exists an $r > 0$ such that in ${D_2}^x \times {D_r}^y - \{x = 0\}$, where for instance ${D_r}^y = \{y \mid |y| < r\}$, the following initial-boundary problem is satisfied,

$$\begin{aligned} \frac{\partial \xi}{\partial x}(x, y) + \frac{\partial \xi}{\partial y}(x, y) f^{\omega'}(x, y) &= f^{\omega}(x, \xi(x, y)), \\ \xi(1, y) &= \phi(y), \\ \xi_\gamma(1, y) &= \phi(y), \end{aligned} \tag{46}$$

where $\phi(y)$ is the conjugation of the holonomies, i.e. it satisfies

$$\phi(y) = h_\omega \circ \phi^{-1} \circ {h_{\omega'}}^{-1} \tag{47}$$

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and $\xi_\gamma(1, y)$ is obtained through continuation along a path in ${D_2}^x \times {D_r}^y - \{x = 0\}$ which projects via $\pi : (x, y) \mapsto x$ onto γ. It is well-known that (46) is equivalent to the uncoupled system

$$\begin{aligned} \frac{dy}{dx} &= f^{\omega'}(x, y), \\ \frac{du}{dx} &= f^{\omega}(x, u(x, y(x; y_0))), \\ u(1, y_0) &= \phi(y_0), \quad |y_0| < r. \end{aligned} \tag{48}$$

The equivalence between (46) and (48) can be made explicit in a more suggestive way, closely related to the geometric construction of the suspension of the conjugacy given in [**5**]. Let g_ω, $g_{\omega'}$ be the flows of the equation of the characteristics

$$\frac{dy}{dx} = f^{\omega'}(x, y)$$

and of

$$\frac{dy}{dx} = f^{\omega}(x, y).$$

Then we get

$$\xi(x, y) = g_\omega(x, 1, \phi({g_{\omega'}}^{-1}(x, 1, y))), \tag{49}$$

where flows are relative to any path based at $x_0 = 1$ and homotopic to γ. The rest of the proof splits into five steps.

First step: saturation of ${D_2}^x \times {D_r}^y - \{x = 0\}$ by characteristic curves. This part of the proof can be found, for instance, in [**5**]. We report it here only to show that the constant r appearing in the proof of Lemma 3.3 actually depends on the first jet only, therefore permitting the application of the same proof in the case of jets. We will prove that for any initial value problem of type

$$\begin{aligned} \frac{dy}{dx} &= -\frac{py}{qx}(1 + A(x, y)), \\ y(1) &= y_0, \quad |y_0| < r \end{aligned} \tag{50}$$

with $A(0, 0) = 0$ and $A(x, y)$ analytic, there exists a positive constant r such that ${D_2}^x \times {D_r}^y - \{x = 0\}$ is saturated by the solutions of (50). This is equivalent to proving that for every $(x, y) \in {D_2}^x \times {D_r}^y - \{x = 0\}$ the following equality holds

$$(x, y) = g_R\left(s(x, y), \frac{x}{|x|}, g_C(t(x, y), 1, y_0(x, y))\right), \tag{51}$$

where g_R and g_C stay for the lifting through the flow of (50) of the radial path $t \mapsto (x/|x|)(1 - t)$, and of the circular path $t \mapsto e^{2\pi i t}$, $0 \le t < 1$, and the functions $(x, y) \mapsto y_0(x, y)$, $(x, y) \mapsto t(x, y)$, $(x, y) \mapsto s(x, y)$ are well-defined in $({D_2}^x \times {D_r}^y - \{x = 0\}) - S_1$, $S_1 = \{(1, y) \quad ||y| < r\}$, and satisfy

$$\begin{aligned} |y_0(x, y)| &< r \\ 0 \le t(x, y) &< 1 \\ 0 < s(x, y) &\le 1. \end{aligned}$$

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To prove (51) we need to have an estimation from above of the growth of the solutions of (50) with respect to an initial datum y_0 and to the homotopy class of the path γ. Let us choose $r > 0$ sufficiently small to have

$$|A(x, y)| < \tfrac{1}{2} \tag{52}$$

holding true in ${D_2}^x \times {D_r}^y$. Let γ be a given path with initial point in S_1. From (50) and (52) we get

$$\left|\frac{dy}{y}\right| < \frac{3p}{2q}\left|\frac{dx}{x}\right| \tag{53}$$

which is valid in ${D_2}^x \times {D_r}^y - \{x = 0\} \cup \{y = 0\}$. Let us observe explicitly that such estimates depend on the first jet only of (50): hence they carry on unaltered in the proof of Lemma 3.3. Therefore, along the path obtained from the composition of the above-defined circular and radial paths the following inequality holds

$$|y(t)| < |y_0| 2\pi e^{(3p/2q)t} \tag{54}$$

with $0 \leq t < 2$. This inequality guarantees the existence of a set R in $\mathbb{C}^2$ filled by solutions of (50). It is clear that, possibly reducing the size of r, we have

$${D_2}^x \times {D_r}^y - \{x = 0\} \subset R.$$

Second step: boundness of $\xi(x, y)$. Applying the conclusion of the previous step to

$$\frac{dy}{dx} = f^{\omega'}(x, y),$$

we obtain that (49) defines a function in $D_2^x \times {D_r}^y - \{x = 0\}$. From (54) applied to

$$\frac{dy}{dx} = f^{\omega}(x, y)$$

and from (49) we get

$$|\xi(x, y)| < |\phi({g_{\omega'}}^{-1}(x, 1, y))| 2\pi\, e^{3p/2q},$$

and therefore by supposing $|y| < r$ and using the fact that $|\phi(y)| < L|y\,|$ we obtain the boundness of ξ in ${D_2}^x \times {D_r}^y - \{x = 0\}$.

Third step: ξ *is a solution of (46).* We only need to prove that the two different definitions of $\xi|_{S_1}$, namely the one given by the initial datum

$$\xi(1, y) = \phi(y)$$

and the one given by (49)

$$\xi(1, y) = g_{\omega}(e^{2\pi i}, 1, \phi(g_{\omega'}(e^{-2\pi i}, 1, y))),$$

coincide.

From the definition of holonomy the last equality becomes

$$\xi(1, y) = \phi(y) = h_{\omega} \circ \phi^{-1} \circ {h_{\omega'}}^{-1}$$

and from (47) $\xi(x, y)$ is well-defined.

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Fourth step: Φ is a germ of a diffeomorphism. As

$$\det \frac{\partial \Phi}{\partial (x, y)}\bigg|_{(0,0)} = \frac{\partial \xi}{\partial y}(0, 0)$$

the claim of this step will be proved if we prove that

$$\frac{\partial \xi}{\partial y}(0, 0) = \phi'(0). \tag{55}$$

Let us introduce the following function, expressing the fact that the conjugation of the holonomies depends on the base point

$$\phi_{x_0}(y) = \xi(x_0, y), \tag{56}$$

where now $x_0 < 1$. From (47) we get

$$\phi_0(y) = h_{\omega, x_0} \circ \phi_{x_0}^{-1} \circ {h_{\omega', x_0}}^{-1},$$

where h_{ω,x_0}, h_{ω',x_0} are the holonomies with respect to the transversal $S_{x_0} = \{(x_0, y) \mid |y| < r(x_0)\}$. From (56) we get

$$\frac{\partial \xi}{\partial y}(0, 0) = {\phi_{x_0}}'(0)$$

and (55) will follow if we prove that $x_0 \mapsto {\phi_{x_0}}'(0)$ is constant. Let x_0, ${x_0}'$ be such that $|x_0|$, $|{x_0}'| < 1$ and let $t \mapsto \tilde{\gamma}(t)$, $0 \le t \le 1$, be such that $\tilde{\gamma}(0) = x_0$, $\tilde{\gamma}(1) = {x_0}'$. We have

$$\phi_{{x_0}'}(y) = \xi({x_0}', y)$$

and also

$$\xi({x_0}', y) = g_\omega({x_0}', x_0, \phi_{x_0}({g_{\omega'}}^{-1}({x_0}', x_0, y))),$$

where the flows are relative to the path $\tilde{\gamma}$. Therefore,

$$\phi_{{x_0}'}(y) = g_\omega({x_0}', x_0, \phi_{x_0}({g_{\omega'}}^{-1}({x_0}', x_0, y)))$$

and so

$${\phi'}_{{x_0}'}(0)y + \mathcal{O}(y^2) = e^{-2\pi p/q}({\phi_{x_0}}' e^{-2\pi p/q})y + \mathcal{O}(y^2)$$

and finally

$${\phi'}_{{x_0}'}(0) = {\phi'}_{x_0}(0).$$

Fifth step: an analytic division lemma. We have already proved that

$$\Phi^* \omega \wedge \omega' = 0$$

holds in ${D_2}^x \times {D_r}^y$. We prove now that this is equivalent to

$$\Phi^\star \omega - u\omega = 0 \tag{57}$$

for $\Phi \in \mathrm{Diff}(\mathbb{C}^2, 0)$ and u a germ at the origin of a unit. This conclusion follows from De Rham's division lemma, but we report here another proof because it can be adapted,

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in an obvious way, to the n-jet case of Lemma 3.3. Making (57) explicit we get the system

$$\omega_1(x,\xi) + \omega_2(x,\xi)\frac{\partial \xi}{\partial x} = u\omega_1'(x,y) \tag{58}$$

$$\omega_2(x,\xi)\frac{\partial \xi}{\partial y} = u\omega_2'(x,y). \tag{59}$$

Hence, dividing for $\omega_2'(x,y) \equiv x$ we obtain

$$u(x,y) = \frac{\partial \xi}{\partial y}(x,y) \tag{60}$$

which substituted in (58) gives exactly the differential equation in (46). Moreover, $u(x,y)$ is a unit as follows from (55) and (60). This ends the proof. □

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