Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2017.s07

Vol. **10**, No. 2, pp. 331-350
May 2017

High Order Mass-Lumping Finite Elements on Simplexes

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Received 19 October 2016; Accepted (in revised version) 17 January 2017

Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. This paper is concerned with the construction of high order mass-lumping finite elements on simplexes and a program for computing mass-lumping finite elements on triangles and tetrahedra. The polynomial spaces for mass-lumping finite elements, as proposed in the literature, are presented and discussed. In particular, the unisolvence problem of symmetric point-sets for the polynomial spaces used in mass-lumping elements is addressed, and an interesting property of the unisolvent symmetric point-sets is observed and discussed. Though its theoretical proof is still lacking, this property seems to be true in general, and it can greatly reduce the number of cases to consider in the computations of mass-lumping elements. A program for computing mass-lumping finite elements on triangles and tetrahedra, derived from the code for computing numerical quadrature rules presented in [7], is introduced. New mass-lumping finite elements on triangles found using this program with higher orders, namely 7, 8 and 9, than those available in the literature are reported.

AMS subject classifications: 65M60, 65D05, 65D32

Key words: Finite element, mass-lumping, simplex, numerical quadrature, multivariate polynomial interpolation, unisolvence.

1. Introduction

In this paper we refer by mass-lumping finite elements or simply mass-lumping elements, also known as spectral elements, to a special class of H^1-conforming Lagrangian finite element bases with (approximately) orthogonal basis functions [1, 2].

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Mass-lumping elements may have advantages over traditional elements when solving some types of problems. For example, for time dependent problems, such as time domain wave equations, when explicit time stepping is used, one obtains diagonal mass matrices with mass-lumping elements, thus avoiding solutions of large sparse linear systems of equations. Another example is eigenvalue problems arising from partial differential equations, the resulting discrete algebraic systems are standard eigenvalue problems when using mass-lumping elements, instead of generalized eigenvalue problems as obtained using traditional elements. For the case of one dimension, it is easy to construct mass-lumping elements using Gauss-Lobatto quadrature points. For two or higher dimensions and tensor-product elements such as quadrilaterals and hexahedra, mass-lumping elements can be constructed using tensor products of one dimensional elements. For two or higher dimensions and simplicial elements such as triangles (2D) and tetrahedra (3D), the construction of mass-lumping elements of orders higher than 1 becomes highly non trivial and there is still no systematic way of doing it. In the past two decades, numerical and symbolic algorithms have been designed to compute or search high order mass-lumping elements. So far, mass-lumping elements up to order 6 for triangles and order 3 for tetrahedra have been reported [1–5]. A 7-th order basis for triangles has been reported in [6], but its numerical quadrature precision is not high enough to ensure the approximation order of numerical solutions.

The purpose of this paper is to address problems related to the construction of mass-lumping elements on simplexes. More precisely, we give a systematic presentation of the class of polynomial spaces for constructing mass-lumping elements as proposed in the existing literature and discuss their properties as well as some related issues, including H^1-conformity and unisolvence of symmetric point-sets. An interesting property about unisolvent symmetric point-sets for the polynomial spaces used in the mass-lumping elements is revealed which, we believe, not only is useful in the computation of mass-lumping elements, but also has important implications in the more general field of polynomial interpolations. A program capable of computing high order mass-lumping elements on triangles and tetrahedra is introduced. It is derived from the program for computing symmetric quadrature rules on triangles and tetrahedra presented in [7]. With this program we were able to find out some new mass-lumping elements on triangles with higher orders than those reported in the literature. This work, as well as the work presented in [7], is a part of the project for developing the parallel adaptive finite element toolbox Parallel Hierarchical Grid (PHG) [8, 9].

The rest of the paper is organized as follows. Section 2 presents basic concepts, notations and properties used in this paper which are related to numerical quadrature and Lagrangian interpolation on simplexes. In Section 3, we discuss the problem of constructing mass-lumping elements, address some related issues including unisolvence of symmetric point-sets on simplexes, and present our numerical algorithm for computing mass-lumping elements on triangles and tetrahedra. In Section 4, we present the mass-lumping elements found using our program. The new elements, including elements of 7, 8 and 9-th order, are validated with numerical tests. Finally in Section 5, we conclude the paper with some remarks.

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2. Preliminaries

We first introduce some notations and definitions used in this paper.

$\mathbb{I}$ is used to denote the set of integers. d, $d \in \mathbb{I}$ and $d \geq 0$, is used to denote the space dimensions, and $\mathbf{S}_d$ is used to denote a d-simplex in $\mathbb{R}^d$, which is defined by $d+1$ affinely independent points $\{v_0, \cdots, v_d\} \subset \mathbb{R}^d$, called vertices, and is regarded in this paper as an open set of $\mathbb{R}^d$:

$$\mathbf{S}_d = \Big\{ x = \sum_{i=0}^{d} \lambda_i v_i \in \mathbb{R}^d \Big| \lambda_i \in (0,1), \ \sum_{i=0}^{d} \lambda_i = 1 \Big\},$$

in which $(\lambda_0, \cdots, \lambda_d)$ are called the barycentric coordinates of $\mathbf{S}_d$.

$\mathbb{P}(\mathbf{S}_d; k)$, $k \in \mathbb{I}$, is used to denote the vector space consisting of all d-variate polynomials on $\mathbf{S}_d$ with degrees less than or equal to k if $k \geq 0$, and an empty set if $k < 0$. For convenience of discussions, $\mathbb{P}(\mathbf{S}_d; k)$ will also be regarded as the space of $(d+1)$-variate polynomials in the barycentric coordinates $\lambda_0, \cdots, \lambda_d$ in this paper. It is not difficult to verify that the homogeneous set of monomials of order k in the barycentric coordinates,

$$\Big\{ \prod_{i=0}^{d} \lambda_i^{k_i} \Big| k_i \in \mathbb{I}, \ 0 \leq k_i \leq k, \ \sum_{i=0}^{d} k_i = k \Big\}, \tag{2.1}$$

forms a basis for $\mathbb{P}(\mathbf{S}_d; k)$. This basis is very convenient in both analysis and computation, especially when symmetry is involved.

Given l, $l \in \mathbb{I}$ and $0 \leq l \leq d$, we denote by $\mathbb{S}_d^l$ the set of l-subsimplexes of $\mathbf{S}_d$. Note that a 0-subsimplex is a vertex of $\mathbf{S}_d$, and $\mathbb{S}_d^0$ is the set of the four vertices of $\mathbf{S}_d$. A subsimplex $s_l \in \mathbb{S}_d^l$ is identified by an l-component index-set $\{i_0, \cdots, i_l\} \subset \{0, \cdots, d\}$, with $0 \leq i_0 < \cdots < i_l \leq d$. For convenience, when no confusion may arise, a subsimplex $s_l \in \mathbb{S}_d^l$ will be indifferently regarded as a subset of $\mathbb{R}^d$ or as an index-set, such that the notations $s_l \subset \mathbb{R}^d$ and $s_l = \{i_0, \cdots, i_l\} \in \mathbb{S}_d^l$ both make sense, and $\mathbb{P}(s_l; k)$ will be regarded as the space of $(l+1)$-variate polynomials in $\lambda_{i_0}, \cdots, \lambda_{i_l}$ of degrees less than or equal to k.

With the notations above, we give the definition of hierarchical polynomial space, which is used as the polynomial spaces for mass-lumping elements.

Definition 2.1. *The hierarchical polynomial space $\mathbb{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ of orders $k_0, \cdots, k_d$, where $k_i \in \mathbb{I}$ and $1 \leq k_0 \leq \cdots \leq k_d$, is defined as follows:*

$$\mathbb{P}(\boldsymbol{S}_d; k_0, \cdots, k_d) = \bigcup_{l=0}^{d} \bigcup_{s_l \in \mathbb{S}_d^l} \big(B(s_l) \cdot \mathbb{P}(s_l; k_l - l - 1) \big), \tag{2.2}$$

where $B(s_l)$ and $\mathbb{P}(s_l; k_l - l - 1)$ denote respectively the bubble function and the polynomial space on the subsimplex s_l, i.e., if $s_l = \{i_0, \cdots, i_l\}$, then

$$\begin{cases} B(s_l) = \Pi_{j=0}^{l} \lambda_{i_j}, \\ \mathbb{P}(s_l; k_l - l - 1) = \textit{Span} \Big\{ \lambda_{i_0}^{p_0} \cdots \lambda_{i_l}^{p_l} \mid p_k \in \mathbb{I}, \ p_k \geq 0, \ p_0 + \cdots + p_l = k_l - l - 1 \Big\}. \end{cases}$$

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Let $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ be a hierarchical polynomial space, then k_0 is called its vertex order, k_1 its edge order, k_2 its face order, and so on. It is clear that

$$\mathbb{P}(\mathbf{S}_d; k_0) \subset \mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d) \quad \text{and} \quad \mathbb{P}(\mathbf{S}_d; \underbrace{k, \cdots, k}_{d+1}) = \mathbb{P}(\mathbf{S}_d; k).$$

It can be verified that the following set of monomials forms a basis for $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$:

$$\bigcup_{l=0}^{d} \bigcup_{\{i_0,\cdots,i_l\}\in\mathbb{S}_d^l} \left\{ \prod_{j=0}^{l} \lambda_{i_j}^{m_j} \;\middle|\; m_j \in \mathbb{I},\ 1 \le m_j \le k_l,\ \sum_{j=0}^{l} m_j = k_l \right\}. \tag{2.3}$$

Remark 2.1. In the existing literature, the vertex order and the edge order of the polynomial spaces for mass-lumping elements are usually the same, i.e., $k_0 = k_1$ (see, e.g., [1, 2]). Here $k_0 \neq k_1$ is allowed to provide more flexibility in the designing of mass-lumping elements. In fact, the 9-th order element found by us and reported later in this paper does have different vertex and edge orders.

2.1. Symmetric quadrature rules on a simplex

Let $\mathbf{S}_d$ be a d-simplex. A numerical quadrature rule or simply quadrature rule (also called a cubature rule for $d > 1$ in the literature) on $\mathbf{S}_d$ is defined to be a set of (point, weight) pairs $\{(p_i, w_i) \mid i = 1, \cdots, n\}$, such that for any function $f(x)$ defined on a domain containing $\mathbf{S}_d \cup \{p_i \mid i = 1, \cdots, n\}$, its integral on $\mathbf{S}_d$ can be approximately computed by:

$$\int_{\mathbf{S}_d} f(x) \approx |\mathbf{S}_d| \sum_{i=1}^{n} f(p_i) w_i, \tag{2.4}$$

where $|\mathbf{S}_d|$ denotes the measure (length if $d = 1$, area if $d = 2$ and volume if $d = 3$) of $\mathbf{S}_d$, and $n \in \mathbb{N}$ is the number of points of the quadrature rule. p_i and w_i are respectively called the points and associated weights of the quadrature rule.

A quadrature rule is said to be of (algebraic) order p if (2.4) is exact for all polynomials of degrees less than or equal to p.

A quadrature rule on a simplex is said to be symmetric if it is invariant under permutations of the barycentric coordinates. More precisely, let $(\lambda_0, \cdots, \lambda_d)$ be a quadrature point associated with the weight w, then for any permutation $(i_0, \cdots, i_d)$ of $(0, \cdots, d)$, the point $(\lambda_{i_0}, \cdots, \lambda_{i_d})$ is also a quadrature point associated with the same weight. For a symmetric quadrature rule, the set of quadrature points can be grouped into symmetry orbits, with each orbit containing all the points generated by permuting the barycentric coordinates of a single point. The symmetry orbits can be classified into a number of orbit types, which are summarized in Tables 1 and 2 for the cases of triangles ($d = 2$) and tetrahedra ($d = 3$). Please see [7] for more details and discussions on symmetric quadrature rules.

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Table 1: Orbit types for $d = 2$ (triangle), a, b and c must be strictly distinct numbers.

Orbit type	Barycentric coordinates	# points
$S_3(\frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	1
$S_{21}(a)$	(a, a, b), with $b = 1 - 2a$	3
$S_{111}(a, b)$	(a, b, c), with $c = 1 - a - b$	6

Table 2: Orbit types for $d = 3$ (tetrahedron), a, b, c and d must be strictly distinct numbers.

Orbit type	Barycentric coordinates	# points
$S_4(\frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	1
$S_{31}(a)$	(a, a, a, b), with $b = 1 - 3a$	4
$S_{22}(a)$	(a, a, b, b), with $b = (1 - 2a)/2$	6
$S_{211}(a, b)$	(a, a, b, c), with $c = 1 - 2a - b$	12
$S_{1111}(a, b, c)$	(a, b, c, d), with $d = 1 - a - b - c$	24

2.2. Lagrangian elements on uniform grids

Let $\mathbf{S}_d$ be a d-simplex. A Lagrangian element on $\mathbf{S}_d$ is defined by a polynomial space $\mathbb{P}$ and a point-set $\mathcal{P} = \{p_1, \cdots, p_n\} \subset \overline{\mathbf{S}}_d$ with $n = \dim(\mathbb{P})$, which satisfy the following conditions:

Condition 1. The point-set $\mathcal{P}$ should be $\mathbb{P}$ unisolvent. More precisely, let $\{f_1, \cdots, f_n\}$ be a basis of $\mathbb{P}$, then $\det[f_i(p_j)]_{n\times n} \neq 0$ (the matrix $[f_i(p_j)]_{n\times n}$ is called the interpolation or Vandermonde matrix).

Condition 2. For any subsimplex $s_l \in \mathbb{S}_d^l$, $0 \leq l < d$, the point-set $\mathcal{P} \cap \overline{s}_l$ should be fully symmetric with respect to s_l, i.e., invariant under permutations of the barycentric coordinates of s_l, and contain necessary number of points to ensure H^1-conformity.

Condition 3. $\mathbb{P}$ should contain $\mathbb{P}(\mathbf{S}_d; k)$ as its subspace for some $k \geq 1$.

Condition 1 is a basic requirement of Lagrange interpolation, Condition 2 is necessary for H^1-conformity, and Condition 3 ensures approximation property of the element.

The most popular class of Lagrangian elements is $\mathbb{P}(\mathbf{S}_d; k)$ on the order k uniform grid on $\mathbf{S}_d$. Using barycentric coordinates, the order k uniform grid, denoted by $\mathcal{P}(\mathbf{S}_d; k)$ in this paper, also called a principal lattice in the literature [10, 11], is defined by

$$\mathcal{P}(\mathbf{S}_d; k) = \left\{ \left(\frac{k_0}{k}, \cdots, \frac{k_d}{k} \right) \middle| k_i \in \mathbb{I},\ k_i \geq 0,\ \sum_{i=0}^{d} k_i = k \right\}. \tag{2.5}$$

The $\mathbb{P}(\mathbf{S}_d; k)$ unisolvence of $\mathcal{P}(\mathbf{S}_d; k)$ is well known (see, e.g., [10, 11]), which we state as the following lemma for later reference.

Lemma 2.1. *For any $k \geq 1$, $\mathcal{P}(\mathbf{S}_d; k)$ is $\mathbb{P}(\mathbf{S}_d; k)$ unisolvent.*

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The Conditions 2 and 3 for a Lagrangian element can also be readily verified for $\mathbb{P}(\mathbf{S}_d;k)$ and $\mathcal{P}(\mathbf{S}_d;k)$. Thus they form a well defined Lagrangian element, which is the most commonly used "P_k" element in the finite element literature.

Now we extend the notion of uniform grid. Let $k>d$. We denote by $\mathcal{P}_0(\mathbf{S}_d;k)$ the set of interior points of $\mathcal{P}(\mathbf{S}_d;k)$. It is easy to see that $\mathcal{P}_0(\mathbf{S}_d;k)$ is given by

$$\mathcal{P}_0(\mathbf{S}_d;k)=\left\{\left(\frac{k_0}{k},\cdots,\frac{k_d}{k}\right)\middle| k_i\in\mathbb{I},\ k_i\geq 1,\ \sum_{i=0}^{d}k_i=k\right\}. \tag{2.6}$$

For the convenience of discussions, we set $\mathcal{P}_0(\mathbf{S}_d;k)=\emptyset$ for $k\leq d$.

Lemma 2.2. *For any $k>d$, $\mathcal{P}_0(\boldsymbol{S}_d;k)$ is $\mathbb{P}_0(\boldsymbol{S}_d;k)$ unisolvent, where the polynomial space $\mathbb{P}_0(\boldsymbol{S}_d;k)$ is defined by*

$$\mathbb{P}_0(\boldsymbol{S}_d;k)=B(\boldsymbol{S}_d)\cdot\mathbb{P}(\boldsymbol{S}_d;k-d-1).$$

Proof. First, one can easily verify that $\dim\big(\mathbb{P}_0(\mathbf{S}_d;k)\big)=\dim\big(\mathcal{P}_0(\mathbf{S}_d;k)\big)$.

Let $m=\dim\big(\mathbb{P}_0(\mathbf{S}_d;k)\big)$ and $n=\dim\big(\mathbb{P}(\mathbf{S}_d;k)\big)$. Let $\{f_1,\cdots,f_m,f_{m+1},\cdots,f_n\}$ be the basis functions (2.1) for $\mathbf{S}_d$, where $\{f_i\}$ are monomials in the barycentric coordinates ordered in a way such that for $i>m$ at least one exponent in f_i is zero and for $i\leq m$ all the exponents in f_i are nonzero.

Let $\{p_1,\cdots,p_m,p_{m+1},\cdots,p_n\}$ be the points of $\mathcal{P}(\mathbf{S}_d;k)$, ordered in a similar way as the basis functions $\{f_i\}$, i.e., for $i>m$ at least one barycentric coordinate of p_i is zero and for $i\leq m$ all the barycentric coordinates are nonzero.

Then it is clear that $\{f_1,\cdots,f_m\}$ forms a basis for $\mathbb{P}_0(\mathbf{S}_d;k)$ and $\mathcal{P}_0(\mathbf{S}_d;k)=\{p_1,\cdots,p_m\}$. Let $A=[f_i(p_j)]_{n\times n}$ $(1\leq i,j\leq n)$ and $A_0=[f_i(p_j)]_{m\times m}$ $(1\leq i,j\leq m)$, then the matrix A can be written in the following block-triangular form:

$$A=\begin{pmatrix}A_0 & 0\\ * & *\end{pmatrix}.$$

By Lemma 2.1 we have $\det(A)\neq 0$, thus $\det(A_0)\neq 0$, and Lemma 2.2 follows. □

The hierarchical uniform grid, defined below, extends the notion of uniform grids.

Definition 2.2. *The hierarchical uniform grid $\mathcal{P}(\mathbf{S}_d;k_0,\cdots,k_d)$ of orders $k_0,\cdots,k_d$, where $k_i\in\mathbb{I}$ and $1\leq k_0\leq\cdots\leq k_d$, is recursively defined as follows:*

$$\mathcal{P}(\boldsymbol{S}_0;k_0)=\big\{(\lambda_0=1)\big\}, \tag{2.7a}$$

$$\mathcal{P}(\boldsymbol{S}_d;k_0,\cdots,k_d)=\mathcal{P}_0(\boldsymbol{S}_d;k_d)\bigcup\left(\bigcup_{s\in\mathbb{S}_d^{d-1}}\mathcal{P}(s;k_0,\cdots,k_{d-1})\right),\quad d>0, \tag{2.7b}$$

where $\mathcal{P}_0(\cdot;\cdot)$ is defined by (2.6).

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Theorem 2.1. *For any $d \geq 0$ and $1 \leq k_0 \leq \cdots \leq k_d$, the point-set $\mathcal{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ is $\mathbb{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ unisolvent.*

Proof. It is not difficult to see that by properly ordering the basis functions in (2.3) and the points in $\mathcal{P}(\mathbf{S}_d; k_0, \cdots, k_d)$, in a similar way as in the proof of Lemma 2.2, the matrix $[f_i(p_j)]$ is a block-triangular one with each diagonal block corresponding to $\mathbb{P}_0(s_l; k_l)$ and $\mathcal{P}_0(s_l; k_l)$ for some subsimplex $s_l \in \mathbb{S}_d^l$. By Lemma 2.2 all the diagonal blocks have nonzero determinant, thus the whole matrix has nonzero determinant, and the theorem follows. □

It can be readily verified that $\mathcal{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ and $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ also satisfy Conditions 2 and 3. Thus they form a well-defined Lagrangian element, which is an extension to the traditional Lagrangian element P_k.

Remark 2.2. The point-set $\mathcal{P}(\mathbf{S}_d; k_0, k_1, \cdots, k_d)$ is independent of k_0 for $1 \leq k_0 \leq k_1$, while the corresponding polynomial space $\mathbb{P}(\mathbf{S}_d; k_0, k_1, \cdots, k_d)$ does depend on k_0.

3. Mass-lumping finite elements

When the basis functions of a Lagrangian element are mutually orthogonal, the element is said to be mass-lumped or mass-lumping, and called a mass-lumping finite element, or simply mass-lumping element.

Let $\{f_1, \cdots, f_n\}$ and $\{p_1, \cdots, p_n\}$ be respectively the basis functions and interpolation points of a Lagrangian element on $\mathbf{S}_d$ such that $f_i(p_j) = \delta_{ij}$. The element is mass-lumping if its (element) mass matrix $[(f_i, f_j)]_{n\times n}$ is a diagonal matrix, here

$$(f_i, f_j) = \int_{\mathbf{S}_d} f_i(x) f_j(x) \mathrm{d}x.$$

Let $\{q_1, \cdots, q_m\}$ and $\{w_1, \cdots, w_m\}$ be respectively the points and weights of the numerical quadrature rule used to compute the mass matrix, then we have

$$(f_i, f_j) \approx |\mathbf{S}_d| \sum_{k=1}^{m} w_k f_i(q_k) f_j(q_k).$$

In this case it is clear that the element is mass-lumping if the interpolation points coincide with the numerical quadrature points, i.e., $n = m$ and $\{p_1, \cdots, p_n\} = \{q_1, \cdots, q_m\}$. For numerical stability all the weights w_k should be strictly positive for a mass-lumping element.

Thus, for a given polynomial space $\mathbb{P}$, the problem of constructing a mass-lumping element can be transformed to the problem of finding a numerical quadrature rule satisfying the following properties:

Property 1. The point-set should satisfy Conditions 1 and 2 for a Lagrangian element, i.e., it is $\mathbb{P}$ unisolvent, is symmetric and has required number of points on any subsimplex of $\mathbf{S}_d$ for H^1-conformity.

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Property 2. The order of the numerical quadrature rule should be high enough to preserve the precision of the finite element solution, and all the weights should be positive to ensure numerical stability of finite element computations.

For the polynomial space $\mathbb{P}(\mathbf{S}_d; k)$, exact computation of the mass matrix requires the order of the numerical quadrature rule to be at least $2k$, but it is in general impossible for $d > 1$ and unnecessary to find a quadrature rule of order $2k$ satisfying Properties 1 and 2.

It is proved that for many 2nd order partial differential equations it is sufficient to use a numerical quadrature rule of order $2k - 2$ to preserve the precision of the finite element solutions, see, e.g., [1] and the references therein. But even with this awakened requirement, one still cannot find a numerical quadrature rule with required order and satisfying Properties 1 and 2 for $k > 1$.

In order to overcome this difficulty encountered with traditional full-order polynomial spaces, people have proposed to use the hierarchical polynomial spaces for mass-lumping elements [1, 2]. More precisely, for the polynomial space $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$, one looks for a numerical quadrature rule of order $k_0 + k_d - 2$, which has all positive weights and satisfies Property 1, i.e., Conditions 1 and 2, for $\mathbb{P} = \mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$. In the following discussions, we will first deal with Condition 2 in Subsection 3.1, then return to Condition 1 in Subsection 3.2.

3.1. Constrained orbits for mass-lumping finite elements

Let $\mathcal{P}$ be the point-set of a mass-lumping element for $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$. It is not difficult to see that Condition 2 is equivalent to:

For any subsimplex $s_l \in \mathbb{S}_d^l$ ($l < d$) the point-set

$$\mathcal{P}_l \stackrel{\text{def}}{=} \overline{s}_l \cap \mathcal{P}$$

satisfies the following conditions:

(a) $\mathrm{card}(\mathcal{P}_l) = \dim\big(\mathbb{P}(s_l; k_0, \cdots, k_l)\big)$;

(b) $\mathcal{P}_l$ is fully symmetric on s_l;

(c) $\mathcal{P}_l$ is $\mathbb{P}(s_l; k_0, \cdots, k_l)$ unisolvent.

where $\mathrm{card}(\mathcal{P}_l)$ denotes the number of elements in the set $\mathcal{P}_l$.

Thanks to the block-triangular structure nature of the Vandermonde matrix and using similar arguments as in the proofs of Lemma 2.2 and Theorem 2.1, it can be shown that the condition (c) above will be automatically satisfied if the whole point-set $\mathcal{P}$ is $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ unisolvent. Evidently the condition (b) will be satisfied if the whole point-set $\mathcal{P}$ is symmetric on $\mathbf{S}_d$. Thus we only need to look for a symmetric numerical quadrature rules which has a $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ unisolvent point-set $\mathcal{P}$ satisfying the condition (a), i.e.,

$$\mathrm{card}(\overline{s}_l \cap \mathcal{P}) = \dim\big(\mathbb{P}(s_l; k_0, \cdots, k_l)\big), \quad \forall l < d \text{ and } s_l \in \mathbb{S}_d^l. \tag{3.1}$$

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Table 3: Constrained orbits for the case of triangles ($d = 2$).

Orbit type	Position	Number of orbits	Restrictions on the abscissas
$S_{21}(0)$	vertex	1	*
$S_{21}(1/2)$	edge center	$c_{21} = 0$ or 1	*
$S_{111}(0, a)$	edge	c_{111}	$a \in (0, 1)$ and $a \neq 1/2$
Requirement for H^1-conformity: $c_{21} + 2c_{111} = k_1 - 1$			

Table 4: Constrained orbits for the case of tetrahedra ($d = 3$).

Orbit type	Position	Number of orbits	Restrictions on the abscissas
$S_{31}(0)$	vertex	1	*
$S_{31}(1/3)$	face center	$c_{31} = 0$ or 1	*
$S_{22}(0)$	edge center	$c_{22} = 0$ or 1	*
$S_{211}(0, a)$	edge	c_{211e}	$a \in (0, 1)$ and $a \neq 1/2$
$S_{211}(a, 0)$	face	c_{211f}	$a \in (0, 1/2)$
$S_{1111}(0, a, b)$	face	c_{1111}	a, b and $1 - a - b$ are all distinct and $\in (0, 1)$
Requirement for H^1-conformity: $\begin{cases} c_{22} + 2c_{211e} = k_1 - 1 \\ c_{31} + 3c_{211f} + 6c_{1111} = (k_2 - 1)(k_2 - 2)/2 \end{cases}$			

Using recurrence on l, it can be shown that the above condition is equivalent to

$$\operatorname{card}(s_l \cap \mathcal{P}) = \dim\big(\mathbb{P}_0(s_l; k_l)\big), \quad \forall l < d \text{ and } s_l \in \mathbb{S}^l_d. \tag{3.2}$$

Note that $s_l \cap \mathcal{P}$ is the set of interior points of $\overline{s}_l \cap \mathcal{P}$ (in $\mathbb{R}^l$).

When using symmetry orbits to describe a point-set, the condition in (3.2) can be transformed into the existence in the point-set of certain special orbits, called constrained orbits. Tables 3 and 4 list the types of constrained orbits, as well as the requirements on the number of each type of constrained orbits in order to satisfy (3.2), for the cases of triangles ($d = 2$) and tetrahedra ($d = 3$) respectively. The number of each type of constrained orbits is denoted by a variable of the form c_{xxx} in the tables.

Remark 3.1. c_{21}, c_{31} and c_{22} can only take the value 0 or 1.

For the case $d = 2$, c_{21} and c_{111} are uniquely determined by k_1.

For the case $d = 3$, c_{22} and c_{211e} are uniquely determined by k_1, c_{31} is uniquely determined by k_2, but there may be multiple solutions of c_{211f} and c_{1111} for a given value of k_2, and each solution, combined with the values of c_{31}, c_{22} and c_{211e}, defines a symmetric H^1-conforming point-set for $\mathbb{P}(\mathbf{S}_3; k_0, k_1, k_2, k_3)$.

3.2. Unisolvence of symmetric point-sets

We now address the problem of unisolvence, i.e., Condition 1. The following theorem states existence of unisolvent H^1-conforming symmetric point-sets for any hierarchical polynomial space, which is a direct consequence of Theorem 2.1.

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Theorem 3.1. *Let $\boldsymbol{S}_d$ be any d-simplex for $d \geq 1$ and $1 \leq k_0 \leq \cdots \leq k_d$. Then the set of symmetric point-sets satisfying Property 1 with $\mathbb{P} = \mathbb{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ is non empty.*

Each symmetric point-set has an associated orbit pattern, represented by the numbers of different orbit types. For example, the orbit pattern of the uniform grid $\mathcal{P}(\mathbf{S}_2; 4)$ consists of 0 S_3 orbit, 3 S_{21} orbits and 1 S_{111} orbit, which is denoted as 0:3:1, while the orbit pattern of the uniform grid $\mathcal{P}(\mathbf{S}_3; 4)$ consists of 1 S_4 orbit, 1 S_{31} orbit, 1 S_{22} orbit, 2 S_{211} orbits and 0 S_{1111} orbit, which is denoted as 1:1:1:2:0.

For a given hierarchical polynomial space $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$, H^1-conforming symmetric point-sets with different orbit patterns exist, and the unisolvence of a symmetric point-set is clearly related to its orbit pattern. A natural question one can ask is the following one:

Among the orbit patterns satisfying H^1-conformity for $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$, which ones admit unisolvent point-sets?

By Theorem 2.1 we know that the orbit pattern of $\mathcal{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ admits a non empty set of $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ unisolvent point-sets which contains $\mathcal{P}(\mathbf{S}_d; k_0, \cdots, k_d)$. An interesting observation is that, even without the requirement for H^1-conformity, it is the only orbit pattern admitting $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ unisolvent point-sets. This statement can be directly verified for given space dimensions d and orders $k_0, \cdots, k_d$, either numerically or symbolically using a symbolic mathematics software like Mathematica (`http://www.wolfram.com`) or Sage (`http://www.sagemath.org`), but we don't know yet whether it can be proved for the general cases. Since theoretical analysis of this problem is out of the scope of this paper, we leave it here as a conjecture.

Conjecture 3.1. *For any given space dimensions $d \geq 0$ and orders $k_0, \cdots, k_d$ with $1 \leq k_0 \leq \cdots \leq k_d$, among all the symmetric point-sets, only those with the same orbit pattern as $\mathcal{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ may be $\mathbb{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ unisolvent. In other words, among all the possible orbit patterns, only the orbit pattern of $\mathcal{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ admits $\mathbb{P}(\boldsymbol{S}_d; k_0, \cdots, k_d)$ unisolvent symmetric point-sets.*

Remark 3.2. When H^1-conformity is satisfied, the Vandermonde matrix $[f_i(p_j)]$ has a block-triangular form if the basis functions and the points are properly ordered, with each diagonal block corresponding to the interior basis functions of a subsimplex s_l and the point-set $\mathcal{P}_0(s_l; k_l)$ (see the proofs of Lemma 2.2 and Theorem 2.1). In this case, Conjecture 3.1 is equivalent to:

For any $d \geq 1$ and $k > d$, among all symmetric point-sets, only those with the same orbit pattern as $\mathcal{P}_0(\mathbf{S}_d; k)$ may be $\mathbb{P}_0(\mathbf{S}_d; k)$ unisolvent.

The above condition suffices for our purpose here, and is much easier to check, whether numerically or symbolically, than Conjecture 3.1.

Remark 3.3. Symmetric quadrature rules of order $k_0 + k_d - 2$ with different orbit patterns than that of $\mathcal{P}(\mathbf{S}_d; \cdots, k_d)$ and satisfying H^1-conformity for $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ do exist.

For example, the orbit pattern of $\mathcal{P}(\mathbf{S}_2; 6, 6, 9)$ is 1:5:5, while when searching for symmetric quadrature rules of order 13 satisfying H^1-conformity for $\mathbb{P}(\mathbf{S}_2; 6, 6, 9)$, solutions with the orbit pattern 1:7:4 have been found using our program presented in

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the next subsection, and a solution with the orbit pattern $1{:}3{:}6$ has also been reported in [5] which corresponds to the case $\{6, 9, 13, 1\}$ there. All these solutions are of course not $\mathbb{P}(\mathbf{S}_2; 6, 6, 9)$ unisolvent according to Conjecture 3.1, and are thus not useful for mass-lumping elements (this answers a related question raised by the author in [5], that is, there would be no useful solution for the case $\{6, 9, 13, 1\}$).

Remark 3.4. M. J. Roth observed in his PhD thesis that for the polynomial space $\mathbb{P}(\mathbf{S}_2; k)$, regardless of quality measure, the best configurations of symmetric point-sets for Lagrangian interpolation satisfy the condition $s_{21} + s_3 = \lfloor \frac{k}{2} \rfloor + 1$, where s_{21} and s_3 denote respectively the number of S_{21} and S_3 orbits (see [12], (2.26) on page 15), which is in accordance with Conjecture 3.1.

3.3. A program for computing mass-lumping finite elements

From now on we restrict ourselves to the cases $d = 2$ and $d = 3$. As discussed in Subsection 3.1, the problem of finding a mass-lumping element with a symmetric point-set for $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ is equivalent to finding a symmetric quadrature rule of order $k_0 + k_d - 2$ with all positive weights, whose point-set satisfies the following conditions:

- The number of points must equal to $\dim\big(\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)\big)$.
- It must include exactly the required constrained orbits as listed in Tables 3-4, and the remaining points must all be interior points of $\mathbf{S}_d$.

In [7] we have presented a program called "`quadrule`" for computing symmetric numerical quadrature rules of arbitrary orders on triangles and tetrahedra. This program has been extended to allow specifying the number of orbits for each constrained orbit type, thus can be used to compute mass-lumping elements. The steps for computing mass-lumping elements for given space dimensions d and orders $k_0, \cdots, k_d$ using this program are given in Algorithm 3.1.

The program `quadrule` allows to specify the orbit pattern "`s#`" and constrained orbits "`c#`" in the command-line arguments in the form

```
s3 s21:(1+2c21) s111:c111
```

for $d = 2$, and

```
s4 s31:(1+2c31) s22:c22 s211:c211e:c211f s1111:c1111
```

for $d = 3$. Below are two examples of its usage.

Example 3.1. Suppose $d = 2$ and $(k_0, k_1, k_2) = (4, 4, 5)$, then the quadrature order is 7, the number of points is 18, the numbers of constrained orbits are $c_{21} = 1$, $c_{111} = 1$, and the orbit pattern for the hierarchical uniform grid is $s_{21} = 4$, $s_{111} = 1$, thus `quadrule` can be called with

```
quadrule 2 7 18 0 4:3 1:1 100
```

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Algorithm 3.1 Steps for computing mass-lumping finite elements for the polynomial space $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ using the `quadrule` program.

1. Determine the quadrature order and number of points.
2. Determine the numbers of constrained orbits according to Tables 3-4.
3. Determine the orbit pattern of the hierarchical uniform grid $\mathcal{P}(\mathbf{S}_d; k_0, \cdots, k_d)$. By Conjecture 3.1 this is the only orbit pattern admitting $\mathbb{P}(\mathbf{S}_d; k_0, \cdots, k_d)$ unisolvent point-sets.
4. Call the `quadrule` program to search for numerical quadrature rules with the given order, number of points, orbit pattern and constrained orbits.
5. If a solution is found by the `quadrule` program, check whether it's unisolvent.

Example 3.2. Suppose $d = 3$ and $(k_0, k_1, k_2, k_3) = (2, 2, 4, 4)$, then the quadrature order is 4, the number of points is 23, the numbers of constrained orbits are $c_{31} = 0$, $c_{22} = 1$, $c_{211e} = 0$, $c_{211f} = 1$, and the orbit pattern for the hierarchical uniform grid is $s_4 = 1$, $s_{31} = 1$, $s_{22} = 1$, $s_{211} = 1$, thus `quadrule` can be called with

```
quadrule 3 4 23 1 1:1 1:1 1:0:1 0:0 100
```

Since it is not easy to form the appropriate command-line arguments for `quadrule` (especially for the orbit pattern and constrained orbits), a script called "`quadrule-ml`" is provided to simplify this task, which takes as command-line arguments $d, k_0, \cdots, k_d$, and optional extra parameters to pass to the `quadrule` program, and calls the latter with appropriate arguments. For examples, for the case in Example 3.1 one just needs to call

```
quadrule-ml 2 4 4 5 -- 100
```

and for the case in Example 3.2 one just needs to call

```
quadrule-ml 3 2 2 4 4 -- 100
```

In the above two examples the number "100" corresponds to the argument "`ntries`" of the `quadrule` program, i.e., the number of initial guesses to try, which needs to be very large (from 10^3 to 10^9) when computing high order bases.

4. Mass-lumping elements on triangles and tetrahedra

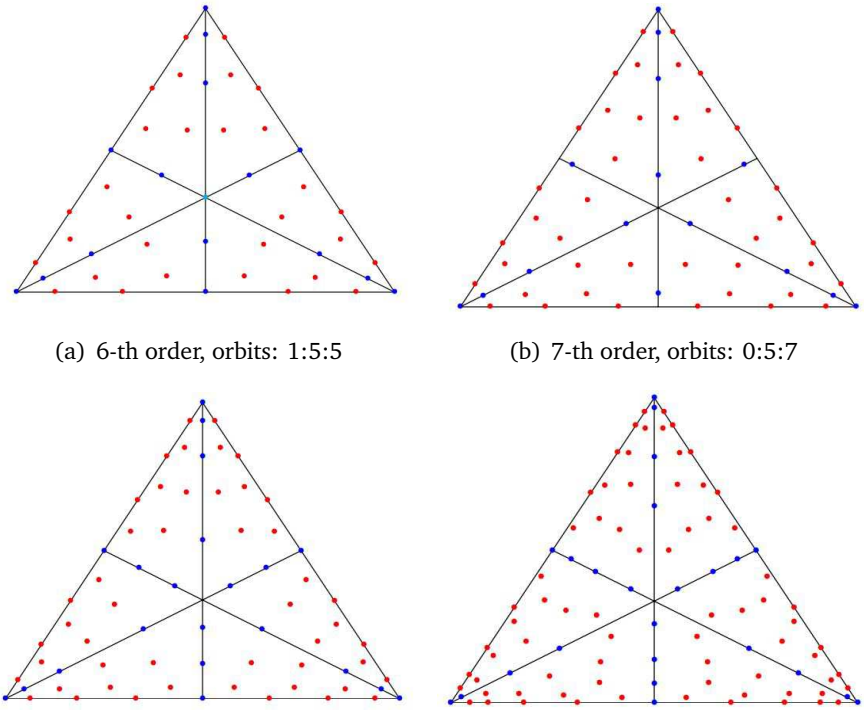
In this section we report the mass-lumping elements so far found using our program. For each element the orbits and weights are listed, with the constrained orbits marked by the superscript "*".

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4.1. Mass-lumping elements on triangles

2D mass-lumping elements so far found using our program are listed in the table below, in which "Orbits" gives total number of orbits for each orbit type in the order of S_3, S_{21} and S_{111}, and if constrained orbits exist their number is listed after the total number, with a ":" between them. Fig. 1 shows the point-sets of the 6, 7, 8 and 9-th order elements. We believe that the 7, 8 and 9-th order elements are new.

For orders ≤ 6 same or similar elements have already been reported in the literature, they are listed here for the convenience of the readers. For the 6, 7 and 9-th orders, multiple solutions were found and the one with largest minimum weight is listed. For the 8-th order several solutions were found and among them only one has all positive weights.



(a) 6-th order, orbits: 1:5:5 (b) 7-th order, orbits: 0:5:7

(c) 8-th order, orbits: 0:7:8 (d) 9-th order, orbits: 0:8:12

Figure 1: Interpolation points for the 6, 7, 8 and 9-th order elements on triangles.

Order 1 element. Points: 3. Orders: 1 1 1. Orbits: 0 1:1 0		
Orbit	Abscissas	Weight
S_{21}^*	.00000000000000000000000000000000000	.33333333333333333333333333333333333
Order 2 element. Points: 7. Orders: 2 2 3. Orbits: 1 2:2 0		
Orbit	Abscissas	Weight

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S_3	.33333333333333333333333333333333	.45000000000000000000000000000000
S_{21}^*	.50000000000000000000000000000000	.13333333333333333333333333333333
S_{21}^*	.00000000000000000000000000000000	.05000000000000000000000000000000
Order 3 element. Points: 12. Orders: 3 3 4. Orbits: 0 2:1 1:1		
Orbit	Abscissas	Weight
S_{21}^*	.00000000000000000000000000000000	.01487291302482058169305106735100205
S_{21}	.20734517566359092426182782125527331	.22077705784041073851802413812368105
S_{111}^*	.00000000000000000000000000000000	.04884168123405100656112906392932512
	.70653044409095980961019599556083747	
Order 4 element. Points: 18. Orders: 4 4 5. Orbits: 0 4:2 1:1		
Orbit	Abscissas	Weight
S_{21}^*	.50000000000000000000000000000000	.02539682539682539682539682539682540
S_{21}^*	.00000000000000000000000000000000	.00634920634920634920634920634920635
S_{21}	.42476396172581058836120087520218113	.15756242893878361853822583683016899
S_{21}	.13079159382974496719435468035337442	.10116772979137511162050432189998974
S_{111}^*	.00000000000000000000000000000000	.02142857142857142857142857142857143
	.21132486540518711774542560974902127	
Order 5 element. Points: 30. Orders: 5 5 7. Orbits: 0 4:1 3:2		
Orbit	Abscissas	Weight
S_{21}^*	.00000000000000000000000000000000	.00141884794135849195859201314216757
S_{21}	.05752768441141010566081751776543190	.02325227091923514227899684743112030
S_{21}	.25685910726195907606389083148220168	.09180247526152571475403829582713371
S_{21}	.45783683807916110193850321742406830	.06906086075456558705677702806082523
S_{111}^*	.00000000000000000000000000000000	.01238113000735325822823625385145974
	.36329807415368604570550633618418105	
S_{111}^*	.00000000000000000000000000000000	.00696115728097842131688535505330661
	.13226458163271398535388822004364736	
S_{111}	.22100121875989000797812820146484192	.05455715193999251909734296553127691
	.07819258362551702199888597846982583	
Order 6 element. Points: 46. Orders: 6 6 9. Orbits: 1 5:2 5:2		
Orbit	Abscissas	Weight
S_3	.33333333333333333333333333333333	.05586538637274356253173477424837725
S_{21}^*	.50000000000000000000000000000000	.00697470012251758743447424062287405
S_{21}^*	.00000000000000000000000000000000	.00092490471543953357135070532616000
S_{21}	.41063851017073795301341212056438007	.04709821701289526827275110758654215
S_{21}	.13218182280398054604355094377286088	.02151016504456748205558643634454797
S_{21}	.04705208731751495655531119156927347	.01549580751390578806503701719807999

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S^*_{111}	.00000000000000000000000000000000000 .28118179917836940000000000000000000	.00612494455362248573296600306883128
S^*_{111}	.00000000000000000000000000000000000 .10224916793261988330009110853449165	.00408418493816979284109251217373712
S_{111}	.18458661094286722140181319344640278 .05033046400253829862069307952587608	.02642569144521976236846984780620857
S_{111}	.16653127942513157927320781829508811 .26283140478643369612728225719849499	.03819460705994430445103585297961422
S_{111}	.05504686319279211012938196328290386 .57509048030744269410336377989347275	.03652444373625689781821356805777719
Order 7 element. Points: 57. Orders: 7 7 10. Orbits: 0 5:1 7:3		
Orbit	Abscissas	Weight
S^*_{21}	.00000000000000000000000000000000000	.00057187603257710553364910400747514
S_{21}	.03831200539348489434936063284054200	.00984431155861776456381405310283263
S_{21}	.27930773520153720457003860574923354	.04837123300802221402790108399159210
S_{21}	.11639788461060549201903665759291354	.01703980478276573686776826336751074
S_{21}	.47802559865964468001101125854349717	.02699063213558926748394131863312944
S^*_{111}	.00000000000000000000000000000000000 .21479412037389310641316988641312571	.00394068080424149158300921571859732
S^*_{111}	.00000000000000000000000000000000000 .07540720721309837766883927085619781	.00243412025191812390820206321942780
S^*_{111}	.00000000000000000000000000000000000 .39964219361398002362656987434459550	.00480107225397601677717051731911991
S_{111}	.36026224663005568929612398542777245 .14183967364361249394883365491143538	.03323316304751691522361203043966463
S_{111}	.14425425864226764503658299629876186 .04093295097394576303833503241936759	.01735079008583329358592834749742437
S_{111}	.29730439276763253813979377673935388 .04340902148622247913867393124892859	.02459099740465178153514672418838990
S_{111}	.13731992147259388145109826854175890 .63370973416435356371591728872815115	.02890691405974299981506085673277271
Order 8 element. Points: 69. Orders: 8 8 11. Orbits: 0 7:2 8:3		
Orbit	Abscissas	Weight
S^*_{21}	.50000000000000000000000000000000000	.00349230635042482070050493767673827
S^*_{21}	.00000000000000000000000000000000000	.00037994199907007578099919493940928
S_{21}	.03123488783242564601024904873035621	.00668635201321269930155855164739580
S_{21}	.09100115583848604780214711967486847	.01270618307414153278582055791906451
S_{21}	.38042917113541587880197811710578371	.04155852613192500870055549763245460
S_{21}	.44093258737717064924419744211516718	.02486365297542547090791656212218097

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S_{21}	.233132086450851003763123866506481 33	.036715904273610219373107044586024 35
S^*_{111}	.000000000000000000000000000000000000 .330294657109078586130667884750762 42	.003239766267894544982827044215844 75
S^*_{111}	.000000000000000000000000000000000000 .818777716708961481793507736399137 84	.002509015557576777900122151203818 78
S^*_{111}	.000000000000000000000000000000000000 .062647088308552351647764474550328 99	.001643607023686500156724790451102 12
S_{111}	.192492192554284550552877530256424 43 .111407718164491467357733399687414 81	.022439763589653862762533672065146 89
S_{111}	.122080420908182854159078093826349 85 .031003660843956088736661254721025 20	.011238289360282611206532210524640 09
S_{111}	.565093455885379191704849091935950 12 .316103620342684352619799843985421 82	.026319382705143560456426271369777 20
S_{111}	.401001585784443849537176841335597 47 .036430308378620995792720606925729 59	.019328875153443981870252854400596 53
S_{111}	.248941026042450802256510269625010 75 .035550031982121308176501243959069 83	.016746533600079913556016499174106 40
Order 9 element. Points: 96. Orders: 9 10 13. Orbits: 0 8:2 12:4		
Orbit	Abscissas	Weight
S^*_{21}	.500000000000000000000000000000000000	.001336692610700910368156928159393 02
S^*_{21}	.000000000000000000000000000000000000	.000118481870030178005888703923751 05
S_{21}	.178086793304375531881186565796992 26	.022332257198472635170826477138545 93
S_{21}	.097364854296494916753449707848713 10	.014938059766254602112867057199137 86
S_{21}	.371767406068474750278995105853929 48	.027774977684427498980375440726108 95
S_{21}	.469397742436583685945500661456123 43	.017218620213271905034330794055749 92
S_{21}	.429736490101558781564317620739589 23	.024921104581314943447772319940121 94
S_{21}	.017167065517571321472500635114041 56	.002568484330518580291690602808738 85
S^*_{111}	.000000000000000000000000000000000000 .311541734415876463105448484399964 22	.001798841260016749094989004320703 84
S^*_{111}	.000000000000000000000000000000000000 .179515297844366693251012464017966 84	.001705347521271880137066899317752 85
S^*_{111}	.000000000000000000000000000000000000 .092105746458433170000000000000000 00	.000947774977876187118859969766801 74
S^*_{111}	.000000000000000000000000000000000000 .047560867362580021161209721647611 22	.000625585320545334650401410022260 35
S_{111}	.302479832380686112653628930798758 15 .132358415038959900000000000000000 00	.019354941416101386749634716726721 51
S_{111}	.200411922565542320001417322954353 22 .085270014275307238114168565849870 46	.017311892923880807636555520284751 68

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S_{111}	.41302861084658998203526825119260858 .01597599097862174107642208861071435	.00932290168272229434999079918770219
S_{111}	.28687593667608651854734035876948463 .21517835964285685580689797366138632	.01920787203828274739647549998767839
S_{111}	.71066464596525716023234676351139562 .26568640019035478131344886664522366	.00996784546376290957624740472535696
S_{111}	.07266445465054350037002741770250724 .02878133698864898097843265295428718	.00650467459050091563262973282905392
S_{111}	.15367244575543926033558398634472357 .81805573369383805443310738637046680	.00897647880253238161210500724013579
S_{111}	.33426289610998681979506369630264371 .06391952769125736390714666353951024	.01533817154167744600575654028197368

4.2. Mass-lumping elements on tetrahedra

3D mass-lumping elements so far found using our code are listed below, in which "Orbits" gives total number of orbits for each orbit type in the order of S_4, S_{31}, S_{22}, S_{211} and S_{1111}, and if constrained orbits exist their number is listed after the total number, with a ":" between them (and in the format "total:c_{211e}:c_{211f}" for the S_{211} orbits). All these elements have already been reported in the literature, they are listed here solely for the convenience of the readers. For the 3-rd order element, two solutions were found and the one with larger minimum weight is listed.

Order 1 element. Points: 4. Orders: 1 1 1 1. Orbits: 0 1:1 0 0 0		
Orbit	Abscissas	Weight
S^*_{31}	.00000000000000000000000000000000000	.25000000000000000000000000000000000
Order 2 element. Points: 23. Orders: 2 2 4 4. Orbits: 1 1:1 1:1 1:0:1 0		
Orbit	Abscissas	Weight
S_4	.25000000000000000000000000000000000	.30476190476190476190476190476190476
S^*_{31}	.00000000000000000000000000000000000	.00129961081762382864323948583189792
S^*_{22}	.00000000000000000000000000000000000	.00751330903878115632153864252437151
S^*_{211}	.18858048469644503927115437402941689 .00000000000000000000000000000000000	.05374664981124274879942069139702288
Order 3 element. Points: 50. Orders: 3 3 5 6. Orbits: 0 2:1 1 3:1:2 0		
Orbit	Abscissas	Weight
S^*_{31}	.00000000000000000000000000000000000	.00139318132340935829885454503458246
S^*_{211}	.00000000000000000000000000000000000 .69474012433043399696452214373799579	.00439720814497923320460068363824448
S^*_{211}	.14804629800083268127155972966379546 .00000000000000000000000000000000000	.00938677154027050153276493215837929
S^*_{211}	.42045997555404366358045983652415104 .00000000000000000000000000000000000	.01517875558886845408624234213923300
S_{31}	.10486452489170353984304348629272939	.04276746867938747299136451172239785
S_{22}	.12587961966825070744487424837645650	.07930076278323240149263804629029959

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4.3. Numerical validation of the new bases

The new bases reported in this paper, namely the 2D bases of orders 7, 8 and 9, have been validated numerically by using them to solve the following Helmholtz equation on the unit square with Dirichlet boundary condition:

$$\begin{cases} \Delta u(x) + k^2 u(x) = f(x), & \text{if } x \in \Omega, \\ u(x) = u_0(x), & \text{if } x \in \partial\Omega, \end{cases} \tag{4.1}$$

where Δ denotes the Laplacian,

$$\Omega = (0,1)^2, \quad k^2 = 500, \quad u_0(x) = \cos(15x + 16y), \quad f(x) = \Delta u_0(x) + k^2 u_0(x).$$

In the numerical tests, a structured triangular mesh containing 13 vertices and 16 elements was used as the initial mesh, which was repeatedly refined to generate a series of meshes with decreasing mesh step size. In order to verify the precision of mass-lumping, diagonal mass matrices computed from the weights in the associated numerical quadrature rules were used in the finite element discretizations. The approximation errors and estimated approximation orders of the finite element solutions are listed in Table 7, which are in consistence with the orders of the bases.

Table 7: Numerical validation of the 7, 8 and 9-th order elements.

7-th order element				
# DOF	L_2 error	L_2 order	H^1 error	H^1 order
757	1.2287e-05	*	1.4864e-03	*
2969	4.7335e-08	8.0	1.1934e-05	7.0
11761	1.6745e-10	8.1	8.5305e-08	7.1
46817	6.4607e-13	8.0	6.6026e-10	7.0
186817	2.5145e-15	8.0	5.1460e-12	7.0
8-th order element				
# DOF	L_2 error	L_2 order	H^1 error	H^1 order
929	1.6399e-06	*	2.1203e-04	*
3649	2.8525e-09	9.2	7.6997e-07	8.1
14465	6.3896e-12	8.8	3.4903e-09	7.8
57601	1.2607e-14	9.0	1.3810e-11	8.0
229889	2.4402e-17	9.0	5.3383e-14	8.0
9-th order element				
# DOF	L_2 error	L_2 order	H^1 error	H^1 order
1321	2.7456e-07	*	2.1056e-05	*
5201	2.1684e-10	10.3	3.6208e-08	9.2
20641	1.9017e-13	10.2	6.4746e-11	9.1
82241	1.8333e-16	10.0	1.2571e-13	9.0
328321	1.7855e-19	10.0	2.4554e-16	9.0

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5. Conclusions

In this paper we have addressed problems related to the construction of mass-lumping finite elements on simplexes, and presented a program for computing mass-lumping elements on triangles and tetrahedra. The source code of the program, together with a set of scripts and tools for analyzing, checking and refining the computed elements, can be freely downloaded at `http://lsec.cc.ac.cn/phg/download/quadrule.tar.bz2`.

The program is put in the public domain in the hope that other people may be interested in using it to search for new mass-lumping elements.

A final remark we would like to make is that only fully symmetric point-sets have been considered in this paper. This is unnecessary since for H^1-conformity, symmetry is required only for the points on the boundaries, i.e., points belonging to $\mathbb{S}_d^l$ for $l < d$. Thus it might be possible to construct mass-lumping elements with fewer points for a given order by allowing asymmetry on the interior points. We refer the readers to [13, 14] for related work on numerical quadrature rules in which asymmetric numerical quadrature rules which have fewer points than the best known symmetric ones, were found and reported.

Acknowledgments We would like to thank Professor Zhiqiang Xu (xuzq@lsec.cc.ac.cn) for discussions on multivariate polynomial interpolation. This work was supported by National Key Research and Development Program under the grant number 2016YFB0201304, China NSF under the grant numbers 91430215, 91530323 and 11321061, State Key Laboratory of Scientific and Engineering Computing (LSEC), and National Center for Mathematics and Interdisciplinary Sciences of Chinese Academy of Sciences (NCMIS).

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