$$\Rightarrow 1 + \phi = GH^2$$

$$\Rightarrow \phi^2 = GH^2$$

$$\Rightarrow GH = \phi.$$

Based on the former proof, a regular pentagon of any size will create a Kepler triangle with geometric progression.

*References*

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## 102.33 A geometrical construction involving two intersecting circles

We are given two circles, intersecting at a point $A$, and a fixed length $\ell$. Is it possible to construct a line through $A$, intersecting the circles at $P$ and $Q$, such that $|AP - AQ| = \ell$?

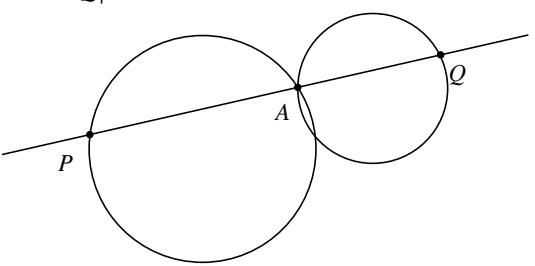



FIGURE 1

This can be done in the following way. Let the centres of the circles be $O_1$, $O_2$ and the diameters through $A$ be $AB_1$, $AB_2$ (see Figure 2). Let $C$ be on $B_2A$ produced so that $B_2A = AC$. Draw a circle with centre $B_1$ and radius $\ell$ and let $CT$, $CT'$ be the two tangents from $C$ to this circle. Draw a line through $A$ perpendicular to $CT$ intersecting the circles at $P$, $Q$ and $CT$ at $D$, and similarly construct $P'$, $Q'$ and $D'$. We claim that these are the points satisfying the requirements.

Note that $B_1TDP$ is a rectangle and that $\Delta AQB_2 \cong \Delta ADC$ (AAS). Hence $AD = AQ$ and so $AP - AQ = PD = B_1T = \ell$, as required. The reader can check, in a similar way, that $AQ' - AP' = \ell$. Hence, in general, two lines satisfying the requirements have been constructed. This procedure also shows that all solutions can be obtained by this method.

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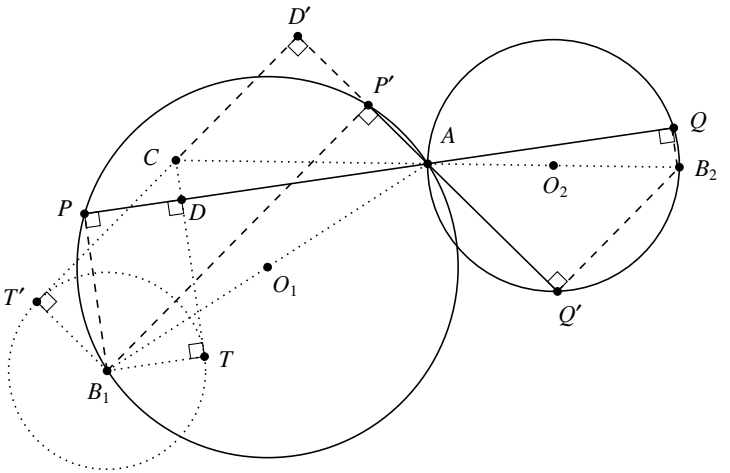



FIGURE 2

*Comments*:

(a) Note that if $E$ is the point of intersection of $CT'$ and $PA$ (as shown in Figure 3), then $\angle T'CT = \frac{1}{2}\pi - \angle CED = \angle P'AP$. This means that the angle between the two solutions $PQ$ and $P'Q'$ is the same as the angle between the two tangents $CT$ and $CT'$. By using standard trigonometric relations in the right triangle $CTB_1$ (or $CT'B_1$), the relation $\sin \frac{1}{2}\angle T'CT = \dfrac{\ell}{CB_1}$ is obtained. Let $R_1, R_2$ denote the radii of the circles with centres $O_1, O_2$ and let $\alpha$ denote $\angle O_1AO_2$. The value of $CB_1$ is given by the cosine rule on the triangle $ACB_1$ which has two sides whose lengths are the diameters of the two circles and $\angle CAB_1 = \pi - \alpha$:

$$CB_1 = 2\sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos\alpha}.$$

It follows that the angle between the two lines $PQ$ and $P'Q'$ is equal to

$$2\arcsin\left(\frac{\ell}{2\sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos\alpha}}\right).$$

In particular, we obtain the inequality

$$\left|AP - AQ\right| \leqslant \left|CB_1\right| = 2\sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos\alpha}.$$

for every line through $A$ intersecting the circles at $P$ and $Q$.

(b) Since $\angle B_1TC$ and $\angle B_1T'C$ are right angles, the points $T$ and $T'$ lie on the circle with diameter $CB_1$ (as shown in Figure 3). The distance from $B_1$ to $T$ and $T'$ is $\ell$. When $\ell = 0$, $T$ and $T'$ coincide with $B_1$. As $\ell$ increases, $T$ and $T'$ move along the two semicircles with diameter $B_1C$ until both coincide with $C$.

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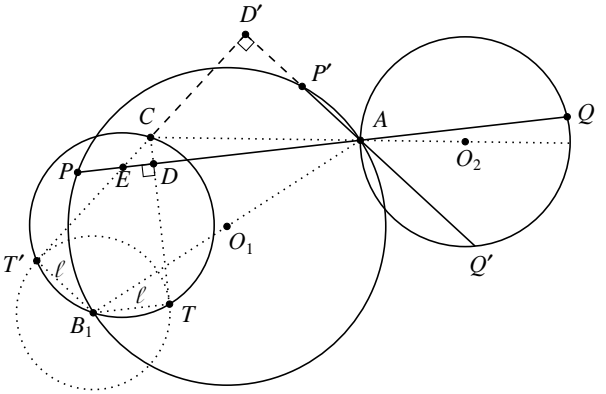



FIGURE 3

*Special cases*

Let us now look at how varying $\ell$ will have the effect of changing the position of the solutions $PQ$ and $P'Q'$. A natural question here is if there are some values of $\ell$ which have a geometrically significant effect on the solutions $PQ$ and $P'Q'$. It turns out that, in general, there are four values of $\ell$ for which an extreme case occurs.

(a) In the case where $\ell = 0$, the circle at $B_1$ reduces to a point and by Thales' theorem $D$ lies on the circle with centre $O_1$, so $D = P$. We have $\Delta ACP \cong \Delta AB_2Q$ (AAS) and so $AP = AQ$ as required. This line is unique (see Figure 4).

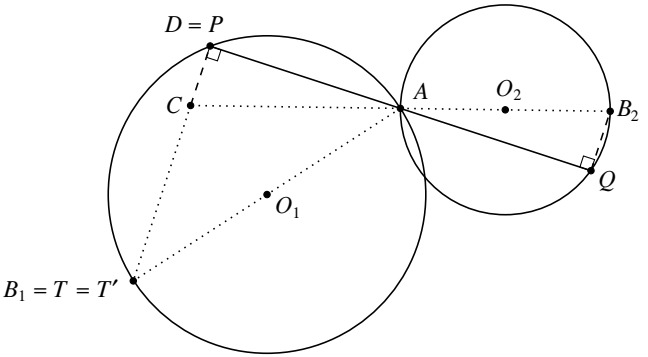



FIGURE 4

(b) As $\ell$ increases, the point $P'$ approaches $A$ until it coincides with $A$ (see Figure 5). In this case $P'Q'$ is tangent to the circle with centre $O_1$ and $P'Q' = \ell$. The triangle $O_2AQ'$ is isosceles with two sides of length $R_2$, so $\ell = 2R_2 \sin\alpha$.

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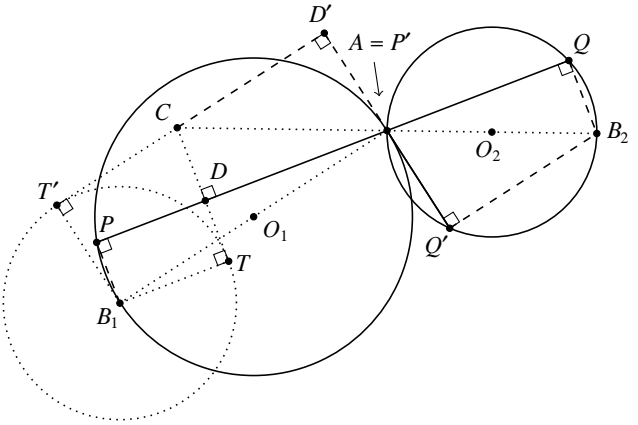



FIGURE 5

(c) With a further increase of $\ell$, $P'$ and $Q'$ are on the same side of $A$, and there is no second solution until $Q'$ coincides with $A$. We now have a situation which is analogous to (b) (see Figure 6). In this case $P'Q'$ is tangent to the circle with centre $O_2$ and $P'Q' = \ell = 2R_1 \sin\alpha$.

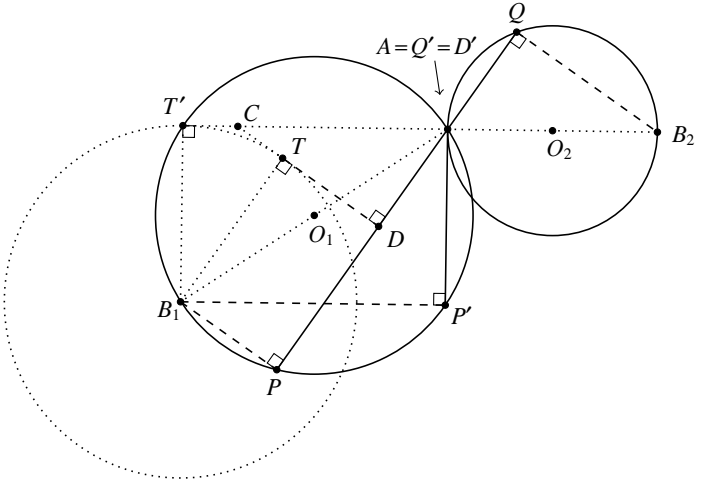



FIGURE 6

(d) If $\ell$ increases further, $P'$ and $Q'$ again lie on opposite sides of $A$. The next and final special case is where the circle centred at $B_1$ passes through $C$ (see Figure 7). Now the points $C$, $T$ and $T'$ coincide. However, the construction still works, although there is only one solution. We have $\ell = B_1C = 2\sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos\alpha}$.

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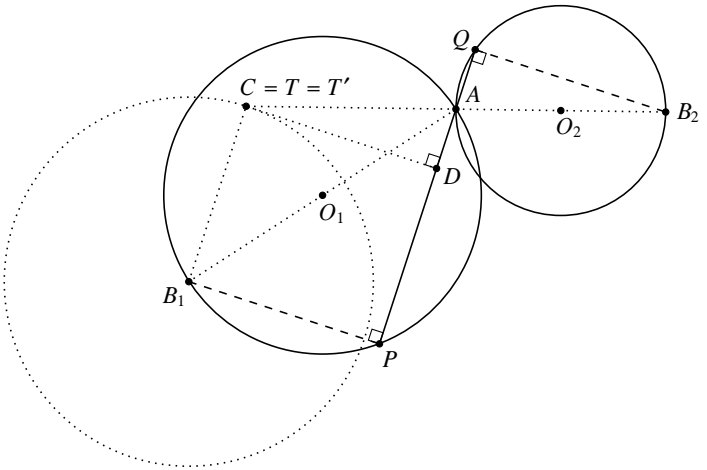



FIGURE 7

It is clear that if $\ell$ increases any more, i.e., if $\ell > 2\sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos\alpha}$, the point $C$ would be an interior point of the circle with centre $B_2$ and radius $\ell$ and we cannot draw tangents from an internal point to a circle; therefore, the proposed method does not work.

The original construction was proposed by my geometry teacher Mr. Mostavi around 1990. The method presented here was my own solution to this problem found around that time. I hope this is of interest to readers.

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## 102.34 On the vertices of the Kiepert hyperbola

If a scalene triangle $ABC$ has externally attached similar isosceles triangles $BCA'$, $CAB'$, $ABC'$, then the joins $AA'$, $BB'$, $CC'$ are concurrent by dint of the converse of Ceva's theorem. The locus of this meet with varying base angle is known as the Kiepert hyperbola.

It follows that the vertices, centroid, orthocentre and Fermat points all lie on this curve for appropriate values of the base angle [1].

Before using areal coordinates to locate the vertices of this circumscribing hyperbola, we will allude to some easier Cartesian coordinate geometry.

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