Ergod. Th. & Dynam. Sys. (2016), **36**, 198–214 doi:10.1017/etds.2014.62
© *Cambridge University Press, 2014*

Ergodic properties of discrete dynamical systems and enveloping semigroups

A. V. ROMANOV

National Research University Higher School of Economics, Moscow, Russia
(e-mail: av.romanov@hse.ru)

(*Received* 17 *March* 2014 *and accepted in revised form* 29 *May* 2014)

Abstract. For a continuous semicascade on a metrizable compact set Ω, we consider the weak* convergence of generalized operator ergodic means in End $C^*(\Omega)$. We discuss conditions under which: every ergodic net contains a convergent sequence; all ergodic nets converge; all ergodic sequences converge. We study the relationships between the convergence of ergodic means and the properties of transitivity of the proximality relation on Ω, minimality of supports of ergodic measures, and uniqueness of minimal sets in the closure of trajectories of a semicascade. These problems are solved in terms of three associated algebraic-topological objects: the Ellis semigroup E, the Köhler operator semigroup $\Gamma \subset \text{End } C^*(\Omega)$, and the semigroup $G = \overline{\text{co}}\,\Gamma$. The main results are stated for semicascades with metrizable E and for tame semicascades.

1. *Introduction*

The main topic of this paper is the weak* convergence of generalized ergodic means along trajectories of discrete compact dynamical systems. In the course of our study, ergodic properties of the semicascade (Ω, φ) generated by a continuous transformation φ of a metrizable compact set Ω are related to characteristics of the Ellis enveloping semigroup $E(\Omega, \varphi)$, the Köhler operator enveloping semigroup $\Gamma(\Omega, \varphi)$, and yet another operator semigroup $G(\Omega, \varphi)$, which was considered in the author's paper [**18**]. We also study relationships between properties of these three semigroups. A considerable part of the results of the present paper were announced in [**19**, **20**].

The transformation $\varphi : \Omega \to \Omega$ induces linear contractions U and $V = U^*$ on the space $X = C(\Omega)$ of continuous scalar functions on Ω and on the dual space X^* (consisting of Borel measures on Ω), respectively. Let End X and End X^* be the normed spaces of bounded linear operators on X and X^*, respectively. The enveloping semigroup $E(\Omega, \varphi)$ of the semicascade (Ω, φ) is defined [**5**] as the closure of the family $\{\varphi^n, n \geq 0\}$ in the Hausdorff topology of pointwise convergence on the space of all mappings $\Omega \to \Omega$. The semigroup $\Gamma(\Omega, \varphi)$ is defined [**14**] as the closure of $\Gamma_0 = \{ V^n,\ n \geq 0\}$ in the weak*

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

operator topology W*O on the operator space End X^*. The operator semigroup $G(\Omega, \varphi)$ is defined [**18**] as the W*O-closed convex hull of Γ_0. The semigroups $E(\Omega, \varphi)$, $\Gamma(\Omega, \varphi)$, and $G(\Omega, \varphi)$ are compact in the respective topologies, and their topological characteristics, as shown in [**18**], are closely related to ergodic properties of the semicascade (Ω, φ). It turns out that such relationships manifest themselves especially strongly in two classes of discrete dynamical systems. These are systems with metrizable enveloping semigroup $E(\Omega, \varphi)$, dubbed *ordinary* in [**19**, **20**], and *tame* semicascades, defined by the relation card $E(\Omega, \varphi) \leq \mathfrak{c}$.

The main results of the paper pertain to the convergence of generalized ergodic means for semicascades of the two above-mentioned classes in the weak* operator topology W*O on End X^*. Let G_0 be the convex hull of the set Γ_0 in End X^*. We say that a net of operators $T_\alpha \in G_0$ is ergodic (averaging) if $T_\alpha(I - V) \xrightarrow{\text{W*O}} 0$, $I = \mathrm{id}$. Then $T_\alpha = R_\alpha^*$ for some $R_\alpha \in \mathrm{End}\, X$, and the corresponding operator net R_α consisting of the elements of the convex hull of the operator set $\{U^n, n \geq 0\}$ in End X is said to be ergodic as well. The sequences of Cesàro means

$$U_n = \frac{1}{n}(I + U + \cdots + U^{n-1}), \quad V_n = \frac{1}{n}(I + V + \cdots + V^{n-1})$$

are ergodic, since $V_n(I - V) = n^{-1}(I - V^n)$, $\|V^n\| = 1$, and $V_n = U_n^*$. The limit element of a convergent ergodic operator net $T_\alpha \in G_0$ is always a bounded projection on X^*. We point out that the W*O-convergence of such nets is equivalent to the convergence, for arbitrary continuous functions $x \in X$, of the means $R_\alpha x \to z$, $z \in X^{**}$, in the w^*-topology of the second dual space $X^{**} = C^{**}(\Omega)$. For ergodic operator *sequences* $T_n \in G_0$, this convergence is equivalent to the pointwise convergence of the functions $R_n x \to z$ on Ω, but in the general case it can be stronger than the latter. Note that the idea of considering W*O-convergence of ergodic means essentially goes back to the classical paper [**15**] by Krylov and Bogolyubov.

Issues related to the W*O-convergence of arbitrary ergodic operator nets or their certain subsequences are considered here in connection with purely dynamic properties of ordinary and tame dynamical systems: transitivity of the proximality relation, uniqueness of a minimum set in the closure of an arbitrary trajectory, and minimality of supports of ergodic measures. The techniques used here include the properties of the enveloping semigroups $E(\Omega, \varphi)$, $\Gamma(\Omega, \varphi)$, and $G(\Omega, \varphi)$ associated with the semicascade (Ω, φ). In the ordinary case, the key point is the metrizability of the semigroup $G(\Omega, \varphi)$, which permits one to pass from the convergence of ergodic nets to that of ergodic sequences. By way of example, let us present some of the results obtained below for the W*O-convergence of ergodic means.

(i) For an ordinary semicascade (Ω, φ), every ergodic operator net T_α contains a convergent sequence $T_{\alpha(k)}$. Moreover, all ergodic operator nets T_α converge if and only if, for each $\omega \in \Omega$, the closure of the trajectory $\{\varphi^n \omega,\ n \geq 0\}$ contains a unique minimal set. The same is true for sequences instead of nets.

(ii) If, for a tame semicascade (Ω, φ), the closure of each trajectory $\{\varphi^n \omega, n \geq 0\}$, $\omega \in \Omega$, contains a unique minimal set, then all ergodic operator sequences T_n converge, and the support of each φ-ergodic measure is a minimal set.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

As was shown in [**18**, Theorem 1.3], for an *arbitrary* semicascade (Ω, φ), the convergence of all ergodic operator nets is equivalent to the condition $\operatorname{card} L = 1$, where L is the kernel (the intersection of all two-sided ideals) of the semigroup $G(\Omega, \varphi)$; it consists of the operators $Q \in G(\Omega, \varphi)$ with the property $VQ = Q$. The convex weakly* compact set $L \subset \operatorname{End} X^*$ of projections was originally considered by Lloyd [**16**]. There are reasons to assume that it is the algebraic-geometric characteristics of this object that are responsible for the W*O-ergodic properties of the semicascade (Ω, φ). Let us also mention one more significant result related to purely dynamic properties of tame semicascades: for these semicascades, the closure of the union of all minimal sets coincides with the minimal attraction center, i.e. with the closure of the union of supports of all φ-ergodic Borel measures on Ω. In this connection, note the strict ergodicity of minimal tame systems, which has recently been established in [**11**, **13**].

Numerous properties of Ellis semigroups of ordinary and tame systems (not necessarily discrete) can be found in Glasner's survey [**8**]. As follows from the results in [**8–10**], a semicascade (Ω, φ) is ordinary if and only if the dynamics is non-chaotic in the sense that each closed semi-invariant $(\varphi\Theta \subset \Theta)$ set $\Theta \subset \Omega$ contains a trajectory that is Lyapunov stable with respect to the restriction (Θ, φ). It is also well known [**8**] that all weakly almost periodic dynamical systems prove to be ordinary.

The present paper suggests an alternative definition for (Ω, φ) being ordinary, which is the requirement that the semigroup $G(\Omega, \varphi)$ is metrizable. Moreover, we show that, in the ordinary case, each operator $T \in G(\Omega, \varphi)$ is uniquely determined by the values it takes on the Dirac measures $\delta(\omega)$, $\omega \in \Omega$.

The class of tame dynamical systems, originally introduced (under a different name) by Köhler [**14**], was comprehensively studied in [**7–10**]. In the present paper, we speak of semicascades (Ω, φ) satisfying any of the following three equivalent conditions: (1) the semigroup $E(\Omega, \varphi)$ is at most of the cardinality of continuum; (2) the compact set $E(\Omega, \varphi)$ is a Fréchet–Urysohn space; (3) all transformations $p \in E(\Omega, \varphi)$ belong to the first Baire class, i.e. are pointwise limits of sequences of continuous mappings $p_n : \Omega \to \Omega$. We see that the class of tame dynamical systems includes the ordinary semicascades. Here we establish one more criterion for tame dynamics: the enveloping semigroup $E(\Omega, \varphi)$ consists of Borel transformations.

The correspondence $V \to \varphi$ generates a continuous algebraic semigroup epimorphism $\Gamma(\Omega, \varphi) \xrightarrow{\pi} E(\Omega, \varphi)$ [**14**]. Moreover, $P\delta(\omega) = \delta(p\,\omega)$ for $P \in \Gamma(\Omega, \varphi)$, $p = \pi P$, $\omega \in \Omega$, and the Dirac measures $\delta(\cdot)$. Let $A(\Omega)$ be the set of Borel probability measures on the compact set Ω. It is well known [**14**], [**8**, p. 2355] that, for a tame system (Ω, φ), the function π is injective and the semigroup $\Gamma(\Omega, \varphi)$ can be identified with $E(\Omega, \varphi)$. It turns out (Corollary 3.6) that in this case all operators $P \in \Gamma(\Omega, \varphi)$ act on the measures $\mu \in A(\Omega)$ by the 'natural' rule $(P\mu)(e) = \mu(p^{-1}e)$ for $p = \pi P$ and Borel sets $e \subset \Omega$ (i.e. by analogy with the operators V^n, $n \geq 0$, generating the semigroup $\Gamma(\Omega, \varphi)$). It is also proved that, for the case of a tame semicascade (Ω, φ), the set of extreme points of the convex compact subset $G(\Omega, \varphi)$ of the operator space $\operatorname{End} X^*$ coincides with the set $\Gamma(\Omega, \varphi)$.

We point out that although Ellis's monograph [**5**] deals with invertible dynamical systems, the results in [**5**] cited in what follows are easily seen to remain valid in the non-invertible case.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

The rest of this paper is organized as follows. Section 2 provides general information about the enveloping semigroups $E(\Omega, \varphi)$, $\Gamma(\Omega, \varphi)$, and $G(\Omega, \varphi)$ and describes the relationships between these semigroups. Section 3 contains a broad hierarchical compendium (in a sense, a classification) of earlier known and some new dependences between various properties of arbitrary semicascades, the corresponding statements being mostly given in terms of the above-mentioned semigroups. The main results pertaining to ergodicity properties of ordinary and tame semicascades are presented in §4.

2. *Preliminaries*

Let φ be a continuous (not necessarily invertible) transformation of a metrizable compact set Ω, and let (Ω, φ) be the corresponding discrete dynamical system (semicascade) on Ω. We define the shift operator $U = U_\varphi$ on the space $X = C(\Omega)$ of continuous scalar functions by the formula $Ux = x \circ \varphi$, $x \in C(\Omega)$. By $A(\Omega)$ and $K(\Omega)$ we denote the sets of Borel probability measures and Dirac measures on Ω, respectively. Recall that every Borel measure on a metrizable compact set is regular. Also, let $\Lambda(\Omega)$ be the class of ergodic measures of the semicascade (Ω, φ), i.e. φ-invariant measures $\lambda \in A(\Omega)$ such that $\lambda(e)$ is either 0 or 1 for every Borel set $e \subset \Omega$ with $\varphi^{-1}e = e$. The set $A(\Omega)$ is convex, compact, and metrizable in the w^*-topology of the dual space $X^* = C^*(\Omega)$, and $K(\Omega)$ is a closed subset of $A(\Omega)$.

By $\mathrm{Im}\,(\cdot)$ and $F(\cdot)$ we denote the range and the subspace of fixed vectors, respectively, of a linear operator. Next, End X and End X^* are the Banach algebras of bounded linear operators on X and on the first dual space X^*, respectively. Let $V = U^*$ be the adjoint operator on X^*; then $\|V^n\| = 1$ for $n \geq 0$. We equip the operator algebra End X^* with the locally convex weak* operator topology (W*O-topology) and single out the multiplicative semigroups $\Gamma_0 = \{V^n, n \geq 0\}$ and $G_0 = \mathrm{co}\,\Gamma_0$ in this algebra. Set $G = \overline{\mathrm{co}}\,\Gamma_0$ and $\Gamma = \overline{\Gamma}_0$, where $\overline{(\cdot)}$ is the operation of W*O-closure of sets in the normed space End X^*; then $\Gamma \subset G$ and $\|T\| = 1$ for $T \in G$. Next, $T : A(\Omega) \to A(\Omega)$ for the operators $T \in G$. Norm-bounded subsets of the space End X^* are relatively compact in the W*O-topology, and hence the sets $\Gamma = \Gamma(\Omega, \varphi)$ and $G = G(\Omega, \varphi)$ are Hausdorff (generally, non-metrizable) separable compact sets. The multiplication in the algebra End X^* is only continuous in the left argument; as a consequence, the semigroup G and the subsemigroup Γ are usually non-commutative, although their elements commute with the operators V^n, $n \geq 0$. Note that $\mathrm{Im}T \supset F(T) \supset F(V)$ for all $T \in G$. As was shown in [**18**, Lemma 1.1], the kernel (the intersection of all two-sided ideals) of $G(\Omega, \varphi)$ coincides with the closed set

$$L = \{Q \in G : VQ = Q\}. \tag{2.1}$$

Furthermore, $TQ = Q$ for all $T \in G$ and $Q \in L$, and the convex compact set L consists of unit-norm projections. One can give an alternative definition, $L = \{T \in G : \mathrm{Im}\,T = F(V)\}$.

The notion of enveloping semigroup of a dynamical system plays an important role in what follows. According to [**5**], the enveloping semigroup $E(\Omega, \varphi)$ of a semicascade (Ω, φ) is the closure of the generating family $E_0(\Omega, \varphi) = \{\varphi^n, n \geq 0\}$ in the Hausdorff topology τ of pointwise convergence on the space Ω^Ω of all mappings $p : \Omega \to \Omega$. The semigroup $E(\Omega, \varphi)$ is compact but non-metrizable and non-commutative in general, and

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

its center contains the set $E_0(\Omega, \varphi)$. Every enveloping semigroup is separable and has non-empty kernel.

Since $(V\mu)(e) = \mu(\varphi^{-1}e)$ for $\mu \in A(\Omega)$ and Borel sets $e \subset \Omega$, it follows that the operator V generates the semicascade (A, V) on the compact set $A = A(\Omega)$. According to [**14**], the semigroup $\Gamma(\Omega, \varphi)$ can be identified with the enveloping semigroup $E(A, V)$ of (A, V), which is the closure of the family $\{V^n, n \geq 0\}$ of continuous transformations of the compact set $A(\Omega)$ in the Cartesian product topology on A^A. On the other hand, it is convenient to interpret the semigroup $G(\Omega, \varphi)$ as the enveloping semigroup of the action $W \times A \xrightarrow{V} A$ on $A = A(\Omega)$ of the abelian semigroup W of non-negative finite numerical sequences with unit sum and with convolution as multiplication. The semigroup of convex combinations of the monomials $\{t^n, n \geq 0\}$ with the usual multiplication may serve as an alternative model for W. Thus, $G(\Omega, \varphi) \simeq E(A, W)$.

Let Σ_b be the σ-algebra of Borel subsets of Ω, and let X_b be the family of scalar bounded Borel functions on Ω. Clearly, $X \subset X_b \subset X^{**}$, and X_b is a norm-closed subspace of the second dual X^{**}. Moreover, $\|x\| = \sup |x(\omega)|$, where the supremum is over $\omega \in \Omega$, for $x \in X_b$. Note that Borel functions are universally measurable, i.e. measurable with respect to the Lebesgue extension of an arbitrary measure $\mu \in A(\Omega)$. We say that a mapping $p : \Omega \to \Omega$ is a *Borel* mapping if $p^{-1}e \in \Sigma_b$ for all closed sets $e \subset \Omega$. Finally, let Π_b be the set of all Borel transformations of Ω. Note that actually $p^{-1}e \in \Sigma_b$ for $p \in \Pi_b$ and $e \in \Sigma_b$.

Next, $V\delta(\omega) = \delta(\varphi\omega)$ for $\omega \in \Omega$, and the set $K = K(\Omega)$ of Dirac measures is closed in the topological space $A(\Omega)$; hence $VK \subset K$ and $PK \subset K$ for all operators $P \in \Gamma(\Omega, \varphi)$. The correspondence $V \to \varphi$ generates a continuous algebraic homomorphism $\pi : P \to p$ of the semigroup $\Gamma(\Omega, \varphi)$ into the semigroup $E(\Omega, \varphi)$, whose action is clearly defined by the relation

$$P\delta(\omega) = \delta(p\omega), \quad \omega \in \Omega. \tag{2.2}$$

Since the set $\Gamma(\Omega, \varphi)$ is compact, $\pi V^n = \varphi^n$, and the generating transformation family $E_0(\Omega, \varphi)$ is τ-dense in $E(\Omega, \varphi)$, we can speak of the *canonical epimorphism* $\pi : \Gamma(\Omega, \varphi) \to E(\Omega, \varphi)$.

For a measure $\mu \in A(\Omega)$ and a set $e \in \Sigma_b$, we have $(V^n\mu)(e) = \mu(\varphi^{-n}e)$, $n \geq 0$. Moreover, $(V^n)^* : x \to x \circ \varphi^n$ for continuous functions $x \in X$. It is useful to find out in what cases arbitrary operators $P \in \Gamma(\Omega, \varphi)$ and their duals $P^* \in \mathrm{End}\, X^{**}$ act in a similar way.

Definition 2.1. We say that an operator $P \in \Gamma(\Omega, \varphi)$ is *regular* if $p \in \Pi_b$ for $p = \pi P$ and

$$(P\mu)(e) = \mu(p^{-1}e) \tag{2.3}$$

for $\mu \in A(\Omega)$ and $e \in \Sigma_b$.

LEMMA 2.2. *If $P \in \Gamma(\Omega, \varphi)$ and $p = \pi P$, then the following conditions are equivalent:*

(a) $P^* : X \to X_b$;

(b) $p \in \Pi_b$ *and* $P^*x = x \circ p$ *for* $x \in X$;

(c) *the operator P is regular.*

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

Proof. Clearly, (b) $\Rightarrow$ (a). If condition (c) holds, then one can deduce from general properties of Lebesgue integral that $(x, P\mu) = (x \circ p, \mu)$ for any $x \in X$ and $\mu \in A(\Omega)$. On the other hand, one always has $(P^*x, \mu) = (x, P\mu)$, hence $P^*x = x \circ p$ and (c) $\Rightarrow$ (b). Now let us establish the implication (a) $\Rightarrow$ (c). Let $x \in X$, $y = P^*x$ and $\omega \in \Omega$; then $y \in X_b$ and $(y, \delta(\omega)) = x(p\omega)$. The elements $y \in X_b$, being functionals in X^{**}, are completely determined by their values on the Dirac measures, and hence $(P^*x)(\omega) = x(p\omega)$ for all $\omega \in \Omega$. Thus, $x \circ p \in X_b$ for each continuous function x. The indicator function x_e of an arbitrary closed set $e \subset \Omega$ is the pointwise limit of some sequence of continuous functions x_n; consequently, $x_n(p\omega) \to x_e(p\omega)$ for all $\omega \in \Omega$, and hence the function $x_e \circ p$ is a Borel function. It follows that $p \in \Pi_b$. Next,

$$(x, P\mu) = (P^*x, \mu) = \int_\Omega x(p\omega)\mu(d\omega) = \int_\Omega x(\omega)\mu_p(d\omega)$$

for any $\mu \in A(\Omega)$, where the Borel probability measure μ_p is defined on the sets $e \in \Sigma_b$ by the formula $\mu_p(e) = \mu(p^{-1}e)$. It follows that $P\mu = \mu_p$, and the proof of the lemma is complete. □

We refer to arbitrary finite linear combinations of Dirac measures as discrete measures. Since Borel functions are universally measurable, we can identify the dual space $X^* = C^*(\Omega)$ with a strongly closed subspace of X_b^*. Let σ be the weak* topology on X_b^*; then $\sigma \geq w^*$ on X^* in the sense of comparison of topologies. Recall that an operator $P \in \Gamma(\Omega, \varphi)$ acting on X^* is continuous in the w^*-topology only if $P = R^*$ for some $R \in \operatorname{End} X$. At the same time, the regularity of an operator $P \in \Gamma(\Omega, \varphi)$ implies its (σ, w^*)-continuity, i.e. continuity with the σ-topology on the domain and the w^*-topology on the range. This readily follows from the identity $(x, Py) = (P^*x, y)$ for $x \in X$ and $y \in X^*$ and from the fact that $P^* : X \to X_b$ by Lemma 2.2.

LEMMA 2.3. *The linear manifold of discrete measures is σ-dense in X^*.*

Proof. The Dirac measures separate the points of X_b. Consequently, according to general facts about weak topologies [**3**, Exercise 5.7.41], the linear manifold generated by these measures is σ-dense in X_b^*. Thus, the discrete measures are also σ-dense in the space X^*. The proof of the lemma is complete. □

Note that every Borel measure on a metrizable compact set is a sequential w^*-limit of discrete measures. (This follows from the Riemann integrability of continuous functions.)

Set $\Phi(P) = \pi^{-1}\pi P$ for $P \in \Gamma(\Omega, \varphi)$. The operator classes $\Phi(P)$ form a partition of $\Gamma(\Omega, \varphi)$, which is generated by the equivalence relation '$P_1 \sim P_2$ whenever $\pi P_1 = \pi P_2$'.

Let $\operatorname{sq}(\cdot)$ be the set of all possible τ-limits of sequences of elements in an arbitrary set of transformations of Ω. Next, let $E_b(\Omega, \varphi)$ be the least sequentially τ-closed subset of all transformations $p : \Omega \to \Omega$ containing the generating family $E_0(\Omega, \varphi)$. In more detail, let us construct an increasing transfinite sequence of transformation classes $E_\nu = E_\nu(\Omega, \varphi)$ by setting $E_\nu = \bigcup_{\eta<\nu} E_\eta$ or $E_\nu = \operatorname{sq}(E_{\nu-1})$ depending on whether ν is a limit ordinal number or not. Then the union $E_b(\Omega, \varphi)$ of classes E_ν over all ordinal numbers ν can be obtained from the generating transformation family $E_0(\Omega, \varphi)$ by a transfinite sequence of sequential passages to the τ-limit. The set Π_b of Borel transformations is stable under

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

such passages, and $E_0(\Omega, \varphi) \subset \Pi_b$, so that $E_b(\Omega, \varphi) \subset E(\Omega, \varphi) \bigcap \Pi_b$; moreover, the inclusion can be proper.

The following claim (which is of interest in itself) is used in the derivation of Theorem 3.5 and Corollary 3.6 in what follows.

PROPOSITION 2.4. *If the function* $\Gamma(\Omega, \varphi) \xrightarrow{\pi} E(\Omega, \varphi)$ *defined in (2.2) is injective and* $E(\Omega, \varphi) = E_b(\Omega, \varphi)$, *then all operators* $P \in \Gamma(\Omega, \varphi)$ *are regular.*

The proof will be given below.

Let End (X, X^{**}) be the Banach space of bounded linear operators from X to X^{**}. Suppose that operators $R \in \text{End}\,(X, X^{**})$ and $T \in \text{End}\, X^*$ are related by the identity $(Rx, y) = (x, Ty)$ for $x \in X$ and $y \in X^*$. Then $R = T^*|_X$ and $T = R^*|_{X^*}$, and moreover, the correspondence $R \xrightarrow{\theta} T$ defines a natural linear isometry between the spaces End (X, X^{**}) and End X^*. To each transformation $p \in \Pi_b$, we assign a linear operator $R \in \text{End}\,(X, X^{**})$ by setting $(Rx)(\omega) = x(p\omega)$ for $x \in X$ and $\omega \in \Omega$. Then $Rx \in X_b$ and $P \in \text{End}\, X^*$ if we set $P = \theta R = \psi p$, where the function $\psi : \Pi_b \to \text{End}\, X^*$, $p \xrightarrow{\psi} P$, is generated by relation (2.3).

LEMMA 2.5. *If* $p \in E_b(\Omega, \varphi)$ *and* $P = \psi p$, *then* $P \in \Gamma(\Omega, \varphi)$.

Proof. Clearly, $P \in \Gamma(\Omega, \varphi)$ for $p \in E_0(\Omega, \varphi)$. Every element $p \in \text{sq}(E_\nu)$ is the τ-limit of a sequence of elements $p_k \in E_\nu$, and hence $x(p_k\omega) \to x(p\omega)$ for all $x \in X$, $\omega \in \Omega$. If we set $(R_k x)(\omega) = x(p_k\omega)$ and $(Rx)(\omega) = x(p\omega)$, then R_k, $R \in \text{End}\,(X, X^{**})$, and by Lebesgue's dominated convergence theorem we have $(R_k x, \mu) \to (Rx, \mu)$ for $\mu \in A(\Omega)$. It follows that $P_k \to P$ in the W*O-topology of the space End X^*, where $P_k = \theta R_k$ and $P = \theta R$. Thus, if $P_k \in \Gamma(\Omega, \varphi)$, then $P \in \Gamma(\Omega, \varphi)$ as well. An application of transfinite induction over ordinal numbers ν to the transformation classes $E_\nu(\Omega, \varphi)$ completes the proof. □

We see that $\pi\psi = \text{id}$ on the set $E_b(\Omega, \varphi)$.

LEMMA 2.6. *Let* $P \in \Gamma(\Omega, \varphi)$ *and* $\pi P \in E_b(\Omega, \varphi)$. *Then the operator set* $\Phi(P)$ *contains exactly one regular element.*

Proof. If $P_0 = \psi\pi P$, then Lemma 2.5 guarantees the inclusion $P_0 \in \Gamma(\Omega, \varphi)$. Moreover, $P_0 \in \Phi(P)$, and the operator P_0 is regular by Definition 2.1. From this definition it also follows that there exists just one such operator in the class $\Phi(P)$. The proof is complete. □

Now Proposition 2.4 directly follows from Lemma 2.6, because the proposition assumes that $\Phi(P) = \{P\}$.

The continuous action $P \to VP$ generates the semicascade (Γ, V) on the W*O-compact set $\Gamma \subset \text{End}\, X^*$, where $\Gamma = \Gamma(\Omega, \varphi)$. We adopt the following notation: $A(\Gamma)$ is the compact (in the w^*-topology on $C^*(\Gamma)$) convex set of regular Borel probability measures on Γ; $\text{Ai}(\Gamma)$ is the closed convex subset of V-invariant measures in $A(\Gamma)$; $\Lambda(\Gamma)$ is the subset of V-ergodic measures in $\text{Ai}(\Gamma)$, i.e. by one of the equivalent definitions in [**17**, Ch. 10], the set $\text{ex}\,\text{Ai}(\Gamma)$ of extreme points.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

Next, since $G = \overline{\text{co}}\,\Gamma_0$ and $\Gamma = \overline{\Gamma}_0$, where $\Gamma_0 = \{V^n,\ n \geq 0\}$, it follows according to [**17**, Proposition 1.5] that $\text{ex}\, G \subset \Gamma$, and since Γ is closed, it follows [**17**, Proposition 1.2] that to each operator $T \in G$ there corresponds a measure $\lambda \in A(\Gamma)$ realizing the Choquet representation

$$T = \int_\Gamma P\, \lambda(dP). \tag{2.4}$$

The integration here can be understood in the sense that

$$(x,\, T\mu) = \int_\Gamma (x,\, P\mu)\, \lambda(dP) \tag{2.5}$$

for all $x \in X$ and $\mu \in A(\Omega)$. We say that λ is a *representing* measure (for the operator T) and sometimes write $T = T_\lambda$. Note that the representing measure is not necessarily unique. The mapping $\gamma : \lambda \to T_\lambda$ of $A(\Gamma)$ onto $G(\Omega, \varphi)$ (the Choquet epimorphism) is continuous with the w^*-topology on the domain and the W*O-topology on the range.

It turns out that the correspondence $\lambda \to T_\lambda$ relates invariant measures on the compact set $\Gamma(\Omega, \varphi)$ to elements of the kernel L of the semigroup $G(\Omega, \varphi)$.

LEMMA 2.7. *The function γ maps* $\text{Ai}(\Gamma)$ *onto the set L.*

Proof. Let $\lambda \in \text{Ai}(\Gamma)$; then the relation

$$\int_\Gamma g(VP)\lambda(dP) = \int_\Gamma g(P)\lambda(dP) \tag{2.6}$$

holds for all functions $g \in C(\Gamma)$, including functions of the form $g(P) = (x,\, P\mu)$ with given $x \in X$ and $\mu \in A(\Omega)$. We find from (2.5) and (2.6) that

$$\begin{aligned}(x,\, T\mu) &= \int_\Gamma (x,\, P\mu)\lambda(dP) = \int_\Gamma (x,\, VP\mu)\lambda(dP)\\ &= \int_\Gamma (Ux,\, P\mu)\lambda(dP) = (Ux,\, T\mu) = (x,\, VT\mu),\end{aligned}$$

and hence $VT = T$ and $T \in L$. Next, the continuous transformation $P \to VP$ of the compact set $\Gamma(\Omega, \varphi)$ induces the (also continuous) self-mapping υ of the convex compact set $A(\Gamma)$ of measures. Moreover, for each operator $Q \in L$ and each $\lambda \in \gamma^{-1}Q$ one has

$$VQ = \int_\Gamma P\lambda_1(dP), \quad \lambda_1 = \upsilon\lambda.$$

When $VQ = Q$, we see that this mapping preserves the convex compact sets $\gamma^{-1}Q \subset A(\Gamma)$, and its fixed points $\lambda \in \gamma^{-1}Q$ prove to be invariant measures in $\text{Ai}(\Gamma)$. The proof of the lemma is complete. □

Thus, for each operator $Q \in L$ there exists at least one representation (2.4) with V-invariant measure λ on the semigroup $\Gamma(\Omega, \varphi)$. Now let us establish a relationship between ergodic measures $\lambda \in \Lambda(\Gamma)$ and extreme points of the kernel L of the semigroup $G(\Omega, \varphi)$.

LEMMA 2.8. *For each operator $Q \in \text{ex}\, L$, the full preimage $\gamma^{-1}Q$ contains an ergodic measure $\lambda \in \Lambda(\Gamma)$. If the Choquet epimorphism $\gamma : \text{Ai}(\Gamma) \to L$ is injective, then $\gamma : \Lambda(\Gamma) \to \text{ex}\, L$.*

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

Proof. Let $Q \in \operatorname{ex} L$; then $\gamma^{-1}Q$ is an extreme subset of $\mathrm{Ai}(\Gamma)$, and hence [**3**, Lemma 5.8.3] the extreme points of $\gamma^{-1}Q$ are extreme points of $\mathrm{Ai}(\Gamma)$, and $\operatorname{ex}\gamma^{-1}Q \subset \gamma^{-1}Q \bigcap \Lambda(\Gamma)$. The opposite inclusion is always true, and it follows that every element of $Q \in \operatorname{ex} L$ has at least one ergodic representing measure. Now let $\lambda \in \Lambda(\Gamma)$ and $Q = \gamma\lambda$. If $2Q = Q_1 + Q_2$ with $Q_i \in L$, then, by Lemma 2.7, $Q_i = \gamma\lambda_i$ for some measures $\lambda_1, \lambda_2 \in \mathrm{Ai}(\Gamma)$. The assumption that the restriction of γ to $\mathrm{Ai}(\Gamma)$ is injective ensures that $2\lambda = \lambda_1 + \lambda_2$, $\lambda = \lambda_i$, and $Q = Q_i$, $i = 1, 2$. The proof of the lemma is complete. □

It follows from formula (2.4) with regard to (2.2) that

$$T_\lambda \delta(\omega) = \int_\Gamma \delta(p\omega)\lambda(dP), \quad \omega \in \Omega,$$

where $\lambda \in A(\Gamma)$ and $T_\lambda \in G$. The extreme points of the convex set $A(\Omega) \subset X^*$ exactly coincide with the set $K(\Omega)$ of Dirac measures $\delta(\omega)$, $\omega \in \Omega$, and hence the Choquet representation for a measure $\mu \in A(\Omega)$ can be written as

$$\mu = \int_\Omega \delta(\omega)\, \mu(d\omega).$$

In particular, this is true for $\mu = T_\lambda \delta(\xi)$. If $e \in \Sigma_b$ and $h = \{P \in \Gamma(\Omega, \varphi) : p\,\xi \in e\}$, where $p = \pi P$, then $\mu(e) = \lambda(h)$. For given $\xi \in \Omega$, the mapping $e \to h$ carries the operations of union, intersection and complement of sets from Ω to $\Gamma(\Omega, \varphi)$ and preserves the classes of closed sets and hence of Borel sets. Moreover,

$$\{P \in \Gamma : VP \in h\} = \{P \in \Gamma : \varphi p\xi \in e\} = \{P \in \Gamma : p\xi \in \varphi^{-1}e\},$$

whence it follows that the correspondence $\lambda \to T_\lambda \delta(\xi)$ preserves the invariance of measures; i.e. $\mathrm{Ai}(\Gamma) \to \mathrm{Ai}(\Omega)$. Finally, to a φ-invariant set $e \subset \Omega$ there corresponds a V-invariant set $h \subset \Gamma$; consequently, ergodic measures $\lambda \in \Lambda(\Gamma)$ are taken to ergodic measures $\mu \in \Lambda(\Omega)$. Thus, we arrive at the following lemma.

LEMMA 2.9. *If $\omega \in \Omega$ and $\lambda \in \Lambda(\Gamma)$, then $T_\lambda \delta(\omega) \in \Lambda(\Omega)$.*

Together with Lemma 2.8, this gives the following proposition.

PROPOSITION 2.10. *If $Q \in \operatorname{ex} L$, then $Q : K(\Omega) \to \Lambda(\Omega)$.*

This claim is used below when proving Proposition 4.2. In particular, if $L = \{Q\}$, then $Q : K(\Omega) \to \Lambda(\Omega)$.

3. *Conditions for ordinary and tame dynamics*

In this section, we describe relationships between properties of the semicascade (Ω, φ). These properties are mostly stated in terms of topological characteristics of the associated semigroups $E(\Omega, \varphi)$, $\Gamma(\Omega, \varphi)$, $G(\Omega, \varphi)$. The base property is that of (Ω, φ) being ordinary.

Definition 3.1. A semicascade (Ω, φ) is said to be *ordinary* if its enveloping semigroup $E(\Omega, \varphi)$ is metrizable.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

It turns out that the metrizability of the enveloping semigroup $E(\Omega, \varphi)$ with the τ-topology of the Cartesian product guarantees that the semigroups $\Gamma(\Omega, \varphi)$ and $E(\Omega, \varphi)$ essentially coincide and that $\Gamma(\Omega, \varphi)$ coincides with the set $\operatorname{ex} G(\Omega, \varphi)$ of extreme points. One more characteristic of the ordinary semicascade (Ω, φ) is related to a slight restatement of the version in [**8**–**10**] of the notion of non-chaotic dynamics.

Definition 3.2. We say that a dynamical system (Ω, φ) is *non-chaotic* if each closed semi-invariant ($\varphi\Theta \subset \Theta$) set $\Theta \subset \Omega$ contains a trajectory $o(\omega)$, $\omega \in \Theta$, that is Lyapunov stable with respect to the semicascade (Θ, φ).

Recall that the trajectory $o(\omega) = \{\varphi^n\omega, n \geq 0\}$ is said to be Lyapunov stable if for any $\varepsilon > 0$ there exists a neighborhood $O(\omega) \subset \Theta$ such that for every $\omega_0 \subset O(\omega)$ we have $\rho(\varphi^n\omega_0, \varphi^n\omega) < \varepsilon$, $n \geq 0$. This property does not depend on the choice of compatible metric ρ on Ω and it is equivalent to the *hereditary almost equicontinuity* of the system (Ω, φ) (see [**8**, Theorem 8.1]).

Consider yet another type of dynamical system starting from the statement due to Köhler [**14**].

Definition 3.3. We say that a semicascade (Ω, φ) is *tame* if, for each continuous function $x \in C(\Omega)$ and an arbitrary increasing subsequence $\{n(k)\}$ of positive integers, one has the relation

$$\inf_a \left\| \sum_{k=1}^{\infty} a_k x_{n(k)} \right\|_{C(\Omega)} = 0,$$

where $x_n(\omega) = x(\varphi^n\omega)$, $\omega \in \Omega$, and the infimum is taken over finite real sequences $a = \{a_k\}$ such that $\sum_{k=1}^{\infty} |a_k| = 1$.

Both classes of dynamical systems are stable (see [**8**]) with respect to passage to closed subsystems. Furthermore, it is well known [**8**, Theorem 6.5] that a dynamical system is tame if and only if $E(\Omega, \varphi) \subset \Pi_1$, where Π_1 is the set of first Baire class transformations of Ω. Let $E_1(\Omega, \varphi)$ be the sequential τ-closure of the set $E_0(\Omega, \varphi) = \{\varphi^n, n \geq 0\}$. Clearly, $E_1(\Omega, \varphi) \subset E(\Omega, \varphi) \bigcap \Pi_1$. Recall that Π_b is the set of Borel mappings $p : \Omega \to \Omega$. For brevity, we sometimes write $\Gamma(\Omega, \varphi) = \Gamma$, $G(\Omega, \varphi) = G$. Just as above, by $A(\cdot)$ and $K(\cdot)$ we denote the spaces, compact in the w^*-topology, of regular Borel probability measures and Dirac measures, respectively, on some Hausdorff compact set.

Now let us present a table of properties, divided into classes $D1$–$D7$, of the semicascade (Ω, φ).

$D1$: (a1) The compact set $E(\Omega, \varphi)$ is metrizable; (b1) the system (Ω, φ) is non-chaotic; (c1) the compact set $G(\Omega, \varphi)$ is metrizable.

$D2$: (a2) The compact set $G(\Omega, \varphi)$ is a Fréchet–Urysohn space; (b2) card $G(\Omega, \varphi) \leq \mathfrak{c}$.

$D3$: (a3) The semicascade (Ω, φ) is a tame system; (b3) the compact set $E(\Omega, \varphi)$ is a Fréchet–Urysohn space; (c3) card $E(\Omega, \varphi) \leq \mathfrak{c}$; (d3) $E(\Omega, \varphi) = E_1(\Omega, \varphi)$; (e3) $E(\Omega, \varphi) \subset \Pi_b$.

$D4$: (a4) The operators $T \in G(\Omega, \varphi)$ are determined by their values on Dirac measures $\mu \in K(\Omega)$.

$D5$: (a5) The correspondence $\lambda \to T_\lambda$ in Choquet representation (2.4) is injective.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

$D6$: (a6) The operators $P \in \Gamma(\Omega, \varphi)$ are determined by their values on Dirac measures $\mu \in K(\Omega)$; (b6) the canonical epimorphism $\pi : \Gamma(\Omega, \varphi) \to E(\Omega, \varphi)$ is injective.
$D7$: (a7) $\operatorname{ex} G = \Gamma$.

Parts of the properties of the dynamical system (Ω, φ) in the classes $D1$, $D3$, and $D6$ were considered earlier in [**7**–**10**]. Although these papers deal with invertible dynamical systems, the passage to the non-invertible case proves to be purely technical in situations of interest. A topological compact set has the Fréchet–Urysohn property if the operation of closure for sets in it can be defined in terms of sequences. Metrizability implies this property, but the converse is not true.

We note that in all cases except the trivial one when $\varphi = \mathrm{id}$, $G(\Omega, \varphi) = \{I\}$, we actually have $\operatorname{card} G(\Omega, \varphi) = \mathfrak{c}$ in property (b2). Property (a5) in particular means that the convex set $G(\Omega, \varphi)$ is Choquet simplex (see [**17**, Ch. 9]). If condition (b6) holds, then the function $\pi : \Gamma(\Omega, \varphi) \to E(\Omega, \varphi)$ defined by (2.2) is simultaneously a homeomorphism and an algebraic isomorphism, which permits complete identification of the enveloping semigroups $\Gamma(\Omega, \varphi)$ and $E(\Omega, \varphi)$. In this situation, we write $\Gamma(\Omega, \varphi) \simeq E(\Omega, \varphi)$.

THEOREM 3.4. *The properties of the semicascade* (Ω, φ) *within each of the classes D1, D2, D3, and D6 are pairwise equivalent.*

Thus, each of properties (b1) and (c1) can be taken for an alternative definition of being ordinary, and each of properties (b3), (c3), (d3), and (e3) completely characterizes tame systems (Ω, φ). We point out that the implications (a1) $\Leftrightarrow$ (b1) and (a3) $\Leftrightarrow$ (b3) $\Leftrightarrow$ (c3) were established in [**7**–**10**]. The equivalence of properties (d3) and (e3) means that if the enveloping semigroup $E(\Omega, \varphi)$ contains transformations that do not belong to the first Baire class, then it also contains transformations that are not Borel.

Proof. The class $D1$. By [**8**, p. 2357], conditions (a1) and (b1) are equivalent. The metrizability of the compact set $G(\Omega, \varphi)$ implies that of its closed subset $\Gamma(\Omega, \varphi)$ and hence the metrizability of the range of the continuous mapping $\Gamma(\Omega, \varphi) \xrightarrow{\pi} E(\Omega, \varphi)$. Thus, (c1) $\Rightarrow$ (a1). Next, (a1) $\Rightarrow$ (c3), and condition (a1) implies the metrizability of the topological space $A(E)$ of measures. It is well known [**8**, Theorem 7.1] that (c3) $\Rightarrow$ (b6), and since, under this condition, $\Gamma(\Omega, \varphi) \simeq E(\Omega, \varphi)$, it follows that condition (a1) also implies the metrizability of the space $A(\Gamma)$ of measures. The operator semigroup $G(\Omega, \varphi)$ is the range of the continuous Choquet surjection $A(\Gamma) \xrightarrow{\gamma} G(\Omega, \varphi)$, and hence the compact set $G(\Omega, \varphi)$ is metrizable as well, and (a1) $\Rightarrow$ (c1).

The class $D2$. As was already noted, the semigroup $G(\Omega, \varphi)$ is enveloping for the action $W \times A \xrightarrow{V} A$ on the compact set $A = A(\Omega)$ of the abelian semigroup W of non-negative finite numerical sequences with unit sum and with convolution as multiplication. Thus, the equivalence of conditions (a2) and (b2) follows from the results in [**8**, §6] for the enveloping semigroup $G = E(A, W)$.

The class $D3$. The pairwise equivalence of conditions (a3)–(c3) is a well-known fact [**8**, pp. 2345, 2353], [**7**, Theorem 1.2], and the implications (b3) $\Rightarrow$ (d3) $\Rightarrow$ (e3) are trivial. To establish the implication (e3) $\Rightarrow$(c3), we slightly generalize the scheme in [**1**, §2.12(ii)] and identify Borel transformations $p : \Omega \to \Omega$ with their graphs $(\omega, p\omega) \subset \Omega \times \Omega, \omega \in \Omega$. The transformation $(\omega_1, \omega_2) \to (p\omega_1, \omega_2)$ of the metrizable compact set $\Omega \times \Omega$ proves to

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

be a Borel transformation, and hence so is the scalar function $g(\omega_1, \omega_2) = \rho(p\omega_1, \omega_2)$ on $\Omega \times \Omega$ for any of the metrics ρ compatible with the original topology on Ω. The graph of p coincides with $g^{-1}(0)$ and hence is a Borel subset of $\Omega \times \Omega$. It is well known (see [**2**, §9.6, Exercise 4c]), that the set of such sets is at most of the cardinality of continuum. Thus, if $E(\Omega, \varphi) \subset \Pi_b$, then card $E(\Omega, \varphi) \leq \mathfrak{c}$, as desired.

The class $D6$. Property (a6) is a useful restatement of (b6) and the proof of the theorem is complete. □

Thus, now we can speak of the classes $\mathcal{D}1$–$\mathcal{D}7$ of discrete dynamical systems (Ω, φ) possessing any of the properties in the respective classes $D1$–$D7$. Note that $\mathcal{D}1$ is the class of ordinary semicascades, and $\mathcal{D}3$ is the class of tame semicascades.

THEOREM 3.5. *One has the embeddings $\mathcal{D}1 \subset \mathcal{D}2 \subset \mathcal{D}3 \subset \mathcal{D}6$, $\mathcal{D}1 \subset \mathcal{D}4 \subset \mathcal{D}6 \subset \mathcal{D}7$ and $\mathcal{D}5 \subset \mathcal{D}7$.*

Proof. It suffices to establish the logical chains (a1) ⇒ (b2) ⇒ (c3) ⇒ (b6), (a1) ⇒ (a4) ⇒ (a6) ⇒ (a7) and (a5) ⇒ (a7). It has already been mentioned above that (c3) ⇒ (b6) and shown that (a1) ⇔ (c1). The implication (c1) ⇒ (b2) is trivial, and since the set $E(\Omega, \varphi)$ is a continuous image of the compact set $\Gamma(\Omega, \varphi)$ and $\Gamma(\Omega, \varphi) \subset G(\Omega, \varphi)$, it follows that (b2) ⇒ (c3), and hence (a1) ⇒ (b2) ⇒ (c3), which proves the first part of the theorem.

Now let us show that (a1) ⇒ (a4). We use the already known implications (a1) ⇒ (c3) ⇒ (b6) and (c3) ⇒ (d3). We rewrite formula (2.5) for the operator $T \in G(\Omega, \varphi)$ in the form

$$(T^*x, \mu) = \int_\Gamma (P^*x, \mu)\lambda(dP)$$

for any $x \in X$ and $\mu \in A(\Omega)$ and note once more that the function $g(P) = (P^*x, \mu) = (x, P\mu)$ is continuous on $\Gamma(\Omega, \varphi)$. Since $\Gamma(\Omega, \varphi) \simeq E(\Omega, \varphi)$ in this case and the compact topological space $E(\Omega, \varphi)$ is metrizable by property (a1), it follows that the compact set $\Gamma(\Omega, \varphi)$ is metrizable as well, and hence $\lambda_k \xrightarrow{w^*} \lambda$ for some sequence λ_k of convex combinations of Dirac measures on Γ. Thus,

$$(T^*x, \mu) = \lim_{k\to\infty} \int_\Gamma (P^*x, \mu)\lambda_k(dP) = \lim_{k\to\infty} (z_k, \mu),$$

where $z_k = \int_\Gamma P^*x\lambda_k(dP)$, $z_k \in X^{**}$. By using property (b6) and Proposition 2.4, we find that all operators $P \in \Gamma(\Omega, \varphi)$ are regular and, by Lemma 2.2, $P^*x \in X_b$ whenever $x \in X$. Thus, $z_k \in X_b$, and since one can set $\mu = \delta(\omega)$, $\omega \in \Omega$, in formula (2.5), we see that $z_k \to T^*x$ pointwise on Ω and $T^*x \in X_b$ for each $x \in X$. Now the identity $(T^*x, \mu) = (x, T\mu)$ implies that all operators $T \in G(\Omega, \varphi)$ are determined by their values on the Dirac measures $\delta(\omega)$ and that the semicascade (Ω, φ) has property (a4); i.e. (a1) ⇒ (a4).

Clearly, (a4) ⇒ (a6). Now we shall show that (a6) ⇒ (a7). Suppose that $P \in \Gamma$ and $2P = T_1 + T_2$, where $T_i \in G$ for $i = 1, 2$. Then $2\delta(p\,\omega) = T_1\delta(\omega) + T_2\delta(\omega)$ for every $\omega \in \Omega$, and taking into account that ex $A(\Omega) = K(\Omega)$, we obtain $T_i|_K = P|_K$. For an arbitrary operator $T \in G$ with $TK \subset K$ we set $H(T) = \{P \in \Gamma : P|_K = T|_K\}$. Then (see [**18**, p. 1180])

$$T = \int_{H(T)} P\lambda(dP),$$

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

where $\lambda \in A(\Gamma)$ is a measure realizing Choquet representation (2.4). It follows from condition (a6) that $H(T_i) = H(P) = \{P\}$, so $T_1 = T_2 = P$. Note that we always have $\operatorname{ex} G \subset \Gamma$; thus $\operatorname{ex} G = \Gamma$.

It remains to show the implication (a5) $\Rightarrow$ (a7). Assume that $T \in \Gamma(\Omega, \varphi)$ but $T \notin \operatorname{ex} G$. Then, by [**4**, Theorem 10.1.7], there exists a measure $\lambda \in A(\Gamma)$ representing the operator T and different from the Dirac measure $\delta(T)$. This, however, contradicts condition (a5). The proof of Theorem 3.5 is complete. □

The implications (a4) $\Rightarrow$ (a5) and (a7) $\Rightarrow$ (b6) were announced without proofs by the author in [**19**, **20**]. The proofs that the author assumed to publish later failed. The author does not know if these implications hold†.

The papers [**8**] and [**10**] give examples of compact tame but non-ordinary dynamical systems that are, however, more complicated than cascades or semicascades. According to [**9**, Proposition 10.5], the semicascades generated by homeomorphisms of the segment or the circle are ordinary. On the other hand [**9**, Example 10.6.1, Theorem 9.14], an arbitrary semicascade corresponding to a hyperbolic diffeomorphism of the two-dimensional torus is not ordinary. Finally, there exists an example [**8**, p. 2356] of a minimal distal cascade on the two-dimensional torus such that this cascade does not belong class $\mathcal{D}6$ and hence is not tame. The classification of semicascades presented here arose solely in connection with the problems of ergodic theory considered in the next section and is of preliminary character. It would be useful to find out whether all classes $\mathcal{D}1$–$\mathcal{D}7$ of discrete dynamical systems are pairwise distinct and whether they form an increasing chain.

Theorem 3.4 shows that for a tame semicascade (Ω, φ) the relations $E(\Omega, \varphi) = E_1(\Omega, \varphi) = E_b(\Omega, \varphi)$ hold. Thus, using also the implication (a3) $\Rightarrow$ (b6), from Proposition 2.4 we derive the following important corollary.

COROLLARY 3.6. *For a tame semicascade (Ω, φ), all operators $P \in \Gamma(\Omega, \varphi)$ are regular.*

The following assertion shows that the enveloping semigroups of non-ordinary dynamical systems have a rather intricate structure.

PROPOSITION 3.7. *If for a semicascade (Ω, φ) there exists a countable set $e \subset \Omega$ such that, for any transformations $p_1, p_2 \in E(\Omega, \varphi)$, their coincidence on this set implies that $p_1 = p_2$, then the semigroup $E(\Omega, \varphi)$ is metrizable.*

Proof. The assumption says that the natural continuous map $E(\Omega, \varphi) \to \Pi_{\omega \in e}\, \Omega$ is injective. Since $E(\Omega, \varphi)$ is compact and $\Pi_{\omega \in e}\, \Omega$ is Hausdorff, this map is a topological embedding. Because e is countable and Ω is metrizable, $\Pi_{\omega \in e}\, \Omega$ is metrizable. Thus $E(\Omega, \varphi)$ is metrizable. □

We see that in the non-ordinary case, for each countable subset $e \subset \Omega$, the enveloping semigroup $E(\Omega, \varphi)$ contains transformations $p_1 \neq p_2$ coinciding on e. On the other hand, Proposition 3.7 has the following corollary.

COROLLARY 3.8. *If all transformations $p \in E(\Omega, \varphi)$ are continuous outside some given countable set $e \subset \Omega$, then the semicascade (Ω, φ) is ordinary.*

† We note also that the proof of Lemma 4.1 in [**18**] is incorrect, though the assertion of the lemma seems plausible.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

It is of interest to compare this assertion with the well-known criterion [**6**] for the continuity of all elements of the enveloping semigroup: $E(\Omega, \varphi) \subset C(\Omega, \Omega)$ if and only if the dynamical system (Ω, φ) is weakly almost periodic. The latter means that the set of translates $\{x \circ \varphi^n, n \geq 0\}$ of an arbitrary continuous function x is relatively compact in the weak topology of the space $C(\Omega)$.

4. *Ergodic properties of semicascades*

The constructions in the preceding sections are used here to study ergodic properties of ordinary and tame discrete dynamical systems. When doing so, we heavily use the results in [**18**]. Just as before, on the space X of continuous functions on a metrizable compact set Ω we consider the shift operator U corresponding to a continuous transformation $\varphi : \Omega \to \Omega$. Let $V = U^*$. In the operator algebra End X^*, we single out the semigroups $\Gamma_0 = \{V^n, n \geq 0\}$ and $G_0 = \operatorname{co} \Gamma_0$ as well as the compact semigroups $G = \overline{\operatorname{co}}\, \Gamma_0$ and $\Gamma = \overline{\Gamma}_0$, where $\overline{(\cdot)}$ is the operation of W*O-closure. The notation of these semigroups will include the dependence on Ω and φ where necessary. Next, let $\Lambda(\Omega)$ be the set of all Borel probability φ-ergodic measures on Ω.

Throughout the following, the convergence of nets in End X^* is understood in the sense of the W*O-topology. For an arbitrary semicascade (Ω, φ), we say that an operator net $T_\alpha \in G_0(\Omega, \varphi)$ is ergodic if $T_\alpha(I - V) \xrightarrow{\text{W*O}} 0$, where $I = \mathrm{id}$. Note that $T_\alpha = R_\alpha^*$, where $R_\alpha \in \text{End } X$. The sequence

$$V_n = \frac{1}{n}(I + V + \cdots + V^{n-1})$$

of Cesàro means is ergodic, because $V_n(I - V) = n^{-1}(I - V^n)$ and $\|V_n(I - V)\| \leq 2n^{-1}$. By [**18**, Lemma 1.1], the kernel L of the semigroup $G(\Omega, \varphi)$ is determined by relation (2.1). Moreover, [**18**, Lemma 1.2], every element in L is the limit of some operator ergodic net, and conversely, if an ergodic net T_α converges to an operator Q, then $Q \in L$. Recall that the kernel L consists of projections $Q \in \text{End } X^*$.

THEOREM 4.1. *If a semicascade (Ω, φ) is ordinary, then every ergodic operator net T_α contains a convergent ergodic sequence $T_{\alpha(k)}$.*

Proof. Since $G(\Omega, \varphi)$ is compact, we can assume that $T_\alpha \xrightarrow{\text{W*O}} Q$, $Q \in L$. From Theorem 3.5 it follows that the set $G(\Omega, \varphi)$ has the Fréchet–Urysohn property. Hence, the net T_α contains a sequence $T_\alpha(k) \xrightarrow{\text{W*O}} Q$. Taking into account that $Q = VQ$ for all $Q \in L$, we see that the sequence $T_\alpha(k)$ is ergodic. The proof is complete. □

Thus, in the case of ordinary dynamics, for each projection $Q \in L$ there exists an ergodic sequence of operators $T_n \in G_0$ converging to Q. We see that actually this result (as well as Theorem 4.1) holds for the class $\mathcal{D}2$ of compact discrete dynamical systems, which is apparently wider than the class $\mathcal{D}1$ of ordinary semicascades.

As a straightforward consequence of the last theorem and Proposition 2.10, we obtain the following assertion.

PROPOSITION 4.2. *If the semicascade (Ω, φ) is ordinary, then for each projection $Q \in \operatorname{ex} L$ there exists a convergent ergodic sequence of operators $T_n \to Q$ such that, for*

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

each $\omega \in \Omega$, the measures $T_n\delta(\omega)$ converge as $n \to \infty$ in the w^*-topology to the ergodic measure $\mu = Q\delta(\omega)$.

In the conventional terminology [**12**, Ch. 4], this means that the asymptotic (with respect to the corresponding operator ergodic sequence $\{T_n\}$) distribution of trajectories of the dynamical system (Ω, φ) is determined by ergodic measures. If, in the ordinary case, card $L = 1$, then, for every ergodic operator net $T_\alpha \in G_0$ and for each $\omega \in \Omega$, one has the convergence $T_\alpha\delta(\omega) \xrightarrow{w^*} \mu_\omega$, where $\mu_\omega \in \Lambda(\Omega)$.

The convergence of the ergodic operator net $T_\alpha = R_\alpha^*$ in the W*O-topology of the space End X^* implies the pointwise convergence on Ω of the net $R_\alpha x$ of functions for an arbitrary continuous function $x \in X$. In the sequential case, the converse assertion follows from the Lebesgue dominated convergence theorem, but this is no longer the case in the general situation. However, it turns out that, when studying the convergence of ergodic operator nets T_α for ordinary dynamical systems, we can restrict ourselves to verifying the pointwise convergence of the functions $R_\alpha x$ continuous on Ω for all $x \in X$.

PROPOSITION 4.3. *If the semicascade (Ω, φ) is ordinary, then a sufficient condition for the* W*O*-convergence of the ergodic operator net $T_\alpha \in G_0$ is given by the pointwise convergence on Ω of the function nets $R_\alpha x$ for all $x \in X$.*

The proof readily follows from the inclusion $\mathcal{D}1 \subset \mathcal{D}4$ in Theorem 3.5, so that actually the class $\mathcal{D}4$ of discrete dynamical systems has this property.

In what follows, we denote by $M(\Omega)$ and $Z(\Omega)$ the union of all minimal sets and the minimal attraction center, respectively, of the semicascade (Ω, φ). Recall that $Z(\Omega)$ coincides with the closure of the union of supports of all measures $\mu \in \Lambda(\Omega)$, and $M^c(\Omega) \subset Z(\Omega)$, where $(\cdot)^c$ is the operation of closure in Ω; in the general case, the inclusion may be proper. As was already noted in §2, the enveloping semigroup $E(\Omega, \varphi)$ for a tame semicascade (Ω, φ) coincides with the transformation class $E_1(\Omega, \varphi)$. This is even more than sufficient for [**18**, Theorem 2.4] to hold, which implies that the sets $M^c(\Omega)$ and $Z(\Omega)$ coincide in this case.

PROPOSITION 4.4. *If the semicascade (Ω, φ) is a tame system, then $M^c(\Omega) = Z(\Omega)$.*

Note that the relation $M^c(\Omega) = Z(\Omega)$ does not mean yet that all ergodic measures $\mu \in \Lambda(\Omega)$ are concentrated on minimal sets.

For a point $\omega \in \Omega$, we denote by $\overline{o}(\omega)$ the closure of the trajectory $\{\varphi^n\omega, n \geq 0\}$ in the topology of Ω.

Slightly varying the statements in [**18**, pp. 1176–1177], consider the following properties of an arbitrary semicascade (Ω, φ).

(A) $E(\Omega, \varphi) = E_1(\Omega, \varphi)$.

(A_1) The support of every Borel probability φ-ergodic measure is a minimal set.

(B) Every minimal set $m \subset \Omega$ supports exactly one Borel probability φ-invariant measure.

(C) For each $\omega \in \Omega$, the closure of the trajectory $\overline{o}(\omega)$ contains a unique minimal set.

(D) All ergodic operator sequences $T_n \in G_0$ converge.

(E) All ergodic operator nets $T_\alpha \in G_0$ converge.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

By [**18**, Theorem 3.2], for an arbitrary semicascade (Ω, φ) one has the implications (E) $\Rightarrow$ (B) + (C), (A_1) + (B) + (C) $\Rightarrow$ (D). Furthermore [**18**, Theorem 1.3], condition (E) is equivalent to the condition that the kernel L of the semigroup $G(\Omega, \varphi)$ is a singleton. Condition (A) is a restatement of the notion of tame system, thus it is true for an ordinary semicascade. It is also well known (see [**11**, **13**], [**8**, p. 2345]) that, for the case of a tame semicascade, every minimal set supports exactly one invariant probability measure; i.e. condition (B) is satisfied. Moreover, all operators $P \in \Gamma(\Omega, \varphi)$ are regular in this case (see Corollary 3.6). Since $P\mu = \mu$ for ergodic measures $\mu \in \Lambda(\Omega)$, it follows that $\mu(p^{-1}e) = \mu(e)$ for all transformations $p \in E(\Omega, \varphi)$ and Borel sets $e \subset \Omega$. As the proof of [**18**, Theorem 3.2] shows, this suffices for deriving the implication (A) + (C) $\Rightarrow$ (A_1). Thus, in the tame situation we have (A_1) + (C) $\Rightarrow$ (D) and hence eventually (C) $\Rightarrow$ (A_1) and (C) $\Rightarrow$ (D). Since one always has (E) $\Rightarrow$ (B) + (C) and Theorem 4.1 implies that (D) $\Leftrightarrow$ (E) in the case of ordinary dynamics, it follows that the equivalences (C) $\Leftrightarrow$ (D) $\Leftrightarrow$ (E) hold in this class of discrete dynamical systems. Let us summarize the preceding in the form of two theorems.

THEOREM 4.5. *If the semicascade (Ω, φ) is ordinary, then all ergodic operator nets T_α converge if and only if, for each $\omega \in \Omega$, the closure of the trajectory $\overline{o}(\omega)$ contains a unique minimal set. The same is true with nets replaced by sequences.*

THEOREM 4.6. *If, for a tame semicascade (Ω, φ), the closure of each trajectory $\overline{o}(\omega)$ contains a unique minimal set, then all ergodic operator sequences T_n converge, and the support of each φ-ergodic measure is a minimal set.*

We say that a pair $\omega_1, \omega_2 \in \Omega$ of points is proximal and write $(\omega_1, \omega_2) \in \mathrm{P}$ if $\inf_{n\geq 0} \rho(\varphi^n\omega_1, \varphi^n\omega_2) = 0$ for a metric ρ compatible with the given topology on Ω. This property does not depend [**5**, Ch. 5] on the choice of compatible metric ρ on Ω.

LEMMA 4.7. *If the proximality relation $\mathrm{P}(\Omega, \varphi)$ is transitive, then for each point $\omega \in \Omega$ there exists exactly one minimal set $m \subset \overline{o}(\omega)$.*

Proof. Assume that there exists a point $\omega \in \Omega$ such that the closure $\overline{o}(\omega)$ of the trajectory $o(\omega)$ contains two distinct minimal sets m_1 and m_2. Then there exist two minimal left ideals I_1 and I_2 in $E(\Omega, \varphi)$ such that $I_1\omega = m_1$ and $I_2\omega = m_2$ (see [**18**, Lemma 3.2]). Since proximality relation is transitive, there exists [**5**, Proposition 5.16] only one left minimal ideal in $E(\Omega, \varphi)$, so $m_1 = m_2$. □

By combining Theorems 4.5 and 4.6 with Lemma 4.7, we obtain the following corollary.

COROLLARY 4.8. *If, for an ordinary semicascade (Ω, φ), the proximality relation $\mathrm{P}(\Omega, \varphi)$ is transitive, then all ergodic operator nets T_α converge. In the case of tame dynamics, the transitivity of the relation $\mathrm{P}(\Omega, \varphi)$ implies the convergence of all ergodic operator sequences T_n.*

Since the convergence of Cesàro means is an important problem in ergodic theory, we single out the following result, which is a consequence of Theorem 4.1 and Corollary 4.8.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press

PROPOSITION 4.9. *For an ordinary semicascade* (Ω, φ)*, the sequence of Cesàro means* V_n *always contains a convergent subsequence* $V_{n(k)}$*. If, for a tame semicascade* (Ω, φ)*, the proximality relation* $\mathrm{P}(\Omega, \varphi)$ *is transitive, then the operator means* V_n *converge as* $n \to \infty$*.*

Note that the claims about ordinary semicascades in Propositions 4.2 and 4.9 as well as in Theorem 4.5 and Corollary 4.8 actually remain valid in the formally broader class $\mathcal{D}2$ of compact discrete dynamical systems.

Acknowledgements. The author is grateful to the anonymous referee for numerous useful remarks as well as for the proof of Proposition 3.7 which turned out to be shorter and more elegant than the author's initial proof.

REFERENCES

[**1**] V. I. Bogachev. *Measure Theory, Vol. I*. Springer, Berlin, 2007.
[**2**] N. Bourbaki. *Elements of Mathematics: General Topology, Part 2*. Hermann, Paris, 1966.
[**3**] N. Dunford and J. T. Schwartz. *Linear Operators General Theory, Part I*. Interscience Publishers, New York, 1958.
[**4**] R. E. Edwards. *Functional Analysis Theory and Applications*. Holt, Rinehart and Winston, New York, 1965.
[**5**] R. Ellis. *Lectures on Topological Dynamics*. Benjamin, New York, 1969.
[**6**] R. Ellis and M. Nerurkar. Weakly almost periodic flows. *Trans. Amer. Math. Soc.* **313** (1989), 103–119.
[**7**] E. Glasner. On tame dynamical systems. *Colloq. Math.* **105** (2006), 283–295.
[**8**] E. Glasner. Enveloping semigroups in topological dynamics. *Topology Appl.* **154** (2007), 2344–2363.
[**9**] E. Glasner and M. Megrelishvili. Hereditarily non-sensitive dynamical systems and linear representations. *Colloq. Math.* **104** (2006), 223–283.
[**10**] E. Glasner, M. Megrelishvili and V. V. Uspenskij. On metrizable enveloping semigroups. *Israel J. Math.* **164** (2008), 317–332.
[**11**] W. Huang. Tame systems and scrambled pairs under an abelian group action. *Ergod. Th. & Dynam. Sys.* **26** (2006), 1549–1567.
[**12**] A. Katok and B. Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995.
[**13**] D. Kerr and H. Li. Independence in topological and C^*-dynamics. *Math. Ann.* **338** (2007), 869–926.
[**14**] A. Köhler. Enveloping semigroups for flows. *Proc. R. Irish Acad.* **95A** (1995), 179–191.
[**15**] N. Kryloff and N. Bogoliouboff. La théorie générale de la mesure dans son application à l'étude des systèmes dynamiques de la mécanique non linéaire. *Ann. of Math.* **38** (1937), 65–113.
[**16**] S. P. Lloyd. On the ergodic theorem of Sine. *Proc. Amer. Math. Soc.* **56** (1976), 121–126.
[**17**] R. R. Phelps. *Lectures on Choquet's Theorem*. Van Nostrand, Princeton, NJ, 1966.
[**18**] A. V. Romanov. Weak* convergence of operator means. *Izv. Math.* **75** (2011), 1165–1183.
[**19**] A. V. Romanov. Enveloping semigroups and ergodic properties of compact semi-cascades. *International Conference on Differential Equations and Dynamical Systems (Suzdal 2012), Abstracts*. MIAN, Moscow, 2012, pp. 151–152.
[**20**] A. V. Romanov. Ordinary semicascades and their ergodic properties. *Funct. Anal. Appl.* **47** (2013), 160–163.

https://doi.org/10.1017/etds.2014.62 Published online by Cambridge University Press