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ON A PARTITIONED INVERSION FORMULA HAVING USEFUL APPLICATIONS IN ECONOMETRICS

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In this paper a novel partitioned inversion formula is obtained in terms of the orthogonal complements of off-diagonal blocks, with the emblematic matrix of unit-root econometrics emerging as the leading diagonal block of the inverse. The result paves the way to a straightforward derivation of a key result of vector autoregressive econometrics.

1. THE PARTITIONED-INVERSION THEOREM

Partitioned inversion formulas have been, for a long time, a basic tool of the standard algebraic equipment of econometricians. In this section we establish a new result in partitioned inversion that may prove useful in unit-root and classical econometrics also.

THEOREM 1. *Let* $\boldsymbol{A}$ *be a square matrix of order* $n \times n$ *and* $\boldsymbol{B}$ *and* $\boldsymbol{C}$ *two rectangular matrices of order* $n \times m$ *and full column rank. Further let* $\boldsymbol{B}_\perp$ *and* $\boldsymbol{C}_\perp$ *denote the orthogonal complements of* $\boldsymbol{B}$ *and* $\boldsymbol{C}$*. Consider the partitioned matrix*

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C}' & \boldsymbol{O} \end{bmatrix}. \tag{1.1}$$

If $\boldsymbol{B}_\perp'\boldsymbol{A}\boldsymbol{C}_\perp$ *is of full rank, then*

(a) the matrix $\boldsymbol{P}$ *is nonsingular;*
(b) its partitioned inverse is

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{E} & (\boldsymbol{I}_n - \boldsymbol{E}\boldsymbol{A})\cdot(\boldsymbol{C}')^- \\ \boldsymbol{B}^-\cdot(\boldsymbol{I}_n - \boldsymbol{A}\boldsymbol{E}) & \boldsymbol{B}^-\cdot(\boldsymbol{A}\boldsymbol{E}\boldsymbol{A} - \boldsymbol{A})\cdot(\boldsymbol{C}')^- \end{bmatrix}, \tag{1.2}$$

where

$$\boldsymbol{E} = \boldsymbol{C}_\perp\cdot(\boldsymbol{B}_\perp'\boldsymbol{A}\boldsymbol{C}_\perp)^{-1}\cdot\boldsymbol{B}_\perp' \tag{1.3}$$

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and $\boldsymbol{B}^-$ *and* $(\boldsymbol{C}')^-$ *are left and right inverses of* $\boldsymbol{B}$ *and* $\boldsymbol{C}'$*, respectively, namely,* $\boldsymbol{B}^- = (\boldsymbol{B}'\boldsymbol{B})^{-1}\cdot\boldsymbol{B}'$ *and* $(\boldsymbol{C}')^- = \boldsymbol{C}\cdot(\boldsymbol{C}'\boldsymbol{C})^{-1}$.

Proof. Result (a) follows from the rank identity (see Marsaglia and Styan, 1974, Theorem 19)

$$\begin{aligned} rk(\boldsymbol{P}) &= rk(\boldsymbol{B}) + rk(\boldsymbol{C}) + rk\{(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}^-)\cdot\boldsymbol{A}\cdot[\boldsymbol{I} - (\boldsymbol{C}')^-\boldsymbol{C}']\} \\ &= rk(\boldsymbol{B}) + rk(\boldsymbol{C}) + rk(\boldsymbol{B}_\perp'\boldsymbol{A}\boldsymbol{C}_\perp) \end{aligned} \tag{1.4}$$

given that

$$\boldsymbol{I} - \boldsymbol{D}\cdot(\boldsymbol{D}'\boldsymbol{D})^{-1}\cdot\boldsymbol{D}' = \boldsymbol{D}_\perp\cdot(\boldsymbol{D}_\perp'\boldsymbol{D}_\perp)^{-1}\cdot\boldsymbol{D}_\perp' \tag{1.5}$$

for every full column rank matrix $\boldsymbol{D}$ (e.g., Johansen, 1995, p. 39).

To prove (*b*) let the inverse of $\boldsymbol{P}$ be

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{E} & \boldsymbol{F} \\ \boldsymbol{G}' & \boldsymbol{H} \end{bmatrix}, \tag{1.6}$$

where the blocks in $\boldsymbol{Q}$ are of the same order as the corresponding blocks in $\boldsymbol{P}$.

Then, to express the blocks of the former in terms of the blocks of the latter, write $\boldsymbol{Q}\boldsymbol{P} = \boldsymbol{I}$ and $\boldsymbol{P}\boldsymbol{Q} = \boldsymbol{I}$ by equating block to block. That is:

$$\begin{cases} \boldsymbol{E}\boldsymbol{A} + \boldsymbol{F}\boldsymbol{C}' = I_n & (1.7) \\ \boldsymbol{E}\boldsymbol{B} = \boldsymbol{O} & (1.8) \\ \boldsymbol{G}'\boldsymbol{A} + \boldsymbol{H}\boldsymbol{C}' = \boldsymbol{O} & (1.9) \\ \boldsymbol{G}'\boldsymbol{B} = \boldsymbol{I}_m & (1.10) \end{cases} \qquad \begin{cases} \boldsymbol{A}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{G}' = \boldsymbol{I}_n & (1.7') \\ \boldsymbol{A}\boldsymbol{F} + \boldsymbol{B}\boldsymbol{H} = \boldsymbol{O} & (1.8') \\ \boldsymbol{C}'\boldsymbol{E} = \boldsymbol{O} & (1.9') \\ \boldsymbol{C}'\boldsymbol{F} = \boldsymbol{I}_m. & (1.10') \end{cases}$$

Equations (1.8) and (1.9′) have the common solution (Rao and Mitra, 1971, Lemma 2.3.1)

$$\begin{aligned} \boldsymbol{E} &= [\boldsymbol{I} - (\boldsymbol{C}')^-\boldsymbol{C}']\cdot\boldsymbol{Z}\cdot(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{B}^-) \\ &= [\boldsymbol{I} - \boldsymbol{C}\cdot(\boldsymbol{C}'\boldsymbol{C})^{-1}\cdot\boldsymbol{C}']\cdot\boldsymbol{Z}\cdot[\boldsymbol{I} - \boldsymbol{B}\cdot(\boldsymbol{B}'\boldsymbol{B})^{-1}\cdot\boldsymbol{B}'] \end{aligned} \tag{1.11}$$

for some $\boldsymbol{Z}$, which is equivalent to

$$\boldsymbol{E} = \boldsymbol{C}_\perp\boldsymbol{W}\boldsymbol{B}_\perp' \tag{1.11'}$$

for some $\boldsymbol{W}$, in view of (1.5).

Substituting (1.11′) into (1.7) and postmultiplying both sides by $\boldsymbol{C}_\perp$, we obtain

$$\boldsymbol{C}_\perp\boldsymbol{W}\boldsymbol{B}_\perp'\boldsymbol{A}\boldsymbol{C}_\perp = \boldsymbol{C}_\perp. \tag{1.12}$$

Because $\boldsymbol{C}_\perp$ is of full column rank, equality (1.12) holds iff

$$\boldsymbol{W}\boldsymbol{B}_\perp'\boldsymbol{A}\boldsymbol{C}_\perp = \boldsymbol{I} \Leftrightarrow \boldsymbol{W} = (\boldsymbol{B}_\perp'\boldsymbol{A}\boldsymbol{C}_\perp)^{-1}. \tag{1.13}$$

Combining (1.11′) and (1.13) we obtain (1.3).

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Now consider equations (1.10) and (1.7′), with $\boldsymbol{E}$ given by (1.3). Then simple computations show that

$$\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{AE})\cdot\boldsymbol{B}. \tag{1.14}$$

Hence a common solution exists, namely (Rao and Mitra, 1971, Theorem 2.3.3),

$$\boldsymbol{G}' = \boldsymbol{B}^{-}\cdot(\boldsymbol{I} - \boldsymbol{AE}) + \boldsymbol{B}^{-} - \boldsymbol{B}^{-}\boldsymbol{BB}^{-} + (\boldsymbol{I} - \boldsymbol{B}^{-}\boldsymbol{B})\cdot\boldsymbol{Z}\cdot(\boldsymbol{I} - \boldsymbol{BB}^{-}) \tag{1.15}$$

for some $\boldsymbol{Z}$, which, noting that $\boldsymbol{B}^{-}\boldsymbol{B} = \boldsymbol{I}$, can be reduced to

$$\boldsymbol{G}' = \boldsymbol{B}^{-}\cdot(\boldsymbol{I} - \boldsymbol{AE}). \tag{1.16}$$

The expression

$$\boldsymbol{F} = (\boldsymbol{I} - \boldsymbol{EA})\cdot(\boldsymbol{C}')^{-} \tag{1.17}$$

is similarly established by referring to equations (1.7) and (1.10′).

Finally, consider equation (1.8′), with $\boldsymbol{F}$ given by (1.17). Premultiplication by $\boldsymbol{B}^{-}$ gives

$$\boldsymbol{H} = \boldsymbol{B}^{-}\cdot(\boldsymbol{AEA} - \boldsymbol{A})\cdot(\boldsymbol{C}')^{-}, \tag{1.18}$$

which completes the proof. ■

The theorem may prove useful in several econometric frameworks, such as restricted estimation, where inversion of bordered matrices arises quite naturally, and for a straightforward derivation of classical results of unit-root econometrics, as will become clear in the next section. The proposed formula improves the classical inversion formula

$$\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{A}^{-1} - \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}'\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}'\boldsymbol{A}^{-1} & \boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{C}'\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \\ (\boldsymbol{C}'\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}'\boldsymbol{A}^{-1} & -(\boldsymbol{C}'\boldsymbol{A}^{-1}\boldsymbol{B})^{-1} \end{bmatrix} \tag{1.19}$$

insofar as no full-rank assumption on $\boldsymbol{A}$ is required.

2. AN APPLICATION TO VAR ECONOMETRICS

In this section, after sketching the analytical tool kit relating matrix polynomial inversion to Laurent series expansion, we derive the matrix residue at a simple pole, and we eventually come to a short-cut proof of a crucial representation theorem of unit-root econometrics.

Consider the matrix

$$\boldsymbol{S}(z) = \boldsymbol{A}(z)\cdot(1 - z) + \boldsymbol{BC}'. \tag{2.1}$$

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Simple computations show that $(1-z)^{-1}\cdot\boldsymbol{S}(z)$ plays the role of the Schur complement of $(z-1)\cdot\boldsymbol{I}$ in the composite matrix $\begin{bmatrix}\boldsymbol{A}(z) & \boldsymbol{B}\\ \boldsymbol{C}' & (z-1)\cdot\boldsymbol{I}\end{bmatrix}$ and that, accordingly, the identity

$$(1-z)\cdot\boldsymbol{S}^{-1}(z)=[\boldsymbol{I},\boldsymbol{O}]\cdot\begin{bmatrix}\boldsymbol{A}(z) & \boldsymbol{B}\\ \boldsymbol{C}' & (z-1)\cdot\boldsymbol{I}\end{bmatrix}^{-1}\cdot\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{O}\end{bmatrix} \tag{2.2}$$

holds true, provided the inverses exist, by classical partitioned inversion rules.

Observe next that

$$\lim_{z\to 1}\{(1-z)\cdot\boldsymbol{S}^{-1}(z)\}=\boldsymbol{V},\quad \text{where } \boldsymbol{V}\neq\boldsymbol{O}, \tag{2.3}$$

provided that $z=1$ is a simple pole of $\boldsymbol{S}^{-1}(z)$ (e.g., Jeffrey, 1992). The matrix $\boldsymbol{V}$ is the matrix residue in the Laurent series expansion,

$$[\boldsymbol{A}(z)\cdot(1-z)+\boldsymbol{B}\boldsymbol{C}']^{-1}=\boldsymbol{V}\cdot(1-z)^{-1}+\sum_{i=0}^{\infty}\boldsymbol{W}_i(1-z)^i, \tag{2.4}$$

of $[\boldsymbol{A}(z)\cdot(1-z)+\boldsymbol{B}\boldsymbol{C}']^{-1}$ about the singularity located at $z=1$.

LEMMA 2. *Let $\boldsymbol{A}(1)$, $\boldsymbol{B}$, and $\boldsymbol{C}$ be defined as the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ in Theorem 1. Then the matrix $[\boldsymbol{A}(z)\cdot(1-z)+\boldsymbol{B}\boldsymbol{C}']^{-1}$ has a simple pole at $z=1$, whose matrix residue is given by*

$$\boldsymbol{V}=\boldsymbol{C}_\perp\cdot[\boldsymbol{B}_\perp'\boldsymbol{A}(1)\boldsymbol{C}_\perp]^{-1}\cdot\boldsymbol{B}_\perp'. \tag{2.5}$$

Proof. Taking the limit of both sides of (2.2) as z tends to 1, we get

$$\boldsymbol{V}=[\boldsymbol{I},\boldsymbol{O}]\cdot\begin{bmatrix}\boldsymbol{A}(1) & \boldsymbol{B}\\ \boldsymbol{C}' & \boldsymbol{O}\end{bmatrix}^{-1}\cdot\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{O}\end{bmatrix}=\boldsymbol{C}_\perp\cdot[\boldsymbol{B}_\perp'\boldsymbol{A}(1)\boldsymbol{C}_\perp]^{-1}\cdot\boldsymbol{B}_\perp' \tag{2.6}$$

in view of Theorem 1. Reference to the standard inversion formula (1.19) does not represent a valid alternative because $\boldsymbol{A}(1)$ may be singular. ■

Now consider, under the assumption $\boldsymbol{y}_t\sim CI(1,1)$, a vector autoregressive (VAR) model in differences, namely (e.g., Banerjee, Dolado, Galbraith, and Hendry, 1993),

$$\boldsymbol{A}(L)\Delta\boldsymbol{y}_t+\boldsymbol{B}\boldsymbol{C}'\boldsymbol{y}_t=\boldsymbol{\varepsilon}_t, \tag{2.7}$$

where L and $\Delta=I-L$ denote the lag and the difference operators, respectively, $\boldsymbol{\varepsilon}_t$ is a vector of white noises, and $\boldsymbol{A}(L)$ is a matrix polynomial, namely,

$$\boldsymbol{A}(L)=\boldsymbol{I}+\sum_{k=1}^{K}\boldsymbol{A}_kL^k, \tag{2.8}$$

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whose matrix coefficients $\boldsymbol{A}_k$ along with the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ fulfill the following specification requirements:

(i) the matrices $\boldsymbol{B}$ and $\boldsymbol{C}$ have full column rank;
(ii) the square matrix $\boldsymbol{B}_\perp'\boldsymbol{A}(1)\boldsymbol{C}_\perp$—where $\boldsymbol{A}(1) = \boldsymbol{I} + \sum_{k=1}^{K} A_k$—has full rank;
(iii) the roots of the characteristic polynomial

$$\pi(z) = \det[(1-z)\boldsymbol{A}(z) + \boldsymbol{B}\boldsymbol{C}'] \tag{2.9}$$

are either equal to one or, in modulus, greater than one.

Then the following representation theorem holds.

THEOREM 3. *The model (2.7), under the assumptions (i), (ii), and (iii) stated previously, has a solution of the form*

$$\boldsymbol{y}_t = \boldsymbol{C}_\perp \cdot [\boldsymbol{B}_\perp'\boldsymbol{A}(1)\boldsymbol{C}_\perp]^{-1} \cdot \boldsymbol{B}_\perp' \sum_{\tau \le t} \varepsilon_\tau + \sum_{i=0}^{\infty} \Xi_i \cdot \varepsilon_{t-i}, \tag{2.10}$$

which is precisely a CI(1,1) *process, given that*

$$\boldsymbol{y}_t \sim I(1), \tag{2.11}$$

$$\boldsymbol{C}'\boldsymbol{y}_t \sim I(0). \tag{2.12}$$

Proof. Looking at (2.7) as a linear operator equation, the closed form expression for $\boldsymbol{y}_t$,

$$\boldsymbol{y}_t = [\boldsymbol{A}(L)\Delta + \boldsymbol{B}\boldsymbol{C}']^{-1} \cdot \varepsilon_t, \tag{2.13}$$

is a particular solution, provided the inverse operator $[\boldsymbol{A}(L)\Delta + \boldsymbol{B}\boldsymbol{C}']^{-1}$ exists.

Applying the Laurent series expansion (2.4) to the inversion of the operator $\boldsymbol{A}(L)\Delta + \boldsymbol{B}\boldsymbol{C}'$—with L and Δ formally replacing z and $1 - z$, respectively—eventually leads to the expansion (e.g., Engsted and Johansen, 1997)

$$[\boldsymbol{A}(L)\Delta + \boldsymbol{B}\boldsymbol{C}']^{-1} = \boldsymbol{V}\Delta^{-1} + \sum_{i=0}^{\infty} \Xi_i L^i, \tag{2.14}$$

where $\boldsymbol{V}$ is the matrix residue of Lemma 2 and the Ξ_i's turn out to be, under the assumption (iii) given previously, matrices of exponentially decreasing coefficients.

In view of (2.14) the solution (2.13) can be rewritten in the form (2.10), using the formal identity $\Delta^{-1} = \sum_{\tau \le t}$.

The solution (2.10) is the sum of two components: the first is an $I(1)$ process, namely, a linear transformation—through the matrix residue $\boldsymbol{C}_\perp \cdot [\boldsymbol{B}_\perp'\boldsymbol{A}(1)\boldsymbol{C}_\perp]^{-1} \cdot \boldsymbol{B}_\perp'$—of the multivariate random-walk $\sum_{\tau \le t} \varepsilon_\tau$, whereas the second is an $I(0)$ process, namely, a causal infinite vector moving average, VMA (∞), of the multivariate white noise ε_t.

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Hence the result (2.11) holds trivially, and the result (2.12) ensues from the annihilation of the $I(1)$ component of the solution once it is premultiplied by the matrix $\boldsymbol{C}'$. ■

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