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Diffraction of elastic waves by a rigid circular disc*

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Abstract. The problem of the diffraction of axisymmetric harmonic elastic waves by a rigid circular disc is reduced to finding the solution of a pair of integral equations of the second kind suitable for iteration at low frequencies. Using the principle of contraction mapping, the rate of convergence of the iteration procedure is discussed and the error caused by stopping at any particular stage estimated. Detailed calculations are given for plane compressional waves at normal incidence.

Introduction. Diffraction of sound and electromagnetic waves by disc-shaped inhomogeneities has been studied in detail. An extensive review of the subject is given by Jones (4). Because of the complexity of elastic wave propagation in the presence of boundaries, in general the literature dealing with the diffraction of elastic waves is not as extensive as that dealing with the diffraction of scalar waves. Although significant progress has been made in elastodynamic diffraction theory in the case of infinite or semi-infinite obstacles, few successful attempts have been made in the case of obstacles of finite dimensions (5). The diffraction of plane elastic waves by two-dimensional straight strips or cracks of finite length has been treated by Ang and Knopoff (1, 2).

In this paper, we consider the diffraction of axisymmetric harmonic waves by a rigid circular disc embedded in an otherwise homogeneous isotropic infinite elastic medium. The boundary condition is that the displacement vector vanishes on the disc. The case when the elastic field satisfies weak boundary conditions on the disc will be treated later.

We describe the diffracted field in terms of two scalar potential functions. Determination of these unknown functions from the boundary conditions is usually very difficult since they are related through a pair of integral equations. If each of the potential functions is decomposed into symmetric and antisymmetric parts about the plane of the disc, we obtain a pair of decoupled 'dual' integral equations satisfied by the unknown functions. These equations are then converted into integral equations of the second kind particularly suitable for iteration at low frequencies; the latter integral equations are not precisely of Fredholm type. Using the principle of contraction mapping, we prove the existence of the solutions and also make an estimate of the error introduced by stopping the iteration at a particular stage. In the last section, we specialize to the case of normally incident plane waves and carry out the first iteration,

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and obtain an approximate expression for the displacement at large distances from the disc.

1. *The integral equations.* We introduce a cylindrical polar coordinate system (r, ϕ, z) with the origin at the centre of the disc and the z-axis perpendicular to it. We normalize all lengths with respect to the radius of the disc. Thus, the disc is located at $z = 0$, $0 \leqslant r \leqslant 1$. We assume that the incident field is independent of ϕ; then the total field is also independent of ϕ. This condition will occur if the incident wave is generated by a symmetric point source on the z-axis, or if it is a normally incident axisymmetric plane wave. Let $e^{-i\omega t}\mathbf{u}^0(\mathbf{r})$ denote the displacement vector due to the incident wave, ω being the circular frequency. We describe the diffracted field by means of two scalar functions $\Phi(\mathbf{r})\,e^{-i\omega t}$ and $\Psi(\mathbf{r})\,e^{-i\omega t}$. The displacement vector $\mathbf{u}(\mathbf{r})\,e^{-i\omega t}$ in the diffracted field is given by

$$u_r(\mathbf{r}) = \frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Psi}{\partial r\,\partial z}, \tag{1}$$

$$u_z(\mathbf{r}) = \frac{\partial \Phi}{\partial z} + \frac{\partial^2 \Psi}{\partial z^2} + k_2^2 \Psi. \tag{2}$$

The functions Φ and Ψ are solutions of the equations

$$\nabla^2 \Phi + k_1^2 \Phi = 0, \tag{3}$$

$$\nabla^2 \Psi + k_2^2 \Psi = 0, \tag{4}$$

where

$$\left.\begin{aligned} k_1 &= \omega/\alpha, \\ k_2 &= \omega/\beta, \end{aligned}\right\} \tag{5}$$

α and β being the compressional and shear wave velocities in the material.

We postulate solutions of (3) and (4) having the following forms:

$$\left.\begin{aligned} \Phi(\mathbf{r}) &= \frac{1}{\omega^2}\int_0^\infty \zeta\left\{\mp P(\zeta) + \frac{\zeta Q'(\zeta)}{\nu_1}\right\} J_0(\zeta r)\, e^{-\nu_1 |z|}\, d\zeta, \\ \Psi(\mathbf{r}) &= \frac{1}{\omega^2}\int_0^\infty \left\{\mp Q(\zeta) + \frac{\zeta}{\nu_2} P'(\zeta)\right\} J_0(\zeta r)\, e^{-\nu_2 |z|}\, d\zeta, \end{aligned}\right\} \tag{6}$$

where

$$\left.\begin{aligned} \nu_j^2 &= \zeta^2 - k_j^2, \\ \Re \nu_j > 0, \quad \Im \nu_j &\geqslant 0 \quad \text{for} \quad 0 \leqslant \zeta < k_j \quad (j = 1, 2). \end{aligned}\right\} \tag{7}$$

The $\mp$ signs in (6) refer to $z \gtrless 0$ respectively. The path of integration in (6) is chosen to lie in the appropriate Riemann sheet defined by suitable cuts in the complex ζ-plane. P, Q, P', Q' are unknown functions of ζ to be determined from the boundary conditions and subject to additional conditions which are equivalent to an edge condition and which ensure the validity of the analysis. The description of the sufficient conditions to be satisfied by these functions will be given in the next section.

From (6) we obtain, after differentiation under the integrals for $z \neq 0$

$$\left.\begin{aligned} u_r(\mathbf{r}) &= -\frac{1}{\omega^2}\int_0^\infty \left[\zeta\left(\mp P + \frac{\zeta Q'}{\nu_1}\right) e^{-\nu_1 |z|} + \nu_2 \left(Q \mp \frac{\zeta}{\nu_2} P'\right) e^{-\nu_2 |z|}\right] \zeta J_1(\zeta r)\, d\zeta, \\ u_z(\mathbf{r}) &= \frac{1}{\omega^2}\int_0^\infty \left[\nu_1 \left(P \mp \frac{\zeta Q'}{\nu_1}\right) e^{-\nu_1 |z|} + \zeta\left(\mp Q + \frac{\zeta}{\nu_2} P'\right) e^{-\nu_2 |z|}\right] \zeta J_0(\zeta r)\, d\zeta. \end{aligned}\right\} \tag{8}$$

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The boundary conditions on the disc are

$$\left.\begin{aligned}\lim_{z\to 0} u_r(\mathbf{r}) &= -g_0(r)\\ \lim_{z\to 0} u_z(\mathbf{r}) &= -f_0(r)\end{aligned}\right\} \quad (0 \leqslant r < 1), \tag{9}$$

where

$$\left.\begin{aligned}f_0(r) &= u_z^0(r,0),\\ g_0(r) &= u_r^0(r,0).\end{aligned}\right\} \tag{10}$$

Substituting (8) into (9), and noting that the functions (8) have different limits as $z\to 0$ from the positive and negative sides of the disc, we obtain for $0 \leqslant r < 1$,

$$\left.\begin{aligned}\int_0^\infty \left(\nu_1 P + \frac{\zeta^2}{\nu^2} P'\right) \zeta J_0(\zeta r)\, d\zeta &= -\omega^2 f_0(r)\\ \int_0^\infty \left(\nu_2 Q + \frac{\zeta^2}{\nu_1} Q'\right) \zeta J_1(\zeta r)\, d\zeta &= \omega^2 g_0(r)\end{aligned}\right\} \quad (0 \leqslant r < 1); \tag{11}$$

$$\left.\begin{aligned}\int_0^\infty (P + P')\, \zeta^2 J_1(\zeta r)\, d\zeta &= 0\\ \int_0^\infty (Q + Q')\, \zeta^2 J_0(\zeta r)\, d\zeta &= 0\end{aligned}\right\} \quad (0 \leqslant r < 1). \tag{12a}$$

The stresses normal to the plane of the disc are

$$\left.\begin{aligned}\tau_{rz} &= -\frac{\mu}{\omega^2}\int_0^\infty \left[2\zeta\nu_1\left(P \mp \frac{\zeta}{\nu_1} Q'\right) e^{-\nu_1|z|} + (2\zeta^2 - k_2^2)\left(\mp Q + \frac{\zeta}{\nu_2} P'\right) e^{-\nu_2|z|}\right] \zeta J_1(\zeta r)\, d\zeta,\\ \tau_{zz} &= \frac{\mu}{\omega^2}\int_0^\infty \left[(2\zeta^2 - k_2^2)\left(\mp P + \frac{\zeta}{\nu_1} Q'\right) e^{-\nu_1|z|} + 2\zeta\nu_2\left(Q \mp \frac{\zeta}{\nu_2} P'\right) e^{-\nu_2|z|}\right] \zeta J_0(\zeta r)\, d\zeta,\end{aligned}\right\} \tag{13}$$

where μ is the rigidity of the elastic medium.

Since the material is homogeneous outside the disc, the displacements and the stresses must be continuous at all points outside the disc. In particular if we require that the displacements and stresses given by expressions (8) and (13) be continuous across the plane $z = 0$ for $r > 1$, we obtain the functional equations

$$\left.\begin{aligned}\int_0^\infty (P + P')\, \zeta^2 J_1(\zeta r)\, d\zeta &= 0\\ \int_0^\infty (Q + Q')\, \zeta^2 J_0(\zeta r)\, d\zeta &= 0\end{aligned}\right\} \quad (r > 1); \tag{12b}$$

$$\left.\begin{aligned}\int_0^\infty [(2\zeta^2 - k_2^2) Q + 2\zeta^2 Q']\, \zeta J_1(\zeta r)\, d\zeta &= 0\\ \int_0^\infty [(2\zeta^2 - k_2^2) P + 2\zeta^2 P']\, \zeta J_0(\zeta r)\, d\zeta &= 0\end{aligned}\right\} \quad (r > 1). \tag{14}$$

Equations (11), (12*a*), (12*b*), (14) are to be solved for P, Q, P', Q'.

Equations (12*a*, *b*) are satisfied only if

$$P(\zeta) + P'(\zeta) = 0, \tag{15a}$$

$$Q(\zeta) + Q'(\zeta) = 0, \tag{15b}$$

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for all ζ. Thus, equations (14) give

$$\int_0^\infty P(\zeta)\,\zeta J_0(\zeta r)\,d\zeta = 0, \tag{16a}$$

$$\int_0^\infty Q(\zeta)\,\zeta J_1(\zeta r)\,d\zeta = 0 \quad \text{for} \quad r > 1. \tag{16b}$$

Equations (15) and (16) are necessary conditions for the expressions (8) and (13) to be continuous across $z = 0$ for $r > 1$. We show that these are also sufficient conditions for the continuity of all the components of the displacements and the stresses. The only discontinuities in the potential functions (6) and their derivatives occur across $z = 0$. Denoting by $[X]$ the magnitude of the jump in X across $z = 0$ we have

$$[\Phi] = \frac{1}{\omega^2}\int_0^\infty P(\zeta)\,\zeta J_0(\zeta r)\,d\zeta,$$

$$[\Psi] = \frac{1}{\omega^2}\int_0^\infty Q(\zeta)\,\zeta J_0(\zeta r)\,d\zeta,$$

$$\left[\frac{\partial\Phi}{\partial z}\right] = \frac{1}{\omega^2}\int_0^\infty Q'(\zeta)\,\zeta^2 J_0(\zeta r)\,d\zeta,$$

$$\left[\frac{\partial\Psi}{\partial z}\right] = \frac{1}{\omega^2}\int_0^\infty P'(\zeta)\,\zeta J_0(\zeta r)\,d\zeta.$$

From (15a) and (16a), the first and last equations of this set give $[\Phi] = 0$ and $[\partial\Psi/\partial z] = 0$. To prove that $[\partial\Phi/\partial z]$ is zero we differentiate (16b) with respect to r and use (15b) and the relation

$$\frac{d}{dr}\{rJ_1(\zeta r)\} = -\zeta r J_0(\zeta r).$$

The jump in Ψ is also shown to vanish if we integrate (16b) with respect to r between the limits r and ∞ ($r > 1$). The continuity of the other higher order derivatives of Φ and Ψ across $z = 0$ can be similarly proved. This implies that all the components of the displacement and the stress are continuous across $z = 0$, for $r > 1$, provided (15) and (16) are satisfied.

Hence the system of equations (11) and (16) are equivalent to the following pairs of dual integral equations:

$$\int_0^\infty P(\zeta)\,J_0(\zeta r)\,d\zeta = \frac{2\omega^2 f_0(r)}{k_1^2 + k_2^2} + \omega f_1(r) \quad (0 \leqslant r < 1), \tag{17a}$$

$$\int_0^\infty \zeta P(\zeta)\,J_0(\zeta r)\,d\zeta = 0 \quad (r > 1), \tag{17b}$$

$$\int_0^\infty Q(\zeta)\,J_1(\zeta r)\,d\zeta = -\frac{2\omega^2 g_0(r)}{k_1^2 + k_2^2} + \omega g_1(r) \quad (0 \leqslant r < 1), \tag{18a}$$

$$\int_0^\infty \zeta Q(\zeta)\,J_1(\zeta r)\,d\zeta = 0 \quad (r > 1), \tag{18b}$$

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where
$$f_1(r) = \int_0^\infty B(t)\,P(t)\,J_0(tr)\,dt, \tag{19}$$
$$g_1(r) = \int_0^\infty C(t)\,Q(t)\,J_1(tr)\,dt, \tag{20}$$
$$B(\zeta) = \frac{1}{\omega}\left\{1 - \frac{2\zeta}{k_1^2+k_2^2}\left(\frac{\zeta^2}{\nu_2} - \nu_1\right)\right\}, \tag{21}$$
$$C(\zeta) = \frac{1}{\omega}\left\{1 - \frac{2\zeta}{k_1^2+k_2^2}\left(\frac{\zeta^2}{\nu_1} - \nu_2\right)\right\}. \tag{22}$$

We observe that $\omega^2 f_0(r)/(k_1^2+k_2^2)$ and $\omega^2 g_0(r)/(k_1^2+k_2^2)$ remain finite for $\omega \to 0$ and that
$$\int_0^\infty |B(t)|\,dt, \quad \int_0^\infty |C(t)|\,dt$$
are bounded for $\omega \to 0$. The foregoing observations will become relevant when we carry out the iteration procedure for small ω. Furthermore, for $\zeta \to \infty$ we have,
$$B(\zeta) \sim O\left[\frac{k_1^4+3k_2^4}{4\omega(k_1^2+k_2^2)}\,\zeta^{-2}\right], \tag{23}$$
$$C(\zeta) \sim O\left[\frac{3k_1^4+k_2^4}{4\omega(k_1^2+k_2^2)}\,\zeta^{-2}\right]. \tag{24}$$

2. *Solution of the integral equations.* Treating the right-hand sides of (17) and (18) as known functions, we write down formal solutions to the pair of 'dual' integral equations. This yields the following more tractable forms of the integral equations for $P(\zeta)$ and $Q(\zeta)$ (cf. Sneddon (6), p. 65)
$$P(\zeta) = \frac{2\omega}{\pi}\cos\zeta\int_0^1 \frac{y}{\sqrt{(1-y^2)}}F(y)\,dy + \frac{2\omega\zeta}{\pi}\int_0^1 \frac{u\,du}{\sqrt{(1-u^2)}}\int_0^1 F(yu)\,y\sin\zeta y\,dy, \tag{25}$$
$$Q(\zeta) = \frac{2\omega}{\pi}\sin\zeta\int_0^1 \frac{y^2}{\sqrt{(1-y^2)}}G(y)\,dy + \frac{2\omega}{\pi}\int_0^1 \frac{u^2\,du}{\sqrt{(1-u^2)}}\int_0^1 G(yu)\,(\sin\zeta y - \zeta y\cos y)\,dy, \tag{26}$$

where
$$\left.\begin{aligned} F(r) &= \frac{2\omega f_0(r)}{k_1^2+k_2^2} + f_1(r), \\ G(r) &= -\frac{2\omega g_0(r)}{k_1^2+k_2^2} + g_1(r). \end{aligned}\right\} \tag{27}$$

After integration by parts and rearrangement, we obtain from (25) to (27)
$$P(\zeta) = P_0(\zeta) + \frac{\omega}{\pi}\int_0^\infty P(t)\,B(t)\left\{\frac{\sin(t-\zeta)}{t-\zeta} + \frac{\sin(t+\zeta)}{t+\zeta}\right\}dt, \tag{28}$$
$$Q(\zeta) = Q_0(\zeta) + \frac{\omega}{\pi}\int_0^\infty Q(t)\,C(t)\left\{\frac{\sin(t-\zeta)}{t-\zeta} - \frac{\sin(t+\zeta)}{t+\zeta}\right\}dt, \tag{29}$$

where
$$\left.\begin{aligned} P_0(\zeta) &= \frac{4\omega^2}{\pi(k_1^2+k_2^2)}\int_0^1 \cos\zeta y\,\frac{\partial}{\partial y}\left[y\int_0^1 \frac{u}{\sqrt{(1-u^2)}}f_0(yu)\,du\right]dy, \\ Q_0(\zeta) &= -\frac{4\omega^2}{\pi(k_1^2+k_2^2)}\int_0^1 \frac{\sin\zeta y}{y}\,\frac{\partial}{\partial y}\left[y^2\int_0^1 \frac{u^2}{\sqrt{(1-u^2)}}g_0(yu)\,du\right]dy. \end{aligned}\right\} \tag{30}$$

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In order to solve the integral equations (28), (29), it is necessary to make assumptions as to the nature of the functions $P(\zeta)$ and $Q(\zeta)$. We postulate that $P(\zeta)$, $Q(\zeta)$ satisfy the following conditions:

(i) $P(\zeta), Q(\zeta)$ are bounded, continuous in $[0, \infty[$,

(ii) $\zeta P(\zeta)$, $\zeta^{-1}Q(\zeta)$ are bounded in $[0, \infty[$,

(iii)
$$\left.\begin{aligned} P(\zeta) &= \int_0^1 p(t)\cos\zeta t\,dt, \\ Q(\zeta) &= \int_0^1 q(t)\sin\zeta t\,dt, \end{aligned}\right\} \tag{31}$$

where $p(t)$, $q(t)$ are bounded in $[0, 1]$ and differentiable in $0 \leqslant t < 1$. We show in the appendix that the formal analysis presented in section 1 is valid under these assumptions and that a solution satisfying the above conditions is consistent with a form of an edge condition.

We consider equation (29) first. Let E_1 be the space of continuous functions f on $[0, \infty[$ such that $\zeta f(\zeta)$ is bounded Let E_1 be normed by

$$\|f\|_1 = \sup_{\zeta \geqslant 0} |f(\zeta)| + \sup_{\zeta \geqslant 0} |\zeta f(\zeta)|.$$

Then E_1 is a Banach space. Let T_1 be the operator defined on E_1 as follows:

$$(T_1 f)(\zeta) = Q_0(\zeta) + \frac{\omega}{\pi}\int_0^\infty f(t)\,C(t)\left\{\frac{\sin(t-\zeta)}{t-\zeta} - \frac{\sin(t+\zeta)}{t+\zeta}\right\}dt. \tag{32}$$

It can be shown that T_1 takes E_1 into E_1. Furthermore, it is easily shown that for any f, g in E_1

$$\|T_1 f - T_1 g\|_1 \leqslant \omega\alpha_1 \|f-g\|_1, \tag{33}$$

where
$$\alpha_1 = \frac{4}{\pi}\int_0^\infty |C(t)|\,dt. \tag{34}$$

Thus if $\omega\alpha_1 < 1$, T_1 is a contraction and equation (29) can be solved by successive approximations. If we take the first approximation to be equal to $Q_0(\zeta)$, the error involved after the nth iteration is

$$|\epsilon_n| \leqslant \frac{(\omega\alpha_1)^n}{1-\omega\alpha_1}\|Q_0\|_1. \tag{35}$$

It can be verified that the solution $Q(\zeta)$ of (29) satisfies conditions (i), (ii) and (iii) above.

We now turn to equation (28). Let E_2 be the space of bounded continuous functions f on $[0, \infty[$, such that $\zeta f(\zeta)$ is bounded. Let E_2 be normed as follows

$$\|f\|_2 = \sup_{\zeta \geqslant 0} |f(\zeta)| + \sup_{\zeta \geqslant 0} |\zeta f(\zeta)|. \tag{36}$$

Then E_2 is a Banach space. Let T_2 be the operator defined on E_2 as follows,

$$(T_2 f)(\zeta) = P_0(\zeta) + \frac{\omega}{\pi}\int_0^\infty f(t)\,B(t)\left\{\frac{\sin(t-\zeta)}{t-\zeta} + \frac{\sin(t+\zeta)}{t+\zeta}\right\}dt. \tag{37}$$

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Then T_2 takes E_2 into E_2. Furthermore, for any two f, g in E_2

$$\|T_2 f - T_2 g\|_2 \leqslant \omega\alpha_2 \|f-g\|_2, \tag{38}$$

where

$$\alpha_2 = \frac{4}{\pi}\int_0^\infty |B(t)|\,dt. \tag{39}$$

Hence for $\omega\alpha_2 < 1$, T_2 is a contraction and equation (28) can be solved by successive approximations. If we take the first approximation to be equal to $P_0(\zeta)$, the error involved after the nth iteration is

$$|\delta_n| \leqslant \frac{(\omega\alpha_2)^n}{1-\omega\alpha_2}\|P_0\|_2. \tag{40}$$

Again we note that the solution $P(\zeta)$ of (28) satisfies conditions (i), (ii) and (iii).

3. *A particular example.* In the general problem considered above the diffracted field is given by

$$\left.\begin{aligned} u_r(\mathbf{r}) &= -\frac{1}{\omega^2}\int_0^\infty \left[-\left(\frac{\zeta^2}{\nu_1}e^{-\nu_1|z|} - \nu_2 e^{-\nu_2|z|}\right)Q(\zeta) \mp \zeta(e^{-\nu_1|z|}-e^{-\nu_2|z|})P(\zeta)\right]\zeta J_1(\zeta r)\,d\zeta,\\ u_z(\mathbf{r}) &= \frac{1}{\omega^2}\int_0^\infty \left[\mp\zeta(-e^{-\nu_1|z|}+e^{-\nu_2|z|})Q(\zeta) + \left(\nu_1 e^{-\nu_1|z|} - \frac{\zeta^2}{\nu_2}e^{-\nu_2|z|}\right)P(\zeta)\right]\zeta J_0(\zeta r)\,d\zeta, \end{aligned}\right\} \tag{41}$$

where $P(\zeta)$ and $Q(\zeta)$ are solutions of the integral equations (28) and (29). We assume that the incident wave is generated by a compressional wave source located at $z = -h$ $(h > 0)$ on the z-axis. The displacement vector due to the incident wave is given by

$$u_r^0(\mathbf{r}) = h\,e^{-ik_1 h}\frac{\partial}{\partial r}\frac{e^{ik_1 R}}{R},$$

$$u_z^0(\mathbf{r}) = h\,e^{-ik_1 h}\frac{\partial}{\partial z}\frac{e^{ik_1 R}}{R},$$

where $R^2 = r^2 + (z+h)^2$ and the weighting factor $h\exp(-ik_1 h)$ has been introduced for convenience. Then

$$f_0(r) = h\,e^{-ik_1 h}\frac{\partial}{\partial r}\frac{e^{ik_1\sqrt{(r^2+h^2)}}}{\sqrt{(r^2+h^2)}},$$

$$\begin{aligned} g_0(r) &= h\,e^{-ik_1 h}\lim_{r\to 0}\frac{\partial}{\partial z}\frac{e^{ik_1 R}}{R}\\ &= -h\,e^{-ik_1 h}\frac{\partial}{\partial h}\frac{e^{ik_1\sqrt{(r^2+h^2)}}}{\sqrt{(r^2+h^2)}}. \end{aligned}$$

If the distance of the source from the disc is large compared to a, it can be shown that the following expansions hold

$$\left.\begin{aligned} P_0(\zeta) &= -\frac{2\omega^2}{\pi(k_1^2+k_2^2)}\sum_{n=0}^\infty \frac{(2n+1)\,n!\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\left\{\frac{t_n}{h^{2n}}\left(ik_1 + \frac{2n+1}{h}\right) - \frac{1}{h^{2n}}\frac{\partial t_n}{\partial h}\right\}\int_0^1 y^{2n}\cos\zeta y\,dy,\\ Q_0(\zeta) &= -\frac{2\omega^2}{\pi(k_1^2+k_2^2)}\sum_{n=1}^\infty \frac{2n(2n+1)\,n!\,\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}\frac{t_n}{h^{2n}}\int_0^1 y^{2n-1}\sin\zeta y\,dy, \end{aligned}\right\} \tag{42}$$

where $t_n(h)$ is the coefficient in the expansion

$$\frac{e^{+ik_1\sqrt{(r^2+h^2)}}}{\sqrt{(r^2+h^2)}} = \frac{e^{+ik_1 h}}{h}\sum_{n=0}^\infty t_n\left(\frac{r}{h}\right)^n \quad (0 \leqslant r < 1). \tag{43}$$

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We consider the special case in which the incident wave is plane compressional and normal to the plane of the disc. Letting $h \to \infty$ in the results of the general axisymmetric case, the formal solution may be written as

$$\left.\begin{aligned} u_r(\mathbf{r}) &= \frac{1}{\omega^2}\int_0^\infty \zeta^2(e^{-\nu_1 z} - e^{-\nu_2 z})\, P(\zeta) J_1(\zeta r)\, d\zeta \\ u_z(\mathbf{r}) &= \frac{1}{\omega^2}\int_0^\infty \left(\nu_1 e^{-\nu_1 z} - \frac{\zeta^2}{\nu_2} e^{-\nu_2 z}\right) \zeta P(\zeta) J_0(\zeta r)\, d\zeta \end{aligned}\right\} \quad (z > 0), \tag{44}$$

where $P(\zeta)$ satisfies the integral equation

$$P(\zeta) = P_0(\zeta) + \frac{\omega}{\pi}\int_0^\infty P(t) B(t) \left\{\frac{\sin(t-\zeta)}{t-\zeta} + \frac{\sin(t+\zeta)}{t+\zeta}\right\} dt, \tag{45}$$

the first approximation to the solution of (45) being given by

$$P(\zeta) = P_0(\zeta) = -\frac{4ik_1\omega^2}{\pi(k_1^2+k_2^2)} \frac{\sin\zeta}{\zeta}. \tag{46}$$

The second approximation can be written as

$$P(\zeta) = P_0(\zeta) - \frac{4ik_1\omega^2}{\pi^2(k_1^2+k_2^2)} I(\zeta), \tag{47}$$

where

$$\begin{aligned} I(\zeta) &= \omega\int_0^\infty B(\eta)\frac{\sin\eta}{\eta}\left\{\frac{\sin(\eta-\zeta)}{\eta-\zeta} + \frac{\sin(\eta+\zeta)}{\eta+\zeta}\right\} d\eta \\ &= I_0(\zeta) - \frac{2}{1+k^2} I_1(\zeta) - \frac{2}{1+k^2} I_2(\zeta), \end{aligned} \tag{48}$$

$$\begin{aligned} I_0(\zeta) &= 2\int_0^1 dt \cos\zeta t \int_0^\infty \frac{\sin\eta\cos\eta t}{\eta}\, d\eta \\ &= \pi\frac{\sin\zeta}{\zeta}, \end{aligned} \tag{49}$$

$$I_1(\zeta) = \int_0^1 \cos\zeta t\, dt \int_0^\infty \frac{2\sin\eta\cos\eta t}{\sqrt{(\eta^2-k_2^2)}}\, d\eta, \tag{50}$$

$$I_2(\zeta) = k_2^{-2}\int_0^1 \cos\zeta t\, dt \int_0^\infty (\sqrt{(\eta^2-k_2^2)} - \sqrt{(\eta^2-k_1^2)})\, 2\sin\eta\cos\eta t\, d\eta \quad (k = k_1/k_2). \tag{51}$$

In the long wavelength approximation for the far field we assume both k_2 and ζ to be small quantities of the first order and evaluate the above integrals neglecting terms of order $(\zeta, k_1, k_2)^n$ $(n \geqslant 2)$. Recalling (7) we write

$$\sqrt{(\eta^2-k_j^2)} = i\sqrt{(k_j^2-\eta^2)} \quad (0 \leqslant \eta < k_j, \quad j = 1, 2).$$

This yields

$$\int_0^\infty \frac{2\sin\eta\cos\eta t}{\sqrt{(\eta^2-k_2^2)}}\, d\eta = \int_1^\infty \frac{2\sin k_2\eta\cos k_2\eta t}{\sqrt{(\eta^2-1)}}\, d\eta - i\int_0^1 \frac{2\sin k_2\eta\cos k_2\eta t}{\sqrt{(1-\eta^2)}}\, d\eta.$$

Retaining terms of the relevant order

$$I_1(\zeta) = \pi\left(\frac{\sin\zeta}{\zeta} - 2ik_2\right). \tag{52}$$

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We write
$$I_2(\zeta) = \int_0^1 S(t)\cos\zeta t\,dt, \tag{53}$$

where
$$S(t) = \int_1^\infty \{\sqrt{(\eta^2-1)} - \sqrt{(\eta^2-k^2)}\}\,A(\eta,t)\,d\eta - \int_k^1 \sqrt{(\eta^2-k^2)}\,A(\eta,t)\,d\eta$$
$$- i\int_0^k \sqrt{(k^2-\eta^2)}\,A(\eta,t)\,d\eta + i\int_0^1 \sqrt{(1-\eta^2)}\,A(\eta,t)\,d\eta,$$
$$A(\eta,t) = 2\sin k_2\eta\cos k_2\eta t. \tag{54}$$

The last three integrals in (54) are easily evaluated yielding

$$S(t) \approx I_3 + \frac{2i}{3}\left(k_2 - \frac{k_1^3}{k_2^2}\right) - \frac{2}{3}\frac{(k_2^2-k_1^2)^{\frac{3}{2}}}{k_2^2}, \tag{55}$$

where
$$I_3 = \sum_{n=1}^{\infty} c_n(1-k^{2n})\int_1^\infty \eta^{-2n+1}\{\sin k_2(1+t)\eta + \sin k_2(1-t)\eta\}\,d\eta \tag{56}$$

with
$$\left.\begin{aligned} c_1 &= -\tfrac{1}{2}, \\ c_n &= -\frac{1.1.3\ldots(2n-3)}{2.4.6\ldots 2n} \quad (n \geqslant 2). \end{aligned}\right\} \tag{57}$$

Neglecting terms of order higher than the first

$$\int_1^\infty \eta^{-1}\sin k_2(1\pm t)\,\eta\,d\eta = \tfrac{1}{2}\pi - k_2(1\pm t),$$

$$\int_1^\infty \eta^{2n-1}\sin k_2(1\pm t)\,\eta\,d\eta = \frac{k_2(1\pm t)}{2n-3} \quad (n \geqslant 2).$$

Hence,
$$I_3 = -(1-k^2)(\tfrac{1}{2}\pi - k_2) + 2k_2\sum_{n=2}^{\infty}\frac{c_n(1-k^{2n})}{2n-3}.$$

The infinite series on the right-hand side of (58) can be easily evaluated by writing

$$\frac{c_n}{2n-3} = \tfrac{1}{3}(c_{n-1} - c_n) \quad (n \geqslant 3). \tag{58}$$

Thus, to this approximation,

$$I_2(\zeta) = -\tfrac{1}{2}\pi(1-k^2)\frac{\sin\zeta}{\zeta} + \frac{2i}{3k_2^2}(k_2^3 - k_1^3). \tag{59}$$

Using (49), (52), (59) the second approximation to $P(\zeta)$ is written as

$$P(\zeta) = -\frac{4ik_1\omega^2}{\pi(k_1^2+k_2^2)}(1-\zeta^2/6) + \frac{16k_1(k_1^3+2k_2^3)\,\omega^2}{3\pi^2(k_1^2+k_2^2)^2}. \tag{60}$$

To obtain the far field, we write

$$z = R\cos\theta, \quad r = R\sin\theta$$

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in (44) and integrate by the method of steepest descents. This yields the following expressions for the displacements in polar coordinates,

$$\left.\begin{aligned} u_R(R,\theta) &= -\frac{ik_1^2}{\omega^2}\cos\theta P(k_1\sin\theta)\frac{e^{ik_1R}}{R}, \\ u_\theta(R,\theta) &= \frac{ik_2^2}{\omega^2}\sin\theta P(k_2\sin\theta)\frac{e^{ik_2R}}{R}. \end{aligned}\right\} \tag{61}$$

The error involved can be computed from (40) if α_2 given by (39) is evaluated. The integral is best evaluated by numerical methods. We can however make an estimate of the upper bound for α_2. It can be shown that

$$\omega\alpha_2 = \frac{4k_2}{\pi}\left[\int_0^1 \left|1-\frac{2t}{1+k^2}\left(\frac{t^2}{\sqrt{(t^2-1)}}-\sqrt{(t^2-k^2)}\right)\right| dt - \int_1^\infty \left\{1-\frac{2t}{1+k^2}\left(\frac{t^2}{\sqrt{(t^2-1)}}-\sqrt{(t^2-k^2)}\right)\right\} dt\right]$$

$$\leqslant \frac{8k_2}{3\pi(1+k^2)}\{5+3k^2+k^3+2(1-k^2)^{\frac{3}{2}}\}.$$

APPENDIX

For $z \neq 0$, the integral expressions for the displacements and the stress components are convergent due to the choice of the branches in (7). However, during the application of the boundary conditions in the plane of the disc, we have interchanged the order of performing the integration over ζ and taking the limit as $z \to 0$. This interchange is justifiable if the following integrals are convergent

$$\mathscr{I}_1 = \int_0^\infty \zeta P(\zeta) J_0(\zeta r)\, d\zeta,$$

$$\mathscr{I}_2 = \int_0^\infty \zeta Q(\zeta) J_1(\zeta r)\, d\zeta,$$

$$\mathscr{I}_3 = \int_0^\infty \left(\frac{\zeta^2}{\nu_2}-\nu_1\right)\zeta J_0(\zeta r) P(\zeta)\, d\zeta,$$

$$\mathscr{I}_4 = \int_0^\infty \left(\frac{\zeta^2}{\nu_1}-\nu_2\right)\zeta Q(\zeta) J_1(\zeta r)\, d\zeta.$$

Using (31),

$$\mathscr{I}_1 = \frac{p(1)}{\sqrt{(1-r^2)}} - \int_r^1 \frac{p'(t)}{\sqrt{(t^2-r^2)}}\, dt \quad (r<1)$$
$$= 0 \quad (r>1),$$

$$\mathscr{I}_2 = \frac{1}{r}\left[q(0)-q(1)\left(1-\frac{1}{\sqrt{(1-r^2)}}\right)+\int_r^1 q'(t)\left(1-\frac{t}{\sqrt{(t^2-r^2)}}\right)dt\right] \quad (r<1)$$
$$= 0 \quad (r>1),$$

where primes denote differentiation. Thus the representation (31) is consistent with (17b) and (18b). The convergence of $\mathscr{I}_3$ and $\mathscr{I}_1$ is easily verified, since both of

$$\left(\frac{\zeta^2}{\nu_2}-\nu_1\right) \quad \text{and} \quad \left(\frac{\zeta^2}{\nu_1}-\nu_2\right)$$

are of the order of ζ^{-1} as $\zeta \to \infty$.

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In a summary of his earlier work, Jones (3) has shown that two equivalent forms of the edge condition exist in electromagnetic theory: that the electromagnetic energy in any finite region must be finite, and that no net electric charge is induced on the edge of a diffracting screen. We assume that this equivalence holds in elastodynamic diffraction theory as well; hence the requirement that the energy of deformation in any finite region be finite is equivalent to the requirement that there be no net body forces (the source of the field) induced on the edge.

We now show that, if functions $P(\zeta)$ and $Q(\zeta)$ satisfy conditions (i), (ii) and (iii), no net body forces are induced on the edge. We enclose the edge of the disc by a surface S_ϵ formed by the concentric cylinders $r = 1 \pm \epsilon$ and the parallel planes $z = \pm\epsilon$ ($\epsilon > 0$), and compute the integral

$$F_i = \int_{S_\epsilon} \tau_{ij} n_j \, ds \quad (i, j = r, z),$$

n_j being the outward unit normal to S_ϵ. Then

$$F_z = -2\pi \int_{-\epsilon}^{\epsilon} (r\tau_{zr})_{r=1-\epsilon} dz + 2\pi \int_{-\epsilon}^{\epsilon} (r\tau_{zr})_{r=1+\epsilon} dz$$
$$- 2\pi \int_{1-\epsilon}^{1+\epsilon} (\tau_{zz})_{z=-\epsilon} r \, dr + 2\pi \int_{1-\epsilon}^{1+\epsilon} (\tau_{zz})_{z=\epsilon} r \, dr.$$

From equations (41)

$$F_z = \frac{4\pi\mu}{\omega^2} \int_0^\infty [rJ_1(\zeta r)]_{1-\epsilon}^{1+\epsilon} \left\{ \frac{2\zeta^2}{\nu_1}(1 - e^{-\nu_1\epsilon}) - \frac{2\zeta^2 - k_2^2}{\nu_2}(1 - e^{-\nu_2\epsilon}) \right\} \zeta Q(\zeta) \, d\zeta$$
$$- \frac{4\pi\mu}{\omega^2} \int_{1-\epsilon}^{1+\epsilon} r \, dr \int_0^\infty \{(2\zeta^2 - k_2^2) e^{-\nu_1\epsilon} - 2\zeta^2 e^{-\nu_2\epsilon}\} P(\zeta) \zeta J_0(\zeta r) \, d\zeta.$$

The first integral above vanishes as $\epsilon \to 0$. In the limit $\epsilon \to 0$, the second integral becomes

$$\lim_{\epsilon \to 0} F_z = \frac{4\pi\mu}{\beta^2} \int_{1-\epsilon}^{1+\epsilon} r \mathscr{I}_1 \, dr.$$

Applying the result above, we have

$$\lim_{\epsilon \to 0} F_z = \frac{4\pi\mu}{\beta^2} \int_{1-\epsilon}^{1} \left\{ \frac{p(1)}{\sqrt{(1-r^2)}} - \int_r^1 \frac{p'(t)\, dt}{\sqrt{(t^2 - r^2)}} \right\} r \, dr.$$

Carrying out the integration

$$\lim_{\epsilon \to 0} F_z = \frac{4\pi\mu}{\beta^2} \int_{1-\epsilon}^{1} \{p(1)\sqrt{(2\epsilon)} + p'(1) O(\epsilon^{\frac{3}{2}})\}.$$

Hence, in the limit $\epsilon \to 0$, $F_z = O(\epsilon^{\frac{1}{2}})$; the radial component of force F_r can be treated similarly.

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