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A note on complementary spines

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This note is concerned with a result about spines of manifolds embedded in a sphere, which proved useful at one point in (2).

We work in the polyhedral category as defined in (3) and use the definitions of collapsing and regular neighbourhoods as in (1).

Let S^n be an n-sphere and $X \subset S^n$ a compact subset. Let N be a regular n $-$ hd of X in S^n and define $M = \mathrm{cl}\,(S^n - N)$. Suppose $M \searrow Y$. Y is said to be a *complementary spine* to X in S^n.

We write $X \vee Y$ (X wedge Y) for the space formed by identifying a point in X with a point in Y; the common image being called the *wedge point*.

More precisely if $X \vee Y$ is formed by identifying $P \in X$ with $P' \in Y$ we will sometimes write $(X, P) \vee (Y, P')$.

Let $X \vee Y \subset S^n$ and suppose $\exists$ a ball $B^n \subset S^n$ with $X \subset B^n$ and

$$B^n \cap Y = \partial B^n \cap X = \text{the wedge point},$$

then we say $X \vee Y$ is *separately embedded* in S^n.

The result is

LEMMA. *Let $X \vee Y$ be separately embedded in S^n. Suppose X has a complementary spine V in S^n and that Y has a complementary spine W in S^n,*

then $X \vee Y$ has a complementary spine homeomorphic to $(V, P) \vee (W, P')$ for some points $P \in V$, $P' \in W$.

Moreover, if $\dim V$, $W < n$ and $S^n - X \vee Y$ is connected, then $X \vee Y$ has a complementary spine in S^n homeomorphic to $(V, P) \vee (W, P')$ for any points $P \in V$, $P' \in W$.

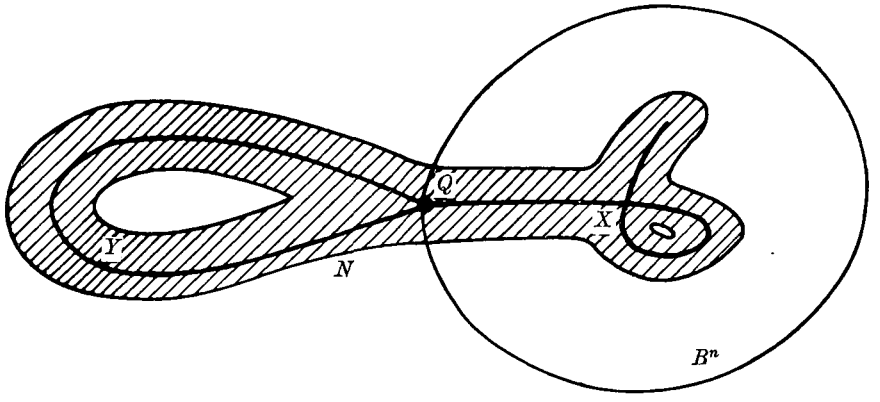
Proof. Let the wedge point of $X \vee Y$ be Q. By the separation condition there is an n-ball B^n with $X \subset B^n$, and $B^n \cap Y = \partial B^n \cap X = Q$.

$\mathrm{cl}\,(S^n - B^n)$ is an n-ball D^n with $Y \subset D^n$, $D^n \cap X = \partial D^n \cap Y = Q$.

Take a regular n $-$ hd N of $X \vee Y$ in S^n which meets $\partial D^n = \partial B^n$ regularly, i.e. in an $n-1-$ball C^{n-1} with $Q \in \mathrm{int}\, C^{n-1}$.

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Now $N \cup D^n$ is a regular n – hd of X in S^n, and so cl $(B^n - N)$ is the complement of a regular n – hd of X in S^n and so has spine V. Similarly cl $(D^n - N) \searrow W$.



Now note that cl $(S^n - N)$ is formed from cl $(D^n - N)$ and cl $(B^n - N)$ by identifying a ball in their boundaries, namely cl $(\partial D^n - C^{n-1})$.

The lemma therefore follows from the following:

Proposition. Let $Q^n = M_1^n \cup M_2^n$, all n-manifolds, where $M_1^n \cap M_2^n =$ common $n-1-$ball C^{n-1} in their boundaries.

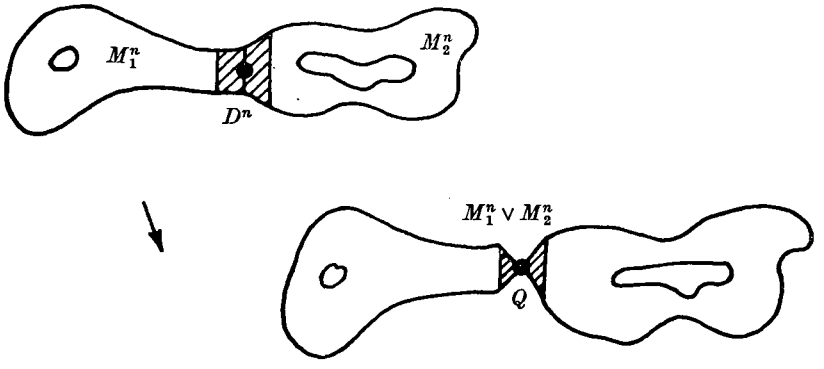
Suppose $M_1^n \searrow V$ and $M_2^n \searrow W$

$$\text{then } Q^n \searrow (V, P) \vee (W, P') \text{ for some points } P \in V,\ P' \in W.$$

Moreover, if dim V, $W < n$ and M_1^n, M_2^n are both connected then $Q^n \searrow (V, P) \vee (W, P')$ for any points $P \in V$, $P' \in W$.

Proof. In four steps.

Step 1. $Q^n \searrow M_1^n \vee M_2^n$, with the wedge point in both ∂M_1^n and ∂M_2^n.



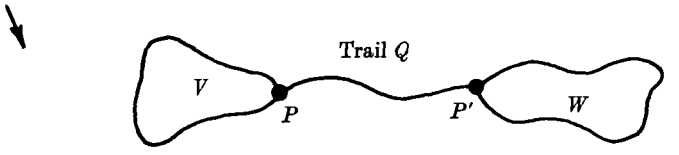
Take D^n a regular n – hd of C^{n-1} in Q^n meeting ∂Q^n regularly and collapse conewise in D^n from $\partial D^n \cap \partial Q^n$ to a vertex in int C^{n-1}.

Step 2. Let $K(P, P')$ be the polyhedron obtained from $V \cup W$ by joining a point $P \in V$ to a point $P' \in W$ by an arc.

$$Q^n \searrow K(P, P') \quad \text{for some points} \quad P, P' \in V, W.$$

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Let the wedge point of $M_1^n \vee M_2^n$ be Q in step 1 then $M_1^n \searrow V \cup \text{trail } Q$ and $M_2^n \searrow Q \cup \text{trail } Q$ (trails as defined in (3).) Therefore $M_1^n \vee M_2^n \searrow K(P, P')$ for $P = \text{trail } Q \cap V$ and P' similarly.

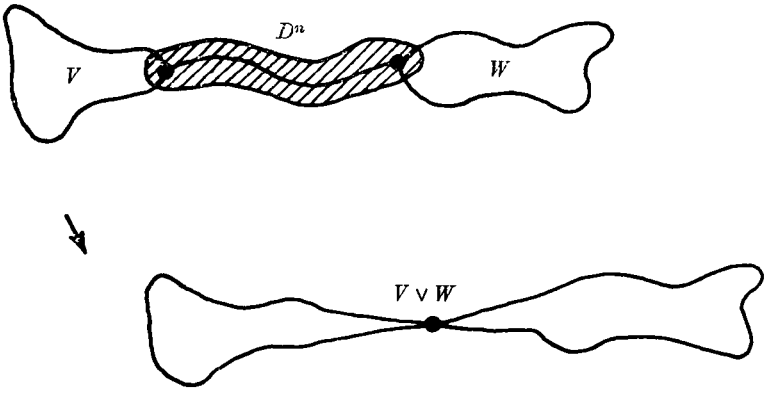


Step 3. Under the special conditions of the second half of the hypothesis $Q^n \searrow K(P, P')$ for any points $P, P' \in V, W$.

For this we use the fact that the property of being a spine in the interior of a manifold is invariant under simple homotopy in the manifold, as proved in (1). Simple homotopy is the relation generated by collapsing and its reverse. It is clear that we can change $K(P, P')$ to any $K(T, T')$ for $T, T' \in V, W$, by a succession of collapses or expansions of dim 2 (over triangles), i.e. by moving the end-points of the arc joining V to W—the extra conditions ensure that we can reach any points of V and W.

Step 4. $Q^n \searrow (V, P) \vee (W, P')$, P, P' as in steps 2 or 3.

We show that $K(P, P')$ has the same simple homotopy type as $(V, P) \vee (W, P')$ by one expansion and collapse in Q^n.



First expand the arc PP' up to a regular neighbourhood D^n which meets V and W regularly (i.e. in cones vertices P, P'), and then collapse D^n conewise to any interior point. (I am grateful to Prof. Zeeman for pointing out this proof of Step 4.)

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