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Limit dynamics of elementary cellular automaton 18

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Abstract. We study the the asymptotic dynamics of elementary cellular automaton 18 through its limit set, generic limit set and μ-limit set. The dynamics of rule 18 are characterized by persistent local patterns known as kinks. We characterize the configurations of the generic limit set containing at most two kinks. As a corollary, we show that the three limit sets of rule 18 are distinct.

Key words: cellular automata, rule 18, limit dynamics, limit sets, generic limit sets
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1. *Introduction*

Cellular automata (CAs) are a class of discrete dynamical systems consisting of an infinite line (or, more generally, a multidimensional grid) of 'cells', each of which holds one of finitely many states. Each cell evolves by an identical rule, and its next state is determined by the current state of itself and some finite set of neighbouring cells. The 256 *elementary* cellular automata (ECAs) [11] are those with only two states and neighbourhood $N = \{-1, 0, 1\}$, meaning that each cell can communicate only with its nearest neighbours. They are arguably the simplest CAs that exhibit complex dynamics.

We focus on the dynamics of elementary CA number 18. It has been studied as a model of mutually annihilating particles with random movement as early as 1983 [3] and by several authors since [5, 6]. Grassberger [3] and Lind [8] conjectured that if the CA is initialized from a uniformly random configuration, then the density of particles almost surely decreases to zero with the same speed as mutually annihilating independent random walks. Progress towards the conjecture has been slow. Kůrka claimed in [6] that the density indeed approaches zero, but we show that the presented proof is faulty (see the discussion after Proposition 4.5).

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The long-term fate of a dynamical system can be studied via its *limit set*, or eventual image. The limit set of a CA often contains transient phenomena, so several variants have been proposed to better capture the intuitive notion of 'what typically happens after a long time'. These include the *μ-limit set* [7] and the *generic limit set*, introduced by Milnor in [9] and studied first in the context of CAs in [2]. As our main results, we characterize the words with at most two particles that occur in the generic limit set of rule 18 and show that the three limit sets are all distinct. We also show that determining the generic limit set of rule 18 is a viable strategy towards (partially) resolving the conjecture of Grassberger and Lind.

2. *Definitions*

Let A be a finite alphabet. The set of finite words over A is denoted by A^* and concatenation of words $v, w \in A^*$ is denoted by either vw or $v \cdot w$. This notation is extended to infinite words: if $v \in A^*$ and $x \in A^{\mathbb{N}}$ is a right-infinite word, then $vx \in A^{\mathbb{N}}$; if $y \in A^{-\mathbb{N}}$ is a left-infinite word, then $yv \in A^{-\mathbb{N}}$; and $yx \in A^{\mathbb{Z}}$ is a bi-infinite word. The *full shift* is the set $A^{\mathbb{Z}}$ of all bi-infinite sequences over A, equipped with the prodiscrete topology, which is generated by the *cylinder sets* $[w]_i = \{x \in A^{\mathbb{Z}} \mid x_{[i,i+|w|-1]} = w\}$ for $w \in A^*$ and $i \in \mathbb{Z}$. The *shift map* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $\sigma(x)_i = x_{i+1}$. A *subshift* is a topologically closed set $X \subseteq A^{\mathbb{Z}}$ that is shift-invariant: $\sigma(X) = X$. A word $w \in A^*$ *occurs* in X if $X \cap [w]_0 \neq \emptyset$. The *language* of X is the set of words occurring in it; a subshift is defined by its language.

A *CA* is a continuous function $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ that commutes with the shift: $f \circ \sigma = \sigma \circ f$. By the Curtis–Hedlund–Lyndon theorem [4], every CA has a finite *radius* $r \geq 0$ and a *local rule* $F : A^{2r+1} \to A$ such that $f(x)_i = F(\sigma^i(x)|_{[-r,r]})$ for all $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. Once r is fixed, we can apply the CA to words $w \in A^n$ as well as configurations: $f(w) = F(w_{[0,2r]}w_{[1,2r+1]} \cdots w_{[n-2r-1,n-1]})$. Note that each application of f shortens the word by $2r$ symbols.

The *ECAs*, defined in [11], are those with alphabet $A = \{0, 1\}$ and radius 1. They are numbered from 0 to 255: the local rule F of ECA n maps $abc \in \{0, 1\}^3$ to bit number $4a + 2b + c$ in the binary representation of n (extended to three digits by prepending zeros if necessary).

A *topological dynamical system* is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a continuous function. CAs are examples of such systems.

A *Borel probability measure* on a topological space X is a function $\mu : \mathcal{B}(X) \to [0, 1]$, where $\mathcal{B}(X)$ is the class of Borel subsets of X, satisfying $\mu(X) = 1$, $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n\in\mathbb{N}} X_n) = \sum_{n\in\mathbb{N}} \mu(X_n)$ whenever the sets X_n are pairwise disjoint. The set of Borel probability measures of X is denoted $\mathcal{M}(X)$. The *support* of $\mu \in \mathcal{M}(X)$ is the smallest closed subset $\mathrm{supp}(\mu) \subseteq X$ with measure 1. If (X, T) is a dynamical system, the dynamics T can be extended to a function $T : \mathcal{M}(X) \to \mathcal{M}(X)$ by $(T\mu)(K) = \mu(T^{-1}(K))$. We denote by $\mathcal{M}_T(X)$ the subset of measures that are T-invariant, meaning $T\mu = \mu$. A measure $\mu \in \mathcal{M}_T(X)$ is *T-ergodic* if $T(A) = A$ for a Borel set $A \subseteq X$ implies $\mu(A) \in \{0, 1\}$.

We are mostly interested in shift-invariant measures on $A^{\mathbb{Z}}$, which are characterized by their values on cylinders of the form $[w]_0$. For convenience, we denote $\mu(w) = \mu([w]_0)$.

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The *uniform Bernoulli measure* is given by $\mu(w) = |A|^{-|w|}$ for all $w \in A^*$ and it is shift-ergodic. It corresponds to the random process where every symbol of an infinite configuration is chosen independently and uniformly at random. The set $\mathcal{M}(X)$ can be endowed with several topologies; we use the weak topology, which for the full shift is equivalent to convergence on cylinder sets: (μ_n) weakly converges to μ if and only if $\mu_n([w]_i) \to \mu([w]_i)$ for all cylinders $[w]_i$.

3. *Limit sets and variants*

We now define the limit sets of a dynamical system that will be used in this paper.

Definition 3.1. [7, 9] Let (X, T) be a topological dynamical system.

- The *limit set* of T is $\Omega(T) = \bigcap_{n\in\mathbb{N}} T^n(X)$.
- The *realm of attraction* of a subset $K \subset X$ is the set

$$D_T(K) = \{x \in X \mid \text{ all limit points of } (T^n(x))_{n\in\mathbb{N}} \text{ are in } K\}.$$

  The *generic limit set* of T is the intersection of all closed subsets $K \subset X$ such that $D_T(K)$ is comeager.
- Let μ be a Borel probability measure on X. The *μ-limit set* of T is the set $\Omega_\mu(T) = \overline{\bigcup_\nu \operatorname{supp}(\nu)}$, where ν ranges over the weak limit points of the sequence $(T^n\mu)_{n\in\mathbb{N}}$.

The limit set can also be called the *eventual image* and it is purely set-theoretic. The generic limit set is its topological refinement that aims to capture the asymptotic dynamics of the system from a generic (in a topological sense) initial condition. The μ-limit set captures those phenomena whose probability of occurrence does not vanish in the limit.

If $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is a CA and μ is a shift-invariant measure on $A^{\mathbb{Z}}$, then a given word w occurs in the μ-limit set $\Omega_\mu(f)$ if and only if $\mu(f^{-n}(w))$ does not converge to zero as n grows. In particular, if μ is the uniform Bernoulli measure, then w occurs in $\Omega_\mu(f)$ if and only if $\limsup_n |f^{-n}(w)|/|A|^{|w|+2nr} > 0$. Generic limit sets of CAs have a combinatorial characterization as well.

LEMMA 3.1. [10] *Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA and $w \in A^*$. Then, w occurs in the generic limit set $\tilde{\omega}(f)$ if and only if there exists a 'seed' word $s \in A^*$ and a position $i \in \mathbb{Z}$ such that for all $u, v \in A^*$, there are infinitely many $n \in \mathbb{N}$ with $f^n([usv]_{i-|u|}) \cap [w]_0 \neq \emptyset$.*

All three limit sets are non-empty subshifts of $A^{\mathbb{Z}}$. The limit set contains the other two, but in general, neither of the generic limit set or the μ-limit set contains the other.

LEMMA 3.2. [7, 9] *Let $f : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be a CA. Then, $\tilde{\omega}(f) \subseteq \Omega(f)$ and $\Omega_\mu(f) \subseteq \Omega(f)$.*

4. *Defect dynamics of rule 18*

The local rule of ECA 18 is given in Table 1: the patterns 001 and 100 produce a 1 and all other patterns produce a 0. In the remainder of this paper, we denote this CA by f_{18}, or just f when there is no danger of confusion. Figure 1 depicts a sample spacetime configuration of f. In this figure and in the following, time advances downward, a black square represents a 1 and a white square represent a 0. Hatched squares are undetermined symbols.

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TABLE 1. The local rule of ECA 18.

000	001	010	011	100	101	110	111
0	1	0	0	1	0	0	0

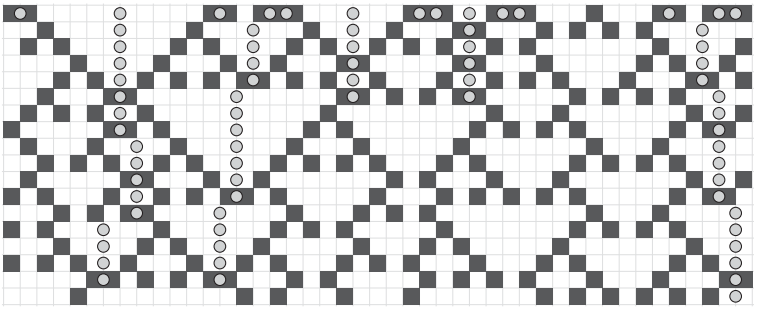


FIGURE 1. An example run of rule 18. The centre of each kink is marked with a circle.

The dynamics of ECA rule 18 are a combination of several interacting processes. On regions where the gaps between 1-symbols have odd length, the dynamics is linear and well understood. A single gap of even length, called a *kink*, is persistent. Its movements are influenced by the linear part of the dynamics, but not *vice versa*: the linear dynamics commutes with a local operation that erases the kink. However, when two kinks collide, both are destroyed and the linear dynamics is perturbed.

Definition 4.1. Words of the form $10^{2n}1$ with $n \in \mathbb{N}$ are called *kinks*. The *left border* of the kink is its leftmost 1-symbol.

Let $x \in \{0,1\}^{\mathbb{Z}}$. We denote $g_x(n) = \#\{-n \le i \le n \mid \text{ there exists } k \in \mathbb{N}, x_{[i,i+2k+2]} = 10^{2k}1\}$. If $\lim_{n\to\infty} g_x(n) = m < \infty$, then we call m the number of kinks in x.

We call $\Sigma \subseteq \{0,1\}^*$ the set of words without kinks and $\tilde{\Sigma} \subseteq \{0,1\}^{\mathbb{Z}}$ the set of configurations without kinks.

PROPOSITION 4.1. [5] *Let f_{90} be the function associated with ECA rule* 90*: $f_{90}(abc) = a \oplus c$ for all $abc \in \{0,1\}^3$, where $\oplus$ is addition modulo 2, or the xor function. Then, for all kinkless words $w \in \Sigma$, we have $f_{18}(w) = f_{90}(w)$ and for all kinkless configurations $x \in \tilde{\Sigma}$, we have $f_{18}(x) = f_{90}(x)$.*

We now investigate the dynamics of kinks under rule 18. When n is even, the following lemma characterizes the preimages of kinks. See Figure 2 for an illustration with $n = 13$ and $n = 14$.

LEMMA 4.2. *Let $u \in \{0,1\}^*$ be a word such that $f(u) = 10^n1$ with $n \in \mathbb{N}$. Then, either $u = 10^{n+2}1$ or $u = 00v00$ for a word $v \in \{0,1\}^*$ that begins and ends in* 1*, does not contain occurrences of* 00*, and contains m occurrences of* 11 *with $m \not\equiv n \bmod 2$.*

Proof. The word u both begins and ends with 001 or 100, *a priori* independently of each other. If u begins with 100, then it must contain 000 at positions $1, 2, 3, \ldots$ to avoid producing a 1 in $f(u)$ and, finally, it ends with 001. Hence, $u = 10^{n+2}1$.

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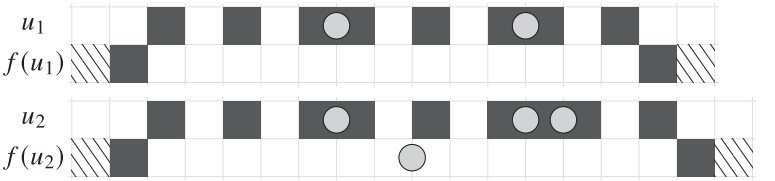


FIGURE 2. The word u_1 above contains an even number of kinks; hence, $f(u_1)$ is not a kink. The word u_2 below contains an odd number of kinks; hence, $f(u_2)$ is a kink.

Suppose then that u begins with 001. We necessarily have $n \geq 1$. Consider the smallest $1 \leq i \leq n+2$ such that u contains 00 at position i; such an i exists since u ends with 100 or 001. Then, u contains 100 at $i-1$, which means $i = n+2$ and u ends with 100. Hence, $u = 001w100$, with $w \in \{0,1\}^{n-2}$ not containing occurrences of 00.

Consider the word $1w1$ as a concatenation of 1s and 01s, starting with a single 1. Each subsequent 1 changes the parity of the length and adds a kink, while each 01 keeps the parity of the length and adds no kinks. Hence, the number m of kinks in u satisfies $m \not\equiv |1w1| = n \bmod 2$. □

In particular, Lemma 4.2 implies that new kinks cannot be created by rule 18.

Consider a kink $K = 10^{2n}1$ occurring in an infinite configuration $x \in \{0,1\}^{\mathbb{Z}}$. In $f(x)$, K becomes $10^{2n-2}1$ if $n \geq 1$. If $n = 0$ (so that $K = 11$) and x is 'typical', K is contained in a subword of the form $u = 001w100$, where w does not contain 00 as a subword. Then, $f_{18}(u) = 10^{|w|+2}1$ and by Lemma 4.2, this word is a kink if and only if the number of kinks in u is odd. Conversely, if the number of kinks in u is even, they are all destroyed. Hence, in 'typical' conditions, kinks can only be destroyed in pairs of 11s.

As an 'atypical' limiting case, consider a configuration $x \in \{0,1\}^{\mathbb{Z}}$ such that some infinite tail $x_{[i,\infty)}$ contains no occurrences of 00. Then, $f(x)$ contains the same number of kinks as $f(y)$, where y is obtained from x by replacing the infinite tail by one of $(01)^\infty$ or $(10)^\infty$ in a way that does not introduce a new 00. Similarly, when f_{18} is applied to finite words, their length decreases, so it is inevitable that all kinks are eventually destroyed as described above, or 'fall off the edge of the word' as in $f(1100) = 01$. The notion of stable words introduced in the following allows us to control this phenomenon.

The following definitions are inspired by [6].

Definition 4.2. Let $w \in \{0,1\}^*$ be a word that contains $m \in \mathbb{N}$ kinks. The *finite kinkless extension* $\Sigma(w) \subseteq \{0,1\}^*$ of w is the set of those words that contains w as a subword and contain exactly m kinks. The *kinkless extension* of w at position $p \in \mathbb{Z}$ is $\tilde{\Sigma}(w,p) = \{x \in [w]_p \mid \text{x contains m kinks}\}$.

Definition 4.3. The (regular) language of *left unstable words* is $(\epsilon + 0)(10)^*11(0+1)^*$ and the language of *right unstable words* is $(0+1)^*11(01)^*(\epsilon + 0)$. A word is *unstable* if it is left or right unstable. If a word is not unstable, it is *stable*. We denote by S the set of stable words.

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In other words, a word is stable if it does not have a prefix of the form $0(10)^n11$ or $(10)^n11$ or, symmetrically, a suffix of the form $11(01)^n0$ or $11(01)^n$ for any $n \geq 0$. For example, 1010110001 is a left unstable word, while 001010110001 is a stable word.

Intuitively, a word u is unstable if it contains an occurrence of the kink 11 and can be extended into a word v such that the 11 evolves into a kink K in $f(v)$, but $f(u)$ does not fully contain K. For example, $u = 1010110001$ can be extended into $v = 00u = 001010110001$. The word $f(v) = 1000000101$ contains the kink $K = 10000001$, but $f(u) = 00000101$ is kinkless. We imagine the kink 11 'falling off the edge' as u evolves into $f(u)$.

Kinkless extensions of stable words are also stable.

PROPOSITION 4.3. *Let $v \in S$ be a stable word. Then, $\Sigma(v) \subseteq S$.*

Proof. Let $uvw \in \Sigma(v)$ be a kinkless extension of v. Suppose for a contradiction that uvw is left unstable (the right unstable case being symmetric), so it has $(10)^n11$ or $0(10)^n11$ as a prefix for some $n \geq 0$. The kink 11 occurs in v, so u is a prefix of $(10)^n$ or $0(10)^n$. Hence, v has a prefix of the form $0(10)^k11$ or $(10)^k11$ for some $k \geq 0$, that is, it is left unstable. □

In stable words, kinks are necessarily erased in pairs. This is depicted in Figure 2. From this and the discussion preceding Definition 4.2, we infer the following proposition.

PROPOSITION 4.4. *Let $w \in \{0,1\}^*$ be a word that contains $n \in \mathbb{N}$ kinks. Then, $f(w)$ contains $m \leq n$ kinks. If $w \in S$, then $m \equiv n \bmod 2$.*

Proof. We first match every kink in $f(w)$ to an odd number of kinks in w so that no kink of w is matched to two kinks of $f(w)$. Namely, suppose $f(w)$ contains a kink $K = 10^{2k}1$ whose left border is at position i and right border at position $i + 2k + 1$. By Lemma 4.2, the subword $w_{[i-1,i+2k+2]}$ contains an odd number of kinks, which we match with K. The first claim follows from this matching.

Suppose then that w is stable. We prove the second claim by showing that in the above matching, the number of kinks of w not matched to a kink of $f(w)$ is even. Suppose w contains a kink $K = 10^{2k}1$ at position i. If $k \geq 1$, then $f(w)$ contains $10^{2(k-1)}1$ at position i, which is matched to K.

Suppose thus $k = 0$, so that $K = 11$. Since w is stable, to the left of K, we must have an occurrence of 00; let the rightmost one of those be at position $i - p$. Then, $p \geq 2$ and w contains 001 at position $i - p$. Similarly, to the right of w, we find the leftmost occurrence of 00 at $i + q$ for some $q \geq 2$ and then 100 occurs at $i + q - 1$. Thus, w contains $001u100$ at position $i - p$, where $u \in \{0,1\}^{q+p-4}$ has no occurrences of 00. Then, $f(w)$ contains $L = 10^{q+p-2}1$ at $i - p$.

If $q + p - 2$ is even, then L is a kink that is matched with K. If it is odd, then the length of u is also odd, so $1u1$ contains an even number of occurrences of the kink 11 that are not matched to any kink of $f(w)$. One of these is K and since K was arbitrary, it follows that every unmatched kink of w is part of such an even-sized collection. □

The most important property of stable words is that they behave nicely with respect to kinkless extensions.

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PROPOSITION 4.5. *Let $w \in S$ be a stable word. Then, $f(\Sigma(w)) \subseteq \Sigma(f(w))$ and $f(\tilde{\Sigma}(w, p)) \subseteq \tilde{\Sigma}(f(w), p+1)$ for all $p \in \mathbb{Z}$. If w is not of the form $0^\alpha(10)^n1^\beta$ with $\alpha, \beta \in \{0, 1\}$ and $n \in \mathbb{N}$, then both inclusions are equalities.*

In the case where w is of the form $0^\alpha(10)^n1^\beta$, the result holds 'up to parity': for each $u = af(w)b \in \Sigma(f(w))$ with $u_i = 0$, whenever $i \not\equiv |a| + \alpha \bmod 2$, we have $u \in f(\Sigma(w))$.

We also explain here the error in [6]. Page 15 of the article contains the claim that rule 18 is Σ_p-permuting, which in our notation means $f(\Sigma(w)) \subseteq \Sigma(f(w))$ for every word $w \in \{0, 1\}^*$ of length at least three. This may be false if w is not stable. For example, take $w = 0011$, for which $f(w) = 10$, and consider $101 \in \Sigma(f(w))$.

Proof. Let $awb \in \Sigma(w)$. We claim that $f(awb) \in \Sigma(f(w))$, that is, all kinks of $f(awb)$ are contained in $f(w)$. Consider any subword u of awb such that $f(u) = 10^{2k}1$ is a kink. By Lemma 4.2, there are two possibilities.

(1) $u = 10^{2k+2}1$ is itself a kink and wider than $f(u)$. If $f(u)$ is not contained in $f(w)$, then u is not contained in w, which contradicts $awb \in \Sigma(w)$.
(2) $u = 00v00$ for some word v that begins and ends with 1, does not contain an occurrence of 00, and contains at least one occurrence of 11 (which must be within w). If $f(u)$ is not contained in $f(w)$, then u is not contained in w, which contradicts the stability of w.

Suppose then that w is not of the form $0^\alpha(10)^n1^\beta$ and let $af(w)b \in \Sigma(f(w))$. We claim that some $cwd \in \Sigma(w)$ satisfies $f(cwd) = af(w)b$. It suffices to consider $|a| = 1$ and $|b| = 0$; the general result follows by iteration and symmetry. There are a few cases.

(1) If $a = 0$ and w does not begin with 01, we can choose $c = 0$.
(2) If $a = 0$ and w begins with 01, we can choose $c = 1$.
(3) If $a = 1$ and w begins with 00, then the assumption $af(w)b \in \Sigma(f(w))$ implies that either $w = 0^{|w|}$ or w begins with $0^{2n+1}1$ for some $n \geq 1$. We can then choose $c = 1$.
(4) If $a = 1$ and w begins with 01, we can choose $c = 0$.
(5) Suppose $a = 1$ and w begins with 1. By stability and the assumption that w is not of the form $0^\alpha(10)^n1^\beta$, it must begin with $1(01)^n00$ for some $n \in \mathbb{N}$. However, then, $af(w)$ begins with $10^{2n}1$, which contradicts $af(w)b \in \Sigma(f(w))$. Hence, this case is impossible.

The claims about $\tilde{\Sigma}$ follow easily. □

From this and Proposition 4.1, we get the following corollary.

COROLLARY 4.6. *Let $w \in S$ be a stable word of length at least 2 and not of the form $0^\alpha(10)^n1^\beta$ for $\alpha, \beta \in \{0, 1\}$. For each $u = af(w)b \in \Sigma(f(w))$, there exist unique $a' \in \{0, 1\}^{|a|}$ and $b' \in \{0, 1\}^{|b|}$ with $f(a'wb') = u$.*

COROLLARY 4.7. *We have $f(\Sigma) = \Sigma$ and $f(\tilde{\Sigma}) = \tilde{\Sigma}$.*

Finally, we make use of the following result to bring down the number of kinks in an initial configuration.

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THEOREM 4.8. [5, Proposition 5] *Let $x \in \{0,1\}^{\mathbb{Z}}$ be a configuration with a finite number of 1-symbols. Then, for all large enough $n \in \mathbb{N}$, the configuration $f^n(x)$ contains at most one kink. Moreover, this $f^n(x)$ is kinkless if and only if x has an even number of kinks.*

COROLLARY 4.9. *Let $w \in \{0,1\}^*$ be a word with an even number of kinks. Then, there exists $n \in \mathbb{N}$ such that $f^n(\Sigma(0^n \cdot w \cdot 0^n)) \subseteq \Sigma$ and $f^n(\tilde{\Sigma}(0^n \cdot w \cdot 0^n), p) \subseteq \tilde{\Sigma}$ for all $p \in \mathbb{Z}$.*

Proof. Let $x \in \{0,1\}^{\mathbb{Z}}$ be the configuration with w at position zero and 0-symbols everywhere else. By Theorem 4.8, for some $n \geq 2$, the configuration $f^n(x)$ is kinkless. The word $u = f^n(0^n \cdot w \cdot 0^n) \in \Sigma$ occurs in $f^n(x)$ at position zero. Since $0^n \cdot w \cdot 0^n$ is stable, by Proposition 4.5, we have $f^n(\Sigma(0^n \cdot w \cdot 0^n)) \subseteq \Sigma(f^n(0^n \cdot w \cdot 0^n)) = \Sigma(u) \subseteq \Sigma$. The second claim is similar. □

5. *The generic limit set of rule 18*

In this section, we characterize the set of words with at most two kinks that occur in the generic limit set of rule 18. In principle, the cases of zero or one kink (Theorems 5.1 and 5.5) are subsumed by the case of two kinks (Theorem 5.9), since all words of at most one kink occur as subwords of two-kink words in the generic limit set. We present proofs for these cases as well to showcase our techniques in a simpler context. In the case of zero kinks, the idea is simple: use Theorem 4.8 to erase all existing kinks, then use the permutivity of f_{18} on kinkless configurations to produce an arbitrary kinkless word.

5.1. *Kinkless words.*

We begin with the easiest case of kinkless words.

THEOREM 5.1. *Every word with no kinks occurs in $\tilde{\omega}(f_{18})$.*

Proof. Let $w \in \Sigma$ be a kinkless word. We consider the seed ϵ at position $i = 0$. Let $q_1, q_2 \in \{0,1\}^*$ be arbitrary words. Our goal is to show that $f^k([q_1 q_2]_{-|q_1|})$ intersects $[w]_0$ for infinitely many k.

There exists a word $r \in \{0,1\}^*$ such that $q_1 q_2 r$ contains an even number of kinks. According to Corollary 4.9, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all configurations $x \in \tilde{\Sigma}(0^n \cdot q_1 q_2 r \cdot 0^n, -|q_1| - n)$, the image $f^n(x)$ contains no kinks. In the following, we call $u = 0^{2n+2} \cdot q_1 q_2 r \cdot 0^{2n+2}$.

For each $m \leq n$, the word $f^m(u)$ begins and ends in 00, so it is stable and not of the form $0^\alpha (10)^k 1^\beta$ for $\alpha, \beta \in \{0,1\}$ and $k \in \mathbb{N}$. By iterating Proposition 4.5, we have

$$\begin{aligned}
\tilde{\Sigma}(f^n(u), -|q_1| - n - 2) &= f(\tilde{\Sigma}(f^{n-1}(u), -|q_1| - n - 3)) \\
&= f^2(\tilde{\Sigma}(f^{n-2}(u), -|q_1| - n - 4)) \\
&\vdots \\
&= f^n(\tilde{\Sigma}(u, -|q_1| - 2n - 2)).
\end{aligned}$$

The word $f^n(u)$ and, hence, each configuration in $\tilde{\Sigma}(f^n(u), -|q_1| - n - 2)$ contain no kinks. Suppose that $|q_1|, |q_2 r|$ are both even, and all 1-symbols in $f^n(u)$ and w occur in even positions (the other cases are handled similarly). Then, all 1-symbols in $f^{|q_2 r| + n + 2}(u)$

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also occur at even positions, so $f^{|q_2r|+n+2}(u)w$ contains no kinks: $f^{|q_2r|+n+2}(u)w \in \Sigma(f^{|q_2r|+n+2}(u))$. Again by iterating Proposition 4.5 (and the remark following it), we obtain $\tilde{\Sigma}(f^{|q_2r|+2n+2}(u), -|q_1q_2r| - 2n - 2) = f^{|q_2r|+n+2}(\tilde{\Sigma}(f^n(u), -|q_1| - 2n - 2))$. Hence, as long as q_1 and q_2 are long enough (as we can choose them to be), there exists a word v of length $|w|$ such that $f^n(u)v$ contains no kinks and $f^{|q_2r|+n+2}(f^n(u)v)$ ends in w. In total, we have

$$\tilde{\Sigma}(w, 0) \subseteq f^{|q_2r|+n+2}(\tilde{\Sigma}(f^n(u)v, -|q_1| - n - 2)) \subseteq f^{|q_2r|+2n+2}(\tilde{\Sigma}(u, -|q_1| - 2n - 2)),$$

which shows that $[w]_0$ intersects $f^k([q_1q_2]_{-|q_1|})$ for some $k \geq n$. Since n can be arbitrarily large, Lemma 3.1 gives that w occurs in $\tilde{\omega}(f_{18})$. □

5.2. *Words with one kink.* In this section, we show that all words with exactly one kink occur in the generic limit set of rule 18. The idea is to again erase all existing kinks except one, then direct it to a specific position and use permutivity to control the kinkless region around it.

LEMMA 5.2. *Let w be a one-kink word. Then, there exists $n \in \mathbb{N}$ such that $f^n(w)$ is a left unstable or right unstable word.*

Proof. We proceed by induction on $|w|$. If $|w| = 2$ or $|w| = 3$, then w must contain 11 as a prefix or suffix, so it is already unstable. Otherwise, if w is stable, it is easy to see that $f(w)$ is also a one-kink word. By the induction hypothesis, there exists some n' such that $f^{n'}(f(w))$ is left or right unstable. Hence, the property is true for $n = n' + 1$. □

Definition 5.1. A *left kink word* is a finite word that begins with a kink and contains no other kinks. The set of left kink words is denoted $L \subset \{0, 1\}^*$.

The following lemma tells us that 11 can be produced from any word in L. As any one-kink word can be produced in one step from 11, it follows from this lemma that any one-kink word can be produced from any word of L.

LEMMA 5.3. *For all left kink words $w \in L$, there exists $n \in \mathbb{N}$ such that $\Sigma(11) \subseteq f^n(\Sigma(w))$.*

Proof. We prove the result by induction on the size of w. There are a few cases.

(1) If $|w| = 2$, then $w = 11$ and the result is immediate. Otherwise, $|w| \geq 3$ and we suppose that the property holds for all shorter words.
(2) If w begins with $10^{2k}1$ for some $k > 0$, then $f(w) \in L$ and $|f(w)| < |w|$. Then, $\Sigma(11) \subseteq f^n(\Sigma(f(w)))$ for some $n \in \mathbb{N}$ by the induction hypothesis and since $w \in S$, Proposition 4.5 gives $f^n(\Sigma(f(w))) \subseteq f^{n+1}(\Sigma(w))$.
(3) If w is of the form $11(01)^k0$ for some $k \in \mathbb{N}$, then we consider $w' = 00w0 \in \Sigma(w)$. We have $f^{k+2}(w') = 11$ and $\Sigma(f^{k+2}(w')) = f^{k+2}(\Sigma(w'))$ by Proposition 4.5, and we can choose $n = k + 2$.
(4) If w is of the form $11(01)^k$ with $k \geq 1$, then we consider $w' = 00w00 \in \Sigma(w)$ and conclude as in the previous case.

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(5) If w is of the form $11(01)^k00u$ for some word u and $k \in \mathbb{N}$, then we consider $w' = 00w$. We have $f(w') = 10^{2(k+1)}1v$ for some word v. Then, we treat $f(w')$ as in case (2) and conclude.

These cases are exhaustive. Namely, we know w begins with $10^{2k}1$. If $k \geq 1$, we are in case (2). If not, then either $w = 11$ (case (1)) or w continues with alternating 0s and 1s until the first occurrence of 00 (case (5)), if any, as it does not contain another 11. Cases (3) and (4) correspond to w containing no 00, so the alternating 0s and 1s continue until the end of w. □

The following lemma is a direct consequence of the previous one; it shows that any one-kink word can be produced at any position from any other one-kink word at position zero.

LEMMA 5.4. *Let w and w' be two words containing one kink. Then, for all $i \in \mathbb{Z}$, there exists $n \in \mathbb{N}$ such that $\tilde{\Sigma}(w', 0) \subseteq f^n(\tilde{\Sigma}(w, i))$.*

The idea is to let the kink of w evolve into a 11 whose context we can freely choose. Then, we can easily produce another 11 at any desired position at a later time step. This 11 can then be transformed into w' in a single step.

Proof. Let $k \in \mathbb{N}$ be the smallest number for which $f^k(w)$ is unstable. We may assume that it is left unstable. By Proposition 4.4, $f^k(w)$ contains one kink, which must be 11. Then, $f(00 \cdot f^k(w))$ is a left kink word.

According to Lemma 5.3, there exists $m \in \mathbb{N}$ such that $\Sigma(11) \subseteq f^m(\Sigma(f(00 \cdot f^k(w))))$ and Proposition 4.5 gives $f^m(\Sigma(f(00 \cdot f^k(w)))) \subseteq f^{k+m+1}(\Sigma(w))$. We can deduce that there exists a position $p \in \mathbb{Z}$ such that $\tilde{\Sigma}(11, p) \subseteq f^{k+m+1}(\tilde{\Sigma}(w, i))$.

Since w' is a one-kink word, it has the form $a10^{2\ell}1b$ for some $a, b \in \{0, 1\}^*$ and $\ell \in \mathbb{N}$. Suppose that $p \leq |a|$, the other case being similar. Consider $w_0 = 0011(01)^{|a|-p+1}00$. We have $f^{|a|-p+3}(w_0) = 11$ and all the intermediate steps are stable; hence, Proposition 4.5 gives $\tilde{\Sigma}(11, |a| + 1) = f^{|a|-p+3}(\tilde{\Sigma}(w_0, p - 2)) \subseteq f^{|a|-p+3}(\tilde{\Sigma}(11, p))$.

The word $u = 0011(01)^\ell 00$ is stable and $f(u) = 10^{2\ell}1$, so we obtain $\tilde{\Sigma}(w', 0) \subseteq \tilde{\Sigma}(10^{2\ell}1, |a|) \subseteq f(\tilde{\Sigma}(u'), |a| - 1) \subseteq f(\tilde{\Sigma}(11, |a| + 1))$ by Proposition 4.5. Putting everything together, we have $\tilde{\Sigma}(w', 0) \subseteq f^{k+m+|a|-p+5}(\tilde{\Sigma}(w, i))$, as claimed. □

THEOREM 5.5. *Every one-kink word occurs in $\tilde{\omega}(f_{18})$.*

Proof. Let $w \in \{0, 1\}^*$ be a one-kink word. We consider the seed ϵ. Let $q_1, q_2 \in \{0, 1\}^*$ be words, and choose $r \in \{0, 1\}^*$ and $n_0 \leq n \in \mathbb{N}$ as in the proof of Theorem 5.1, except that $q_1 \cdot q_2 \cdot r$ should contain an odd number of kinks and every configuration in $f^n(\tilde{\Sigma}(0^n q_1 q_2 r 0^n, -|q_1| - n))$ should contain a single kink. Call $u = 0^{2n+2} \cdot q_1 \cdot \epsilon \cdot q_2 \cdot r \cdot 0^{2n+2}$.

For all $m \leq n$, $f^m(u)$ is a stable one-kink word, so that $\tilde{\Sigma}(f^n(u), -|q_1| - n - 2) = f^n(\tilde{\Sigma}(u, -|q_1| - 2n - 2))$. According to Lemma 5.4, there exists $k \in \mathbb{N}$ such that $\tilde{\Sigma}(w, 0) \subseteq f^k(\tilde{\Sigma}(f^n(u), -|q_1| - n - 2)) = f^{n+k}(\tilde{\Sigma}(u, -|q_1| - 2n - 2))$. This means that $[w]$ intersects $f^{n+k}([q_1 q_2]_{-|q_1|})$. Since n can be arbitrarily large, Lemma 3.1 implies that w occurs in $\tilde{\omega}(f)$. □

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5.3. *Words with two kinks.* In this section, we determine the set P of words with exactly two kinks that appear in the generic limit set of rule 18. The idea is to erase all existing kinks except two, then bring the two kinks close to each other to produce the word 1101001 and then to show that all words of P can be produced from a two-kink configuration that contains 1101001 at any position.

Denote the reversal of a word $w \in \{0, 1\}^*$ by w^R. For a set $L \subseteq \{0, 1\}^*$, we denote $L^R = \{w^R \mid w \in L\}$.

Definition 5.2. We call $B \subset \{0, 1\}^*$ the set of two-kink words that begin with 11, end with a kink (which can overlap with the prefix 11) and are not of the form $11 \cdot (01)^k \cdot 1$ for $k \in \mathbb{N}$.

This definition is useful as B is a set of words containing two kinks that we can prevent from colliding. The following lemma states that we can direct the two kinks of a word of B so that they can be brought close and produce the word 1101001.

LEMMA 5.6. *Let $w \in B \cup B^R$ be a word. Then, there exists $n \in \mathbb{N}$ such that $\Sigma(1101001) \subseteq f^n(\Sigma(w))$.*

The proof of this lemma contains many different cases; however, the cases are quite natural and the proof of each case is easy. The main idea of the proof is that by forbidding the kinks to move away from each other, it is possible to bring them eventually closer. When the two kinks are on the verge of colliding, it is possible to avoid the collision and to instead make them closer to each other as illustrated in Figure 3.

Proof. First, we show that it suffices to prove the result for $w \in B$. Namely, then for $w \in B^R$, we have $\Sigma(1001011) \subseteq f^n(\Sigma(w))$ for some $n \in \mathbb{N}$ by symmetry and a direct computation shows that $f^3(10 \cdot 1001011 \cdot 0100) = 1101001$, so that $\Sigma(1101001) \subseteq f^{n+3}(\Sigma(w))$.

Let then $w \in B$ be arbitrary. We proceed by induction on the length of w.

Case 0: $|w| \leq 7$. There are only a few possibilities for w. If $w = 1101001$, the result is immediate. If $w = 11001$, then $\Sigma(1001011) \subseteq f^3(\Sigma(w))$ as shown in Figure 4 and $\Sigma(1101001) \subseteq f^3(\Sigma(1001011))$ as shown in Figure 5; hence, $\Sigma(1101001) \subseteq f^6(\Sigma(w))$. If $w = 1100001$, then $\Sigma(1101001) \subseteq f^2(\Sigma(1001011)) \subseteq f^5(\Sigma(w))$ as shown in Figure 6. If $w = 1100011$, then $\Sigma(1101001) \subseteq f^8(\Sigma(000010 \cdot w \cdot 0101000100)) \subseteq f^8(\Sigma(w))$ as shown in Figure 7.

For the rest of the proof, we suppose $|w| > 7$ and all shorter words of $B \cup B^R$ satisfy the claim.

Case 1: $w = 110^{2k}1$ *for some* $k \in \mathbb{N}$. Then, $k \geq 3$. We consider $w_0 = 00 \cdot w$. Then, $f^2(w_0) = 11010^{2(k-2)}1$. Evidently, $f^2(w_0) \in B$, $|f^2(w_0)| < |w|$ and $\Sigma(f^2(w_0)) \subseteq f^2(\Sigma(w))$, so we can conclude by induction.

Case 2: $w = 11010^{2k}1$ *for some* $k \in \mathbb{N}$. Then, $k \geq 2$. We consider $w_0 = 00 \cdot w$. If $k = 2$, then $f^2(w_0) = 1001011$, which we treated in Case 0. If $k = 3$, then $f^3(w_0) = 1100011$, which we already treated as well. If $k \geq 4$, then $f^3(w_0) = 1100010^{2(k-3)}1 \in B$ and $|f^3(w_0)| < |w|$. We can then conclude by induction.

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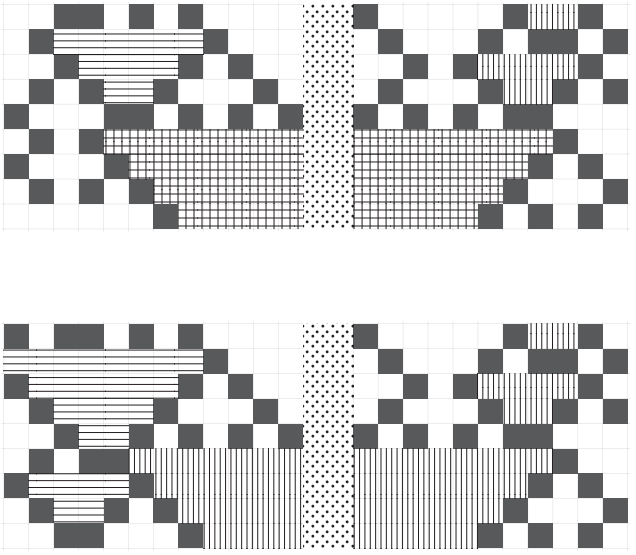


FIGURE 3. Two kinks collide and annihilate each other, but it is possible to avoid collision by making the leftmost kink longer.



FIGURE 4. We can see here that $w = 0010101100101$ is stable, $f(w)$ and $f^2(w)$ are stable, and $f^3(w) = 1001011$. We can deduce that $\Sigma(1001011) = f^3(\Sigma(w)) \subseteq f^3(\Sigma(1001011))$.

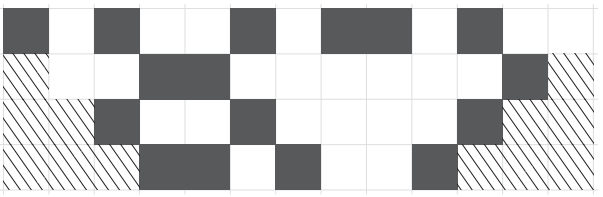


FIGURE 5. We can see here that $w = 1010010110100$ is stable, $f(w)$ and $f^2(w)$ are stable, and $f^3(w) = 1101001$. We can deduce that $\Sigma(1101001) = f^3(\Sigma(w)) \subseteq f^3(\Sigma(1001011))$.



FIGURE 6. We can see here that $w = 00101100001$ is stable, $f(w)$ is stable and $f^2(w) = 1001011$. We can deduce that $\Sigma(1101001) \subseteq f^2(\Sigma(1001011)) = f^5(\Sigma(w)) \subseteq f^5(\Sigma(1100001))$.

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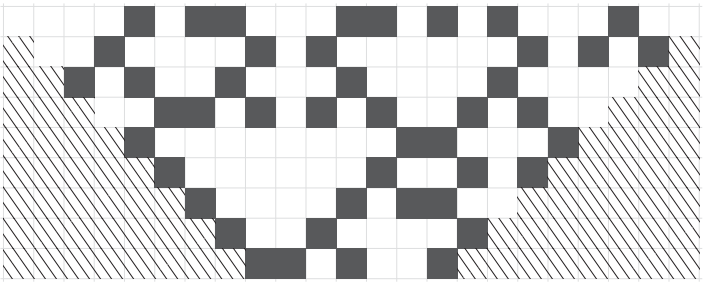


FIGURE 7. We can see here that $w = 000010110001101010000100$ is stable, $f(w)$, $f^2(w)$, $\ldots$, $f^7(w)$ are stable and $f^8(w) = 1101001$. We can deduce that $\Sigma(1101001) = f^8(\Sigma(w)) \subseteq f^8(\Sigma(1100011))$.

Case 3: $w = 11(01)^k0^{2\ell}1$ *for some* $k, \ell \geq 1$. We consider $w_0 = 00 \cdot w$. Then, $f(w_0) = 10^{2(k+1)}10^{2(\ell-1)}1$, so that for $m = \min(k+1, \ell-1)$, the word $w' = f^{m+1}(w_0)$ begins and/or ends with 11. If $\ell = 1$, then $|w'| = |w|$ and we can handle w' as in Case 1. Otherwise, $|w'| < |w|$. If $w' \in B \cup B^R$, then we can conclude by induction. If $w' \notin B \cup B^R$, then $\ell = k+2$ and $w' = 11(01)^{2(k+1)}1$. Then, we instead consider $w_1 = 0010 \cdot w$, for which $f^{k+1}(w_1) = 1001 \cdot u \cdot 11$ for some word u of length $2(k+1)$. This word is in B^R and is shorter than w, so we conclude by induction.

If we are not in the above cases, then the gap between the two kinks is longer than one cell and contains an occurrence of 00. Considering the leftmost occurrence of 00, we see that there exist $k, \ell \in \mathbb{N}$ and a word u such that $w = 11(01)^k00 \cdot u \cdot 10^{2\ell}1$. The remaining cases constrain the values of k and ℓ.

Case 4: $\ell \geq 2$. We again consider $w_0 = 00 \cdot w$ and split into subcases depending on the value of k. If $k = 0$, we consider $w_0' = f^2(w_0)$, which begins with 11 and ends with $10^{2(\ell-2)}1$. If $w_0' \in B$, we can conclude by induction as $|w_0'| < |w|$. Otherwise, $\ell = 2$ and $w_0' = 11(01)^{(|w|-6)/2}011$. Then, we instead consider $v_0 = 0010 \cdot w \cdot 00$, which satisfies $f^3(v_0) = 11(00)^{(|w|-3)/6} \cdot 1$. We can now conclude using Case 1, as $|f^3(v_0)| = |w|$.

If $0 < k < \ell - 2$, then $f^{k+2}(w_0) \in B$ and $|w_0| < |w|$, so we can conclude using induction.

If $k = \ell - 2$, then consider $w' = f^\ell(w_0)$. If $w' \in B$, we have $|w'| < |w|$ and we conclude by induction. If $w' \notin B$, we instead consider $w_1 = 0010 \cdot w$. Then, $f^{\ell+1}(w_1) \in B^R$ and $|f^{\ell+1}(w_1)| < |w|$, so we conclude using induction.

If $k > \ell - 2$, then $f^\ell(w_0) \in B^R$ and $|f^\ell(w_0)| < |w|$. We conclude by induction.

Case 5: $\ell \leq 1$ *and* $k \geq 1$. If $\ell = 0$, then we consider $w_0 = 00 \cdot w \cdot 00$. Now, $f(w_0)$ begins with $10^{2(k+2)}1$ and ends with $10^{2m}1$ for some $m \geq 1$. We proceed as in Case 3, using $w_1 = 0010 \cdot w \cdot 00$ instead of w_0 as needed. If $\ell = 1$, then $f(w_0)$ is in B^R, has the same length as w and belongs to Case 3, which we already solved.

Case 6: $\ell = 1$ *and* $k = 0$. We consider $w_0 = 00 \cdot w$ and denote $w_0' = f(w_0)^R$, which begins with 11 and ends with 1001. If w_0' belongs to one of the already solved Cases 0–5, then we are done. Otherwise, $w_0' \in B$ belongs to Case 6. Then, we repeat this process: for each $i \in \mathbb{N}$, as long as w_i' belongs to Case 6, define $w_{i+1} = 00 \cdot w_i'$ and $w_{i+1}' = f(w_{i+1})^R = 11 \cdot u_{i+1} \cdot 1001$. Suppose that we reach $w_{|w|}$ this way. We can now go backward in time as shown in Figure 8: for each $j \geq 0$, the length-j prefix and suffix

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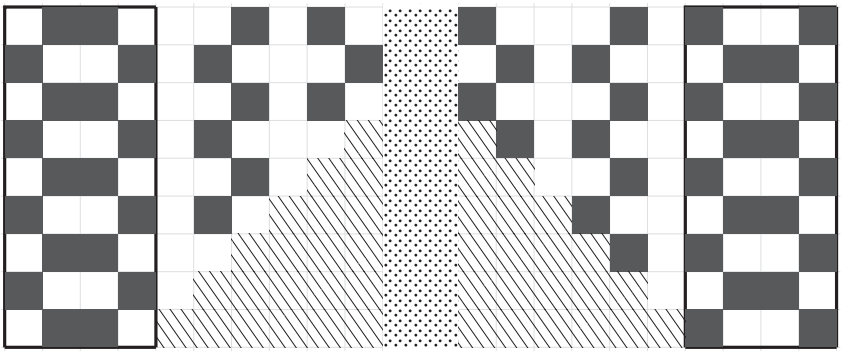


FIGURE 8. At each step of the backward computation, we force the four left and four right symbols to be either 0110 or 1001, this in turn fixes the neighbours symbols to some value. After enough steps, the whole word is forced.

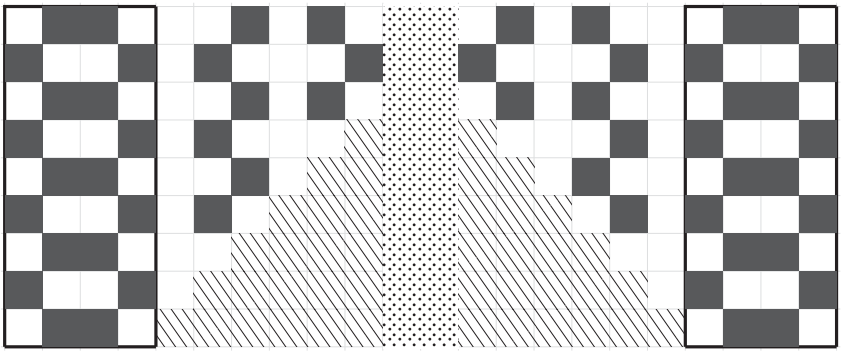


FIGURE 9. In the same way as in Figure 8, after enough steps of backward computation, the whole word is forced.

of $u_{|w|-j}$ are determined locally. In the end, we determine that $w = 1 \cdot (100010)^{(|w|-5)/6} \cdot 1001$.

We now consider $v_0 = 0010 \cdot w \cdot 01$ instead of w_0. Then, $v'_0 = f^3(v_0) = 110 \cdot (101000)^{(|w|-5)/6} \cdot 11$ and $v'_1 = f^4(00 \cdot v'_0 \cdot 0001)$ verifies $v'_1 \in B$ and $|v'_1| < |w|$. We can then conclude using induction.

Case 7: $\ell = k = 0$. We consider $w_0 = 00 \cdot w \cdot 00$ and $w'_0 = f^2(w_0)$. If w'_0 belongs to one of the already solved cases, then since $|w'_0| = |w|$, we are done. If w'_0 belongs to Case 7, we again repeat the process: for each $i \in \mathbb{N}$, as long as w'_i belongs to Case 7, define $w_{i+1} = 00 \cdot w'_i \cdot 00$ and $w'_{i+1} = f^2(w_{i+1}) = 11 \cdot u_{i+1} \cdot 11$. In the same way as in Case 6, if we reach $w_{|w|}$ in this way, we go backward in time as shown in Figure 9 to fully determine that $w = 11000 \cdot (101000)^{(|w|-7)/6} \cdot 11$.

We now consider $v_0 = 0010 \cdot w \cdot 0001$. Then, $v'_0 = f^3(v_0) = 110101 \cdot (000101)^{(|w|-7)/6} \cdot 001$ and $v'_1 = f^4(00 \cdot v'_0 \cdot 0100) = 1100 \cdot (000101)^{(|w|-7)/6} \cdot 001$. We are now in Case 6, which was already solved, with $|w| = |v'_1|$.

Finally, we explain why the cases are exhaustive. Since $w \in B$, it begins with 11, ends with a kink $10^{2\ell}1$ and contains no other kinks. If the two kinks overlap, we are in Case 1. If they are separated by a single 0, we are in Case 2. If they are separated by a longer

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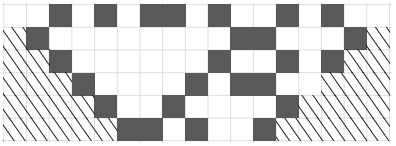


FIGURE 10. Moving a 1101001 one step to the left.



FIGURE 11. Moving a 1101001 one step to the right.

alternating sequence of 0s and 1s, then by the definition of B, we must have $\ell \geq 1$, so we are in Case 3. Otherwise, as explained after Case 3, we have $w = 11(01)^k00 \cdot u \cdot 10^{2\ell}1$ for some word u, and Cases 4–7 cover all values of k and ℓ. This finishes the proof. □

LEMMA 5.7. *Let $i, j \in \mathbb{Z}^2$ be two positions, then there exists $n \in \mathbb{N}$ such that $\tilde{\Sigma}(1101001, i) \subseteq f^n(\tilde{\Sigma}(1101001, j))$.*

Proof. We proceed by induction on $|i - j|$. If $i = j$, we can choose $n = 0$. If $i < j$, then $\tilde{\Sigma}(1101001, i + 1) \subseteq f^5(\tilde{\Sigma}(1101001, j))$ as shown in Figure 10 and we can conclude using induction. If $i > j$, then $\tilde{\Sigma}(1101001, i - 1) \subseteq f^3(\tilde{\Sigma}(1101001, j))$ as shown in Figure 11 and we can conclude using induction. □

The following definition provides us with a set P of words that can be produced from 1101001 and, as we show later, these words constitute exactly the words of the generic limit set that have two kinks.

Definition 5.3. Let w be a word that contains exactly two kinks. We call $b(w)$ the smallest subword of w that contains two kinks. If $b(w) = 10^{2k}1 \cdot u \cdot 10^{2\ell}1$ for some $k, \ell \in \mathbb{N}$ and $u \in \{0, 1\}^*$, then we call $\delta(w) = u$. If $b(w)$ is of the form $10^{2k}10^{2\ell}1$, then we call $\delta(w) = \epsilon$.

Definition 5.4. We define $P \subset \{0, 1\}^*$ as the set of those two-kink words w that satisfy all of the following:

- $b(w) \neq 111$;
- if $b(w) = 11v11$, then v contains an even number of 1s;
- for all $k \in \mathbb{N}$, $b(w) \neq 1(100010)^k1001$ and $b(w) \neq 1001(010001)^k1$.

Our goal is to prove that P coincides with the set of two-kink words of the generic limit set of f. The first two conditions are necessary, since the words 111 and $11v11$ with v containing an odd number of 1s have no preimages under f. Namely, the only preimage of 11 is 1001, which already forbids 111. However, one can prove by induction on the length of w that if w contains an even number of 1s, then all preimages of $11w$ end in 01, 10 or

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11 and, otherwise, they all end in 00. Then, $11v11$ has no preimage, as it would have to end in both 1001 and $00ab$ for some $a, b \in \{0, 1\}$. Similarly, every two-kink word $11w11$ where w contains an even number of 1s has a unique two-kink preimage (and possibly other preimages with more kinks): if $w = 0^{2n_0-1}10^{2n_1-1}1 \cdots 10^{2n_{2k}-1}$, that preimage is $100(10)^{n_0}0^{2n_1}(10)^{n_2}0^{2n_3} \cdots (10)^{n_{2k}}01$.

We now proceed backwards in time and try to find for every word in P a parent that contains 1101001. Just as in Lemma 5.6, the proof is a combination of an induction and a case analysis. There are quite a few cases, but none of them is very difficult. The main idea is that we are first trying to close the gap between the two kinks in the word and then shorten the kinks.

LEMMA 5.8. *Let $w \in P$. Then, there exists $n \in \mathbb{N}$ and a position $p \in \mathbb{Z}$ such that $\tilde{\Sigma}(w, p) \subseteq f^n(\tilde{\Sigma}(1101001, 0))$.*

Proof. We first handle the 'trivial' cases.

Case 0: w contains 1101001 *or its reversal.* If 1101001 is a subword of w, then the property is verified with $n = 0$. If 1001011 is a subword of w, then $\Sigma(1001011) \subseteq f^3(\Sigma(1101001))$, as shown in Figure 5. Hence, there exists a position p such that $\tilde{\Sigma}(w, p) \subseteq f^3(\tilde{\Sigma}(1101001, 0))$.

We proceed by induction on the length of $\delta(w)$. Let w be a word such that $|\delta(w)| = 0$. Then, there exist some words α, β and some integers k, ℓ such that $w = \alpha 10^{2k}10^{2\ell}1\beta$.

Case 1: $k = 0$ or $\ell = 0$. If $k = 0$ (the case $\ell = 0$ is symmetrical), then $w \in P$ implies $\ell \geq 2$. According to Corollary 4.6, there exists a unique word α' and a unique word β' such that $w' = \alpha' \cdot 1001011(01)^{\ell-2}00 \cdot \beta'$ contains two kinks and $f(w') = w$. Since $w' \in \Sigma(1001011)$ and w' is stable, we have $\tilde{\Sigma}(w, 1) = f(\tilde{\Sigma}(w', 0))$. The word w' belongs to Case 0, so $\tilde{\Sigma}(w', p) \subseteq f^n(\tilde{\Sigma}(1101001, 0))$ for some $p \in \mathbb{Z}$. Then, $\tilde{\Sigma}(w, p + 1) \subseteq f^{n+1}(\tilde{\Sigma}(1101001, 0))$ by Proposition 4.5. In the remainder of the proof, we implicitly apply this argument whenever we fall to an earlier case or use the induction hypothesis.

Case 2: $k \geq 1$ and $\ell \geq 1$. Now, there exist unique words α' and β' such that $w' = \alpha' \cdot 10^{2(k+1)}11(01)^{\ell-1}00 \cdot \beta'$ contains two kinks and $f(w') = w$. Since w' is stable, we have $\Sigma(w) = f(\Sigma(w'))$. Hence, we can conclude by Case 1.

In the following, let $w \in P$ be a word with $|\delta(w)| > 0$. We now suppose the property holds for all words $w' \in P$ with $|\delta(w')| < |\delta(w)|$. Let α and β be words, and k, ℓ be integers such that $w = \alpha \cdot 10^{2k}1 \cdot \delta(w) \cdot 10^{2\ell}1 \cdot \beta$. Because of Case 0, we may assume $k \leq \ell$.

Case 3: $|\delta(w)| = 1$. Then, we must have $\delta(w) = 0$. There exist unique words α' and β' such that $w' = \alpha' \cdot 10^{2(k+1)}10^{2(l+1)}1 \cdot \beta'$ contains two kinks and $f(w') = w$. As w' is stable, we can conclude using Case 2.

Case 4: $|\delta(w)| \geq 3$, $k = 0$ and $\ell = 0$. As $w \in P$, the word $\delta(w)$ between the two kinks necessarily contains an even number of 1s. According to the discussion following Definition 5.4, there exists a word of the form $w' = 1001\delta1001$ with two kinks such that $f(w') = w$. Now, $w' \in P$ and $\delta(w') = \delta$ verifies $|\delta(w')| < |\delta(w)|$, so we can conclude by the induction hypothesis.

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Case 5: $k = 0$ *and* $\ell = 1$. Now, there exists a unique word u' such that $1001 \cdot u'$ contains one kink and $f(1001 \cdot u') = 11 \cdot \delta(w)$.

If u' is of the form $v' \cdot 10$ for some word v', then there exist unique words α' and β' such that $w' = \alpha' \cdot 1001 \cdot v' \cdot 100001 \cdot \beta'$ contains two kinks and $f(w') = w$. Now, $\delta(w') = v'$; hence, $|\delta(w')| < |\delta(w)|$ and we can conclude by induction.

Otherwise, u' is of the form $v' \cdot 00$. There exists a word u'' such that $1001 \cdot u''$ contains one kink and $f(1001 \cdot u'')^R = u' \cdot 11$. We now consider u'': if it ends in 10, we conclude as above, otherwise, we continue.

We suppose that we are able to repeat the process an infinite number of times: we can define words $\alpha^{(k)}$, $\beta^{(k)}$, $u^{(k)}$ for all $k \in \mathbb{N}$ such that the word $w^{(k)} = \alpha^{(k)} \cdot 1001 \cdot u^{(k+1)} \cdot 11 \cdot \beta^{(k)}$ contains two kinks, $w^{(0)} = w$ and $f(w^{(k+1)})^R = w^{(k)}$. As in the proof of Case 6 of Lemma 5.6 and in Figure 8, we can go backwards in time to deduce prefixes and suffixes of the $u^{(k)}$, and eventually determine that $w^{(|w|)} = 1 \cdot (100010)^{(|w|-4)/6} \cdot 1001$. Then, by direct computation, we have $w = 1 \cdot (100010)^{(|w|-4)/6} \cdot 1001$ as well. We then proved that $w \notin P$, which is a contradiction.

Case 6: $k = 0$ *and* $\ell > 1$. Again, there exists a unique word u' such that $1001 \cdot u'$ contains one kink and $f(1001 \cdot u') = 11 \cdot \delta(w)$.

If u' is of the form $v' \cdot 10$ for some word v', then there exist unique words α' and β' such that $w' = \alpha' \cdot 1001 \cdot v' \cdot 10^{2(\ell+1)}1 \cdot \beta'$ contains two kinks and $f(w') = w$. Now, $w' \in P$ and $\delta(w') = v'$ implying $|\delta(w')| < |\delta(w)|$, so we conclude by induction.

Otherwise, u' ends in 00. There exist unique words α' and β' such that $w' = \alpha' \cdot 1001 \cdot u' \cdot 11(01)^{\ell-1}00 \cdot \beta'$ contains two kinks and $f(w') = w$. If $w' \in P$, then it is in Case 5 with $|\delta(w')| = |\delta(w)|$ and we can conclude.

Otherwise, $w' \notin P$, that is, there exists $m \in \mathbb{N}$ such that $u' \cdot 1 = (010001)^m$. This implies $w = \alpha \cdot 1(100010)^m 10^{2\ell}1 \cdot \beta$ and we call it the *difficult subcase*. We define

$$u'' = \begin{cases} 001011(000000010101)^{m/2}00001 & \text{if } m \text{ is even,} \\ 1000000(101010000000)^{(m-1)/2}1010100001 & \text{if } m \text{ is odd.} \end{cases}$$

In both cases, it can be checked by a direct computation that $f^2(u'') = 1001 \cdot u' \cdot 1011$. Then, there are unique words α'' and β'' such that $w'' = \alpha'' \cdot u'' \cdot \beta''$ has two kinks and satisfies $f^3(w'') = w$. If m is even, then $|\delta(w'')| = |\delta(w)|$ and w'' belongs to Case 6, and by its form, it cannot be in the difficult subcase. If m is odd, then $|\delta(w'')| < |\delta(w)|$. Hence, we can conclude.

Case 7: $k \geq 1$ *and* $\ell \geq 1$. Again, there exists a unique word u' such that $10^{2(k+1)}1 \cdot u'$ contains one kink and $f(10^{2(k+1)}1 \cdot u') = 10^{2k}1 \cdot \delta(w)$.

If u' is of the form $v' \cdot 10$ for some word v', then there exist unique words α' and β' such that $w' = \alpha' \cdot 10^{2(k+1)}1 \cdot v' \cdot 10^{2(l+1)}1 \cdot \beta'$ contains two kinks and $f(w') = w$. We have $\delta(w') = v'$; hence, $|\delta(w')| < |\delta(w)|$ and we conclude by induction.

Otherwise, u' ends in 00. Then, there exist unique words α' and β' with $w' = \alpha' \cdot 10^{2(k+1)}1 \cdot u' \cdot 11(01)^{\ell-1} \cdot \beta'$ and $f(w') = w$. We have $w' \in P$ and $|\delta(w')| = |\delta(w)|$. Moreover, w' belongs to the already handled Case 6.

Finally, we explain why the cases are exhaustive. Since $w \in P$, it contains exactly two kinks; let them be $10^{2k}1$ and $10^{2\ell}1$ in this order. Cases 1 and 2 cover the cases where

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the kinks overlap. The kinks cannot be directly adjacent, and Cases 0 and 3 cover the cases where they are separated by a single symbol, which must be a 0. Otherwise, the kinks are separated by a word $\delta(w)$ of length at least three. Cases 4–6 cover the subcases where $k = 0$. Case 7 covers the case where $k, \ell \geq 1$. The remaining cases, where $k \geq 1$ and $\ell = 0$, are handled by symmetry: the word $w^R \in P$ falls into one of the above cases, so $\tilde{\Sigma}(w^R, p) \subseteq f^n(\tilde{\Sigma}(1001011, 0))$ for some n and p, and reflecting around position 4, we get $\tilde{\Sigma}(w, 8 - p) \subseteq f^n(\tilde{\Sigma}(1101001, 0))$. This concludes the proof. □

THEOREM 5.9. *Let $w \in \{0,1\}^*$ be a word that contains two kinks. Then, w occurs in $\tilde{\omega}(f_{18})$ if and only if $w \in P$.*

Proof. First, let $w \notin P$ be a word with two kinks. If w has a subword 111 or a subword of the form $11u11$ with u containing an odd number of 1s, then w has no parent and, hence, it does not occur in $\tilde{\omega}(f_{18})$.

Suppose then that w has a subword of the form $u = 1(100010)^k1001$ for some $k \in \mathbb{N}$. We call $u' = 1001(010001)^k1$. These have the property that $f^{-1}([u]_0) \cap f(\{0,1\}^*) \subseteq [u']_{-1}$ and $f^{-1}([u']_{-1}) \cap f(\{0,1\}^*) \subseteq [u]_0$. We suppose for a contradiction that $w \in \tilde{\omega}(f_{18})$. Then, there exist a seed $s \in \{0,1\}^*$ and a position $p \in \mathbb{Z}$ such that for all $a, b \in \{0,1\}^*$, there are infinitely many $n \in \mathbb{N}$ such that $f^n([asb]_{i-|a|}) \cap [w]_0 \neq \emptyset$. We may assume that $p \leq 0$ and $|s| - p \geq |w|$.

There exists a word q such that sq contains an even number of kinks. By Corollary 4.9, there exists $\ell \in \mathbb{N}$ such that $f^\ell(0^\ell sq0^\ell)$ is kinkless. Then, we have $f^\ell(x)_{[0,|w|-1]} \in \Sigma$ for all $x \in [0^\ell sq0^\ell]_{p-\ell}$. However, there should exist $n \geq \ell$ and $x \in [0^\ell sq0^\ell]_{p-\ell}$ with $f^n(x) \in [w]_0$. This implies $f^n(x) \in [u]_j$ for some $0 \leq j < |w| - |u|$. Then, for all $m \leq n$, we have $f^{n-m}(x) \in [u]_j$ if m is even and $f^{n-m}(x) \in [u']_{j-1}$ if m is odd. When $m = n - \ell$, we obtain a contradiction with $f^\ell(x)_{[0,|w|-1]} \in \Sigma$, since u and u' are two-kink words. Hence, w does not occur in the generic limit set.

We now prove the other side of the implication. Let $w \in P$ be a word. We consider the seed ϵ at position $i = 0$. Let $q_1, q_2 \in \{0,1\}^*$ be words. There exists a word $r \in \{0,1\}^*$ such that $q_1 \cdot q_2 \cdot r$ contains an even number of kinks. According to Corollary 4.9, for all large enough $n \in \mathbb{N}$, all words in $f^n(\Sigma(0^n \cdot q_1q_2r \cdot 0^n, -|q_1| - n))$ are kinkless. In the following, we call $s = 10^{2n}10^{3n} \cdot q_1q_2r \cdot 0^{3n}10^{2n}1$.

We have $f^n(\Sigma(s)) = \Sigma(f^n(s))$ by Proposition 4.5. Also, $f^n(s)$ is a two-kink word of the form $11 \cdot v \cdot 11$, where $v \neq (01)^{\lfloor v/2 \rfloor}0$, so that $f^n(s) \in B$. According to Lemma 5.6, there exists $n_0 \in \mathbb{N}$ such that $\Sigma(1101001) \subseteq f^{n_0}(\Sigma(f^n(s))) = f^{n+n_0}(\Sigma(s))$. Hence, there exists a position $j \in \mathbb{Z}$ such that $\tilde{\Sigma}(1101001, j) \subseteq f^{n+n_0}(\tilde{\Sigma}(s, -|q_1| - 5n - 2))$.

According to Lemma 5.8, there exists $n_1 \in \mathbb{N}$ and a position $p \in \mathbb{Z}$ such that $\tilde{\Sigma}(w, p) \subseteq f^{n_1}(\tilde{\Sigma}(1101001, 0))$. According to Lemma 5.7, there exists $n_2 \in \mathbb{N}$ such that $\tilde{\Sigma}(1101001, -p) \subseteq f^{n_2}(\tilde{\Sigma}(1101001, j))$. Hence, $\tilde{\Sigma}(w, 0) \subseteq f^{n_1+n_2}(\tilde{\Sigma}(1101001, j))$. Combining all of this, we obtain

$$\tilde{\Sigma}(w, 0) \subseteq f^{n+n_0+n_1+n_2}(\tilde{\Sigma}(s, -|q_1| - 5n - 2)) \subseteq f^{n+n_0+n_1+n_2}([q_1q_2]_{-|q_1|}).$$

As n can be chosen arbitrarily large, this shows that w occurs in $\tilde{\omega}(f)$. □

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6. *Separation of limit sets*

In this section, we show that the limit set, generic limit set and μ-limit set of f_{18} are distinct for any shift-invariant probability measure μ. In fact, there is a single word that occurs in $\tilde{\omega}(f_{18})$ but does not occur in any $\Omega_\mu(f_{18})$.

THEOREM 6.1. *Rule 18 satisfies* $\tilde{\omega}(f_{18}) \subsetneq \Omega(f_{18})$ *and* $\tilde{\omega}(f_{18}) \setminus \overline{\bigcup_{\mu \in \mathcal{M}_\sigma(\{0,1\}^{\mathbb{Z}})} \Omega_\mu(f_{18})} \neq \emptyset$.

Proof. We first prove that $\tilde{\omega}(f_{18})$ is properly contained in the limit set. The word 10011 occurs in the latter, since the spatially periodic point $x = {}^{\infty}(1001)^{\infty}$ satisfies $f_{18}^2(x) = x$. By Theorem 5.9, it does not occur in $\tilde{\omega}(f_{18})$.

For the other claim, take any shift-invariant measure μ and for $n \in \mathbb{N}$, denote by

$$d_n = \sum_{k \in \mathbb{N}} (f_{18}^n \mu)(10^{2k}1)$$

the probability of having the left border of a kink at a given coordinate after n iterations of f_{18}. By Proposition 4.4, f_{18} cannot create new kinks and by shift-invariance of μ, we then have $d_{n+1} \leq d_n$. The probability of seeing the left border of a kink that will be destroyed at the next step is exactly $d_n - d_{n+1}$.

Consider the word $w = 001101100$. Since $f_{18}(w) = 1000001$ results in the destruction of two kinks, we have $\sum_{n \in \mathbb{N}} (f_{18}^n \mu)(w) \leq \sum_{n \in \mathbb{N}} d_n - d_{n+1} = \lim_{n \to \infty} d_0 - d_n \leq 1$. Hence, $\lim_{n \to \infty} (f_{18}^n \mu)(w) = 0$, so w does not occur in $\Omega_\mu(f_{18})$. By Theorem 5.9, it occurs in $\tilde{\omega}(f_{18})$. □

While our analysis of the generic limit set is enough to obtain this separation result, we are not able to fully characterize its language. Doing so would be interesting for the following reason. Let μ be the uniform Bernoulli measure. In [**8**, Conjecture 3], Lind conjectures that the density d_n of kinks in an $f_{18}^n \mu$-random configuration is approximately $(8\pi Dn)^{-1/2}$, where the 'diffusion coefficient' $D \approx 1/2$.

The weaker (but still open) claim that $d_n \to 0$ is equivalent to the condition that no kinks occur in the μ-limit set $\Omega_\mu(f_{18})$. By [**1**, Proposition 2.10], the μ-limit set of every CA g is *shift-recurrent*, meaning that if w occurs in $\Omega_\mu(g)$, then there exists a word u such that wuw occurs in $\Omega_\mu(g)$ as well. Hence, for the weaker claim, it would be enough to prove that $\Omega_\mu(f_{18}) \subseteq \tilde{\omega}(f_{18})$ and that for some $k \in \mathbb{N}$, no word with k kinks occurs in the generic limit set $\tilde{\omega}(f_{18})$. We have shown that the latter claim is false for $k \leq 2$.

Open problem 6.2. Characterize the language of the generic limit set $\tilde{\omega}(f_{18})$.

Open problem 6.3. Do we have $\Omega_\mu(f_{18}) \subseteq \tilde{\omega}(f_{18})$ for the uniform Bernoulli measure μ?

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