A NOTE ON STOKES'S STREAM FUNCTION FOR MOTION WITH A SPHERICAL BOUNDARY

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The circle theorem of Milne-Thomson (1) connecting the complex potential in a two-dimensional irrotational flow about a circular cylinder with that of the flow when the cylinder is absent has a three-dimensional counterpart in the result due to Weiss (3) for the perturbed velocity potential in an unlimited irrotational flow when the rigid spherical boundary $r = a$ is inserted.

The present note establishes the perturbed Stokes stream function for axisymmetric motion internal or external to the rigid spherical boundary $r = a$. The advantages in using the stream function are twofold. The stream sheets are given directly and the functional form is simpler, since it does not involve an integral term as does the sphere theorem of Weiss. On the other hand, the velocity potential result of Weiss is of wider application, not being restricted to axisymmetric motion.

The proofs of the two sphere theorems depend on an inversion theorem for Stokes stream function $\psi(r, \theta)$, which, for an axisymmetric irrotational motion of an inviscid fluid, satisfies the equation (see (2), p. 422)

$$\left\{r^2 \frac{\partial^2}{\partial r^2} + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)\right\} \psi(r, \theta) = 0, \tag{1}$$

in spherical polar coordinates r, θ, ϕ.

If we put $rr' = \text{constant} = a^2$, so that

$$\frac{\partial}{\partial r} = -\frac{r'^2}{a^2}\frac{\partial}{\partial r'} \quad \text{and} \quad r^2 \frac{\partial}{\partial r} = -a^2 \frac{\partial}{\partial r'},$$

the equation becomes

$$\left\{\frac{\partial}{\partial r'}\left(r'^2 \frac{\partial}{\partial r'}\right) + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)\right\} \psi\left(\frac{a^2}{r'}, \theta\right) = 0,$$

or, dropping the accents and multiplying by r,

$$\left\{r \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right) r\right\} \psi\left(\frac{a^2}{r}, \theta\right) = 0.$$

But
$$r \frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) = r^3 \frac{\partial^2 \psi}{\partial r^2} + 2r^2 \frac{\partial \psi}{\partial r} = r^2 \frac{\partial}{\partial r}\left(r \frac{\partial \psi}{\partial r} + \psi\right) = r^2 \frac{\partial^2}{\partial r^2}(r\psi),$$

so that
$$\left\{r^2 \frac{\partial^2}{\partial r^2} + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)\right\} r\psi\left(\frac{a^2}{r}, \theta\right) = 0. \tag{2}$$

Comparison of (1) and (2) shows that if $\psi(r, \theta)$ satisfies equation (1), then so does $r\psi(a^2/r, \theta)$. This inversion theorem for the Stokes stream function is the analogue of Kelvin's inversion theorem for harmonic functions.

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The sphere theorem for external fluid. Suppose $\psi_0(r,\theta)$ is the Stokes stream function for an unlimited fluid in irrotational motion, all the singularities (sources, vortices, etc.) being in the region external to the sphere $r = a$, and suppose that $\psi_0 = 0$ at the origin. When the rigid boundary $r = a$ is introduced the stream function for the fluid external to this boundary can be written

$$\psi = \psi_0(r,\theta) + \psi_1(r,\theta), \tag{3}$$

ψ_1 being termed the perturbation stream function. The sphere theorem for the fluid motion in the region external to the sphere asserts that the perturbation stream function in this case is

$$\psi_1 = -\frac{r}{a}\psi_0\left(\frac{a^2}{r},\theta\right). \tag{4}$$

The necessary and sufficient conditions which the perturbation stream function must satisfy are

(i) ψ_1 must be a solution of (1),

(ii) since $r = a$ is a stream sheet,

$$\psi_1(a,\theta) = -\psi_0(a,\theta)$$

apart from an additive constant,

(iii) any singularities of ψ_1 must lie outside the fluid, and

(iv) the perturbation velocity resolutes

$$u_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi_1}{\partial\theta}, \quad u_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi_1}{\partial r} \tag{5}$$

must vanish as $r\to\infty$ and there must be no net flow introduced over the sphere at infinity.

Now it is clear from the inversion theorem that (i) is satisfied by the function ψ_1 given by (4), and it is equally clear that this automatically satisfies condition (ii). Moreover, if a singularity of $\psi_0(r,\theta)$ exists at P $(r = r_0)$ then a corresponding singularity of ψ_1 will occur at P' $(r = r' = a^2/r_0)$. In this case, since all such points P are outside the sphere, $r_0 > a$, and so $r' < a$, hence all points P' fall outside the fluid, and condition (iii) is satisfied. Finally, since ψ_0 vanishes at the origin and gives finite velocities there, $\psi_0(r,\theta) = O(r^2)$ at most for small r and so $\psi_1(r,\theta) = O(1/r)$ at most for large r. Hence the outflow over a distant sphere $\int u_r dS = O(1/r)$ at most for large r and so tends to zero as $r\to\infty$, which shows that the remaining condition (iv) is satisfied and establishes the theorem.

The sphere theorem for internal fluid. Suppose now that $\psi_0(r,\theta)$ is the Stokes stream function for an unlimited axisymmetric irrotational flow in which all the singularities lie within the sphere $r = a$, and such that $\psi_0 = O(1/r)$ for large r, so that $u_r = O(1/r^3)$ for large r, and there is accordingly no net outflow over the sphere at infinity.

The sphere theorem for the region internal to the sphere $r = a$ then asserts that the function ψ_1 with the new meaning for ψ_0 is the perturbation stream function for the fluid motion occupying the region $r \leqslant a$ when $r = a$ is made a rigid boundary.

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The necessary and sufficient conditions are as before in (i), (ii) and (iii), and the new condition (iv) will be

(iv) ψ_1 must give finite velocity resolutes at the origin.

It is clear that (i), (ii) and (iii) are satisfied as in the previous case, and from our restriction on ψ_0 for large r, we see that $\psi_1 = O(r^2)$ at most for small r, hence the perturbation velocity resolutes given by (5) at the origin are finite, and the new condition (iv) is satisfied, which completes the sphere theorem when the fluid is inside the sphere.

We conclude with some applications of the two theorems which exhibit their usefulness in dealing with this type of problem. For these we shall need the following results:

Let $$R^2 = r^2 + b^2 - 2br\cos\chi, \quad R'^2 = r^2 + b'^2 - 2b'r\cos\chi, \tag{6}$$

where $bb' = a^2$ and χ is a function of θ only. Then

$$R\left(\frac{a^2}{r}, \theta\right) = \left\{\frac{a^4}{r^2} + b^2 - 2b\frac{a^2}{r}\cos\chi\right\}^{\frac{1}{2}} = \frac{b}{r}\{r^2 + b'^2 - 2b'r\cos\chi\}^{\frac{1}{2}}$$

or $$R\left(\frac{a^2}{r}, \theta\right) = \frac{b}{r}R'(r, \theta). \tag{7}$$

Further, let $A_1(f, 0)$, $A_2(f', 0)$, $A_3(g, 0)$, $A_4(g', 0)$, $A_5(g, \pi)$, $A_6(g', \pi)$ be points on the axis of the axisymmetric motion, such that $ff' = a^2$, $gg' = a^2$, and let (r_s, θ_s, ϕ) be the spherical polar coordinates of the point $P(r, \theta, \phi)$ referred to the points $A_s (s = 1, \ldots, 6)$.

Then from (7), since $$r_1^2 = r^2 + f^2 - 2rf\cos\theta \tag{8}$$

with similar results for $r_2, \ldots, r_6$, we have

$$r_1\left(\frac{a^2}{r}, \theta\right) = \frac{f}{r}r_2(r, \theta), \quad r_3\left(\frac{a^2}{r}, \theta\right) = \frac{g}{r}r_4(r, \theta), \quad r_5\left(\frac{a^2}{r}, \theta\right) = \frac{g}{r}r_6(r, \theta). \tag{9}$$

We have also $$r\sin\theta = r_s\sin\theta_s \quad (s = 1, \ldots, 6), \tag{10}$$

with
$$\begin{aligned} r\cos\theta &= r_1\cos\theta_1 + f &&= r_2\cos\theta_2 + f' \\ &= r_3\cos\theta_3 + g &&= r_4\cos\theta_4 + g' \\ &= r_5\cos\theta_5 - g &&= r_6\cos\theta_6 - g', \end{aligned} \tag{11}$$

and also
$$\begin{aligned} r^2 &= r_1^2 + f^2 + 2fr_1\cos\theta_1 &&= r_2^2 + f'^2 + 2f'r_2\cos\theta_2 \\ &= r_3^2 + g^2 + 2gr_3\cos\theta_3 &&= r_4^2 + g'^2 + 2g'r_4\cos\theta_4 \\ &= r_5^2 + g^2 - 2gr_5\cos\theta_5 &&= r_6^2 + g'^2 - 2g'r_6\cos\theta_6. \end{aligned} \tag{12}$$

(1) *Source outside a sphere.* For a source of strength m at A_1 (with $f > a$), taking $\psi = 0$ at the origin,

$$\psi_0(r, \theta) = m(1 + \cos\theta_1),$$

or, using (11), $$\psi_0(r, \theta) = m\{1 + (r\cos\theta - f)/r_1\}.$$

Hence from (4), using (9), $$\psi_1 = -\frac{m}{a}\left\{r + \frac{r^2}{fr_2}\left(\frac{a^2}{r}\cos\theta - f\right)\right\},$$

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whence, using (11) and (12),

$$\psi_1 = -\frac{m}{a}\left\{r + \frac{a^2}{fr_2}(r_2\cos\theta_2 + f') - \frac{1}{r_2}(r_2^2 + f'^2 + 2f'r_2\cos\theta_2)\right\}$$

or
$$\psi_1 = -\frac{m}{a}(r - r_2) + \frac{ma}{f}\cos\theta_2, \tag{13}$$

which is the stream function for a source of strength ma/f at A_2 together with a line sink of strength m/a per unit length between O and A_2, which is the well-known image system for the source in the presence of the sphere.

The equation of the stream sheets will be $\psi_0 + \psi_1 = \text{constant}$, or

$$\cos\theta_1 + \frac{a}{f}\cos\theta_2 - \frac{1}{a}(r - r_2) = \text{constant}. \tag{14}$$

(2) *Radial line segment source outside sphere.* Consider a uniform line source, of strength m per unit length between A_1 and A_3, with $a < f < g$. Then, with $\psi_0 = 0$ at the origin O,

$$\psi_0(r, \theta) = m(r_1 - r_3) - m(f - g).$$

Hence from (4), using (9),

$$\psi_1 = -\frac{r}{a}\psi_0\left(\frac{a^2}{r}, \theta\right) = m(f-g)\frac{r}{a} - m\frac{r}{a}\left(f\frac{r_2}{r} - g\frac{r_4}{r}\right),$$

or rearranging
$$\psi_1 = -\frac{m}{a}(g - f)(r - r_4) + m\frac{f}{a}(r_4 - r_2), \tag{15}$$

so that the perturbation stream function is that for a line sink of strength $m(g-f)/a$ between O and A_4, together with a line source between A_4 and A_2 of strength mf/a per unit length.

The equation of the stream sheets is

$$a(r_1 - r_3) + f(r - r_2) + g(r_4 - r) = \text{constant}. \tag{16}$$

(3) *Circular vortex filament outside sphere centred on its axis.* For a vortex filament, of strength $4\pi\kappa$, located at $\theta = \alpha$, $r = b(> a)$, the Stokes stream function is known (see (2), p. 493) to be

$$\psi_0(r, \theta) = -b\kappa r\sin\theta\sin\alpha\int_0^{2\pi}\frac{\cos\phi}{R}\,d\phi, \tag{17}$$

where R is given by (6), with

$$\cos\chi = \cos\theta\cos\alpha + \sin\theta\sin\alpha\cos\phi. \tag{18}$$

Note that $\psi_0 = 0$ at the origin O.

Then from (4), using (7),

$$\psi_1 = -\frac{r}{a}\psi_0\left(\frac{a^2}{r}, \theta\right) = -b'\kappa' r\sin\theta\sin\alpha\int_0^{2\pi}\frac{\cos\phi}{R'}\,d\phi, \tag{19}$$

where
$$\kappa' = -\kappa\frac{a}{b'} = -\kappa\left(\frac{b}{b'}\right)^{\frac{1}{2}},$$

and so the image system is a similar circular vortex of opposite sign of circulation, of strength $4\pi\kappa'$, located at $\theta = \alpha$, $r = b'$.

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The stream sheets are given by

$$r\sin\theta\int_0^{2\pi}\left\{\kappa\frac{b}{R}+\kappa'\frac{b'}{R'}\right\}\cos\phi\,d\phi=\text{constant}, \tag{20}$$

assuming the vortex is constrained to be fixed.

The next three applications depend on the sphere theorem for the inside of the sphere $r=a$.

(4) *Radial doublet within a sphere.* Consider a doublet of strength μ at A_1 (with $f<a$), with its axis pointing radially outwards. Then, using (10),

$$\psi_0(r,\theta)=-\mu\frac{\sin^2\theta_1}{r_1}=-\mu\frac{r^2}{r_1^3}\sin^2\theta$$

and $\psi_0=O(1/r)$ as $r\to\infty$.

From (4), using (9), the perturbation stream function is

$$\psi_1=-\frac{r}{a}\left\{-\mu\frac{a^4r}{f^3r_2^3}\sin^2\theta\right\},$$

or
$$\psi_1=\mu'\frac{\sin^2\theta_2}{r_2},\quad\text{where}\quad\mu'=\mu\frac{a^3}{f^3}, \tag{21}$$

on using (10), and so the image system is a doublet of strength μ' at A_2 pointing radially inwards.

The equation of the stream sheets is

$$r^2\sin^2\theta\left\{\frac{f^3}{r_1^3}-\frac{a^3}{r_2^3}\right\}=\text{constant}. \tag{22}$$

(5) *Equal source and sink on same diameter within sphere.* Here there are two cases to consider according as the source and sink are on opposite sides of the origin or are on the same side of it.

(i) For a source of strength m at A_1 and an equal sink at A_3, where $0<g<f<a$, we have, using (11),

$$\psi_0(r,\theta)=m(\cos\theta_1-\cos\theta_3)=\frac{m}{r_1}(r\cos\theta-f)-\frac{m}{r_3}(r\cos\theta-g),$$

and so $\psi_0=O(1/r)$ as $r\to\infty$. Then from (4), using (9),

$$\psi_1=-m\frac{r}{a}\left\{\frac{r}{fr_2}\left(\frac{a^2}{r}\cos\theta-f\right)-\frac{r}{gr_4}\left(\frac{a^2}{r}\cos\theta-g\right)\right\},$$

or, from (11) and (12),

$$\psi_1=\frac{m}{agr_4}\{a^2(r_4\cos\theta_4+g')-g(r_4^2+g'^2+2r_4g'\cos\theta_4)\}$$
$$-\frac{m}{afr_2}\{a^2(r_2\cos\theta_2+f')-f(r_2^2+f'^2+2r_2f'\cos\theta_2)\},$$

whence
$$\psi_1=\frac{m}{a}(r_2-r_4)+\frac{ma}{f}\cos\theta_2-\frac{ma}{g}\cos\theta_4. \tag{23}$$

Hence the image system is a source of strength ma/f at A_2 and a sink of strength ma/g at A_4, together with a line source of strength m/a per unit length between A_2 and A_4.

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The equation of the stream sheets is

$$\cos\theta_1 - \cos\theta_3 + \frac{a}{f}\cos\theta_2 - \frac{a}{g}\cos\theta_4 + \frac{1}{a}(r_2 - r_4) = \text{constant}. \tag{24}$$

(ii) For a source of strength m at A_1 and an equal sink at A_5 on opposite sides of the centre O of the sphere $\psi_0(r,\theta) = m(\cos\theta_1 - \cos\theta_5)$,

which leads in similar fashion to

$$\psi_1 = \frac{m}{a}(r_2 + r_6) + \frac{ma}{f}\cos\theta_2 - \frac{ma}{g}\cos\theta_6, \tag{25}$$

so here the image system is a source of strength ma/f at A_2 and a sink of strength ma/g at A_6, together with a line source from A_2 to infinity on $\theta = 0$ and a line sink from A_6 to infinity on $\theta = \pi$, each of strength m/a per unit length.

The equation of the stream sheets is

$$\cos\theta_1 - \cos\theta_5 + \frac{a}{f}\cos\theta_2 - \frac{a}{g}\cos\theta_6 + \frac{1}{a}(r_2 + r_6) = \text{constant}. \tag{26}$$

(6) *Source and radial line sink on same diameter inside sphere.* Consider a line sink from A_5 to the origin O, of strength m/g per unit length together with a source of strength m at A_1. Then

$$\psi_0(r,\theta) = m\cos\theta_1 - \frac{m}{g}(r_5 - r) = \frac{m}{r_1}(r\cos\theta - f) - \frac{m}{g}(r_5 - r),$$

on using (11), from which it can be seen that $\psi_0 = O(1/r)$ at infinity. From (4), using (9),

$$\psi_1 = -m\frac{r}{a}\left\{\frac{r}{fr_2}\left(\frac{a^2}{r}\cos\theta - f\right) - \frac{1}{g}\left(g\frac{r_6}{r} - \frac{a^2}{r}\right)\right\},$$

or

$$\psi_1 = \frac{m}{a}(r_2 + r_6) + \frac{ma}{f}\cos\theta_2, \tag{27}$$

using (11) and (12), and dropping a constant term. This is the stream function for a line source from A_2 to infinity on $\theta = 0$ and a line sink from A_6 to infinity on $\theta = \pi$, each of strength m/a per unit length, together with a source of strength ma/f at A_2.

The stream sheets are given by

$$\cos\theta_1 + \frac{a}{f}\cos\theta_2 - \frac{1}{g}(r_5 - r) + \frac{1}{a}(r_2 + r_6) = \text{constant}. \tag{28}$$

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