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The singularities of $\mathscr{H}$-space†

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1. *Introduction*

The non-linear graviton construction of Penrose (9) and the $\mathscr{H}$-space construction of Newman (6) are two complementary techniques for constructing complex four dimensional space-times with quadratic metric and anti-self-dual curvature tensor.

In the former, the space-time is the space of holomorphic sections of a complex fibre space obtained by deforming part of flat twistor space. In the latter the space-time is the space of regular solutions of a differential equation, the good cut equation.

Pathologies arise in the non-linear graviton construction when the normal bundle of a holomorphic section changes. This is reflected in the $\mathscr{H}$-space construction by a change in the character of the solutions of the linearized good cut equation, the Newman equation.

In this paper, we begin with a review of the relevant parts of the Penrose construction and a discussion of the possible pathologies in terms of holomorphic geometry. Then we review the Newman construction and investigate the same pathologies in terms of the existence of the solutions to the generalized Newman equation.

While the conclusions from both viewpoints are necessarily the same, one obtains a greater understanding of the occurrence of singularities from the interplay between the holomorphic approach, and the general theorems of Kodaira and Douady, and the non-holomorphic approach which is based solely on the existence theory of elliptic equations.

2. *The deformed twistor space*

In this section, we review Penrose's notion of a deformed twistor space (9). We begin by recalling the definition of flat twistor space $\mathbb{T}$ (2, 8). $\mathbb{T}$ is a four complex dimensional vector space with the zero vector removed and regarded as a complex manifold. It is usually given in terms of coordinates $(\omega^A, \pi_{A'}) = (\omega^0, \omega^1, \pi_0, \pi_{1'})$ and is fibred over $\mathbb{C}_{A'}$, the two dimensional complex vector space of $\pi_{A'}$. $\mathbb{T}$ has the following structures on it:

(i) A projection p: $\mathbb{T} \to \mathbb{C}_{A'}$

$$(\omega^A, \pi_{A'}) \to \pi_{A'}.$$

The base space $\mathbb{C}_{A'}$ has a choice of two form $\epsilon^{A'B'}$ on it. In coordinates,

$$\epsilon^{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

This defines a two-form $\tau = \frac{1}{2}\epsilon^{A'B'} d\pi_{A'} \wedge d\pi_{B'}$ on $\mathbb{C}_{A'}$.

(ii) An Euler vector field or homogeneity operator

$$\Upsilon = \omega^A \frac{\partial}{\partial \omega^A} + \pi_{A'} \frac{\partial}{\partial \pi_{A'}}.$$

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(iii) A Poisson bracket structure

$$\hat{\mu} = \epsilon^{AB} \frac{\partial}{\partial \omega^A} \otimes \frac{\partial}{\partial \omega^B},$$

or equivalently a non-degenerate two form

$$\mu = \tfrac{1}{2} \epsilon_{AB} \, d\omega^A \wedge d\omega^B,$$

on the fibres of p.

There is then a four parameter family of holomorphic sections of p homogeneous of degree one with respect to Υ. In the usual coordinates these are given by

$$\omega^A = i x^{AA'} \pi_{A'}, \tag{2.1}$$

where $x^{AA'}$ are the four parameters.

This family is identified with complexified Minkowski space and the Minkowski metric is recovered from μ and τ (8, 9).

As remarked in (2) this structure is rigid but by removing the origin from $\mathbb{C}_{A'}$, and the fibre above it from $\mathbb{T}$, a space is obtained which admits complex deformations which preserve (i), (ii) and (iii). As long as there is still a suitable four parameter family of sections of p, the construction of the metric can be repeated and the result is a manifold $\mathscr{M}$ with quadratic metric with anti-self-dual curvature tensor (9).

We suppose that the deformation is parameterized by t for t in some neighbourhood Γ of the origin in $\mathbb{C}$. The deformed twistor space may be denoted by $\mathscr{T}_t$ with $\mathscr{T}_0 = \mathbb{T}$ and the anti-self-dual space-time by $\mathscr{M}_t$. There is then a theorem of Kodaira (5) which asserts that an analogue $\mathscr{L}_t$ in $\mathscr{T}_t$ of a particular section $\mathscr{L}_0$ in $\mathscr{T}_0$ exists for small enough t provided

$$H^1(\mathscr{L}_0, \mathscr{N}_0) = 0, \tag{2·2}$$

where $\mathscr{N}_0$ is the normal bundle to $\mathscr{L}_0$ in $\mathscr{T}_0$. Further, a sufficient condition for $\mathscr{L}_t$ to lie in an n-dimensional family in $\mathscr{T}_t$ is

$$H^1(\mathscr{L}_t, \mathscr{N}_t) = 0, \tag{2·3}$$

and then (5)

$$n = \dim H^0(\mathscr{L}_t, \mathscr{N}_t). \tag{2·4}$$

Thus the existence of the four-parameter family of sections, and therefore also of $\mathscr{M}_t$, is tied to the behaviour of $\mathscr{N}_t$. Singularities of various kinds arise in $\mathscr{M}_t$ when the normal bundles $\mathscr{N}_t$ change for varying t. It is these pathologies that we wish to investigate here.

It follows from the work of Douady (1) that the space of compact holomorphic sub-manifolds of a complex manifold forms an analytic space. Since each $\mathscr{M}_t$ is a space of sub-manifolds of $\mathscr{T}_t$ of a particular homology class, this means that each $\mathscr{M}_t$ is an analytic space. In a later section we shall show this directly but this already gives us information about the possible singularities of $\mathscr{M}_t$.

We look first at the normal bundles in the case of vanishing deformation. For a particular fixed section $\mathscr{L}_0(x)$ in $\mathscr{T}_0$ defined by $x^{AA'}$, we see from (2·1) that a section of the normal bundle $\mathscr{N}_0(x)$ is defined by a pair of equations

$$V^A = i V^{AA'} \pi_{A'}, \tag{2·5}$$

where V^A are the fibre coordinates.

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This is a pair of linear functions in $\pi_{A'}$, so

$$\mathscr{N}_0(x) \cong \mathscr{O}(1) \oplus \mathscr{O}(1), \tag{2·6}$$

where $\mathscr{L}_0(x)$ is identified with $\mathbb{P}_1(\mathbb{C})$. Thus $H^1(\mathscr{L}_0, \mathscr{N}_0) = 0$

$$n = \dim H^0(\mathscr{L}_0, \mathscr{N}_0) = 4.$$

As expected there is a four parameter family of sections of p in $\mathscr{T}_0$ and any one of them survives a small deformation.

Further, (2·6) defines a stable bundle so that $\dim H^0(\mathscr{L}_t, \mathscr{N}_t)$ and $\dim H^1(\mathscr{L}_t, \mathscr{N}_t)$ are constant for t in a neighbourhood of zero. Thus conditions (2·3) and (2·4) are satisfied and a whole four parameter family of sections $\mathscr{L}_t$ exists in $\mathscr{T}_t$. (Note that this does not mean that there is a map from $\mathscr{M}_0$ to $\mathscr{M}_t$ identifying points in any canonical way.)

The Chern class of $\mathscr{N}_0$ and so of $\mathscr{N}_t$ is 2. Thus if $\mathscr{N}_t$ changes with t it must jump to one of

$$\mathscr{O}(2) \oplus \mathscr{O}(0), \quad \mathscr{O}(3) \oplus \mathscr{O}(-1), \quad \ldots, \quad \mathscr{O}(n+1) \oplus \mathscr{O}(1-n),$$

which we shall refer to as one 'jump', two 'jumps', ..., n 'jumps'. The effect of these jumps on $\dim H^0$ and $\dim H^1$ is shown in Table 1.

Table 1. $\dim H^0$ *and* $\dim H^1$ *when* $\mathscr{N} = \mathscr{O}(n+1) \oplus \mathscr{O}(1-n)$

No. of jumps	$\dim H^0$	$\dim H^1$
0	2+2	0+0
1	3+1	0+0
2	4+0	0+0
3	5+0	0+1
4	6+0	0+2

Note that there is no change for the first two jumps so by the Kodaira theorems there is still an appropriate family of sections of p. Thereafter both increase, so that $\mathscr{M}$ will be singular in some way, but the difference is still four. We shall return to this point.

The points of $\mathscr{M}_t$ correspond as stated to the sections of p in $\mathscr{L}_t$. Therefore the tangent space $T_x(\mathscr{M}_t)$ at a point x is identified with the space of sections of $\mathscr{N}_t(x)$ provided this space is four dimensional, i.e. provided $\dim H^0 = 4$. If this dimension becomes greater than four, which is the case for three or more jumps, then $H^0(\mathscr{L}_t(x), \mathscr{N}_t(x))$ is identified with the Zariski tangent space to $\mathscr{M}_t$ at x and a tangent cone can be defined. This is done in Section 5.

As well as the tangent bundle, $\mathscr{M}_t$ has two spin bundles $\mathscr{S}'_t$ and $\mathscr{S}_t$, respectively primed and unprimed. Because of the global parallelism of primed spinors in an anti-self-dual space-time, $\mathscr{S}'_t$ is trivial and the fibre $\mathscr{S}'_t(x)$ may be identified with the vector space of sections of the bundle $\mathscr{O}(1)$ over $\mathscr{L}_t(x)$. These sections are linear functions $f(\pi_{A'}) = \alpha^{A'}\pi_{A'}$.

In the case of no jumps, the fibre $\mathscr{S}_t(x)$ of the unprimed spin bundle is identified with the vector space of sections of $\mathscr{N}_t(x) \otimes \mathscr{O}(-1)$. Provided this space has dimension equal to two this identification can be retained in the presence of jumps in $\mathscr{N}_t(x)$.

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In the case of no jumps, the Riemannian conformal structure is defined by the isomorphism

$$T_x(\mathscr{M}_t) \cong \mathscr{S}_t(x) \oplus \mathscr{S}'_t(x),$$

or

$$\Gamma(\mathscr{N}_t(x)) \cong \Gamma(\mathscr{O}(1)) \oplus \Gamma(\mathscr{N}_t(x) \otimes \mathscr{O}(-1)). \tag{2·7}$$

In other words, a tangent vector to $\mathscr{M}_t$ is null if it is a product of a primed and an unprimed spinor. Equivalently, a vector is null if the corresponding section of $\mathscr{N}_t(x)$ has a zero. This is a quadratic condition, and the isomorphism (2·7) is true, only in the case of no jumps. For one jump, every section with one zero has two and for two or more jumps every section has a zero.

In summary then we have a picture of various pathologies in $\mathscr{M}_t$ in terms of the number of jumps in the normal bundles of the sections of $\mathscr{T}_t$: one jump results in the conformal structure becoming singular, two jumps makes the unprimed spin bundle singular and three or more makes the manifold singular.

In the remainder of this paper we shall use the $\mathscr{H}$-space construction to see how these singularities arise in the solution of the good cut equation.

3. *The $\mathscr{H}$-space construction*

The $\mathscr{H}$-space construction of Newman (6) is a technique for producing an anti-self-dual space-time as the space of solutions of a differential equation, the good cut equation. The construction is closely related to the non-linear graviton construction of Penrose with the good cut equation playing the role of Cauchy–Riemann equations: the local solutions of the good cut equation define a deformed twistor space $\mathscr{T}$ and the global solutions define the holomorphic sections of $\mathscr{T}$ which in turn are the points of the anti-self-dual space-time $\mathscr{M}$ (2).

In this section we study the good cut equation and its linearization, the Newman equation. We show how the results of the previous section, obtained by an appeal to the general theorems of Kodaira, are obtained in a more elementary way by a consideration of the existence of solutions to the Newman equation.

The good cut equation was written in (2) as

$$\frac{\partial \omega^A}{\partial \overline{\pi}_B} = \overline{\pi}^A \overline{\pi}^B \sigma^0(\omega^C \overline{\pi}_C, \overline{\pi}_C, \pi_{C'}) \tag{3·1}$$

where ω^A is a function of $\overline{\pi}_A$ and $\pi_{A'}$ and ω^A and σ^0 have homogeneities:

$$\omega^A(\overline{\lambda}\overline{\pi}_B, \lambda\pi_{B'}) = \lambda\omega^A(\overline{\pi}_B, \pi_{B'}),$$

$$\sigma^0(\lambda\overline{\lambda}\omega^A\overline{\pi}_A, \overline{\lambda}\overline{\pi}_C, \lambda\pi_{C'}) = \overline{\lambda}^{-3}\lambda\sigma^0(\omega^A\overline{\pi}_A, \overline{\pi}_C, \pi_{C'}). \tag{3.2}$$

The good cut function $Z(\overline{\pi}_A, \pi_{A'})$ is defined by

$$Z = \overline{\pi}_A \omega^A$$

so that, as shown in (2), (3·1) is just

$$\eth^2 Z = \sigma^0(Z, \overline{\pi}, \pi). \tag{3·3}$$

Note that Z has homogeneity 1 in each of $\overline{\pi}_A$ and $\pi_{A'}$ and so has spin weight zero and conformal weight one (2).

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Written back in terms of the stereographic coordinate ζ on the sphere, (3·3) becomes

$$\eth^2 Z \equiv \frac{\partial}{\partial \zeta}\left((1+\zeta\tilde{\zeta})^2 \frac{\partial Z}{\partial \zeta}\right) = \sigma^0(Z, \zeta, \tilde{\zeta}). \tag{3.4}$$

We assume for simplicity that σ^0 is linear in the deformation parameter t of Section 2:

$$\sigma^0 = t\Sigma,$$

where Σ is independent of t, but it is convenient to omit explicit appearances of t.

We may linearize (3·4) about a particular solution Z_0 by writing $Z = Z_0 + \epsilon V$, whence

$$\eth^2 V = \dot{\sigma}^0(Z_0, \zeta, \tilde{\zeta})\, V = \dot{\sigma}^0 V, \tag{3·5}$$

where

$$\dot{\sigma}^0(u, \zeta, \tilde{\zeta}) = \frac{\partial \sigma^0}{\partial u}.$$

We shall omit the arguments in $\dot{\sigma}^0$ for brevity and write (3·5) as

$$L_0 V \equiv (\eth^2 - \dot{\sigma}^0)\, V = 0, \tag{3·6}$$

which is the Newman equation.

Note that the operator L_0 maps spin weight zero to spin weight two functions. We may extend the definition of the operator to

$$L_s \equiv \eth^2 - \dot{\sigma}^0 : \mathscr{B}(s) \to \mathscr{B}(s+2), \tag{3·7}$$

where $\mathscr{B}(s)$ is the space of real analytic spin weight s functions on the sphere with L^2 norm, and consider a generalized Newman equation:

$$L_s V = 0.$$

By (2), each L_s is elliptic, so that, in particular if $V \in \ker L_s$, V is real analytic. If the particular solution Z_0 of (3·4) is identified with a particular section $\mathscr{L}$ in $\mathscr{T}$, then the solutions of (3·6) may be identified with the sections of the normal bundle $\mathscr{N}$ to $\mathscr{L}$ in $\mathscr{T}$, i.e.

$$\ker L_0 \cong H^0(\mathscr{L}, \mathscr{N}). \tag{3·8}$$

In an appendix we show explicitly how the rank two vector bundle $\mathscr{N}$ may be constructed from solutions of (3·6). Here we remark that, because of the identification (3·8) and the discussion of Section 2, if $\dim \ker L_0 = 4$ then the good cut equation has a four parameter family of solutions in the vicinity of Z_0 (4). We write these as $Z(z^a, \zeta, \tilde{\zeta})$ where z^a are the four parameters. Substituting Z for Z_0 in (3·5) gives a family of elliptic equations parameterized by z^a. By general theory, the solutions define a vector bundle over the parameter space provided the rank is constant, and this is the tangent bundle to $\mathscr{M}$.

If the rank jumps then by (3·8) $\ker L_0$ is the Zariski tangent space to $\mathscr{M}$ and we expect to find a tangent cone in $\ker L_0$. The codimension of the tangent cone turns out to be $\dim H^1(\mathscr{L}, \mathscr{N})$ which gives significance to the observation in Section 2 that $\dim H^0 - \dim H^1 = 4$.

Another vector bundle on $\mathscr{M}$ is defined by $\ker L_{-\frac{1}{2}}$ that is, by the solutions of

$$L_{-\frac{1}{2}} V = 0,$$

again provided the rank is constant. This is the unprimed spin bundle.

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Thus the pathologies discussed in Section 2 are related to the dimensions of the kernels of the various L_s. This question is taken up in Section 4. In Section 5, we investigate the codimension in $\mathscr{M}$ of the sets of points on which these various pathologies arise.

Some concluding remarks are collected in Section 6.

Finally some conventions: we write the subscript s on $\eth_s$ as a reminder of the spin weight involved but we always write just $\eth^2$ for $\eth_{s+1} \circ \eth_s$ regardless of s. The kernel of $\eth_s: \mathscr{B}(s) \to \mathscr{B}(s+1)$ consists of functions of the form $g(\bar{\zeta})\,(1+\zeta\bar{\zeta})^{-s}$ where g is a polynomial of degree $2s$ in $\bar{\zeta}$ and independent of ζ. A convenient basis for $\ker \eth_s$ consists of the spin weighted spherical harmonics ${}_sY_{sm}(\zeta, \bar{\zeta})$ where $m = -s, -s+1, \ldots, s-1, s$. There are $2s+1$ possibilities for m and in expressions like $a^m {}_sY_{sm}$ there is an implied summation over the allowed m values while s is fixed and not summed.

4. *Solutions of the Newman equation*

In this section we study the distribution of the solutions of the generalized Newman equation:

$$L_s V \equiv (\eth^2 - \dot{\sigma}^0)\, V = 0, \quad V \in \mathscr{B}(s). \tag{4·1}$$

Specifically, by a series of lemmas, we obtain information about the dimension of $\ker L_s$ for each s. Defining $N(s) = \dim \ker L_s$ we have first:

LEMMA 1.

$$N(s) = N(-s-2) + 4(s+1) \quad (s \geqslant -1). \tag{4·2}$$

Proof. This is an application of the Atiyah–Singer Index Theorem (7).

The adjoint $L_s^\dagger$ of L_s is

$$L_s^\dagger \equiv \bar{\eth}^2 - \dot{\bar{\sigma}}^0: \mathscr{B}(s+2) \to \mathscr{B}(s). \tag{4·3}$$

The elements of $\ker L_s^\dagger$ are therefore the complex conjugates of the elements of $\ker L_{-s-2}$. Thus the index of L_s is

$$\begin{aligned} \operatorname{ind} L_s &= \dim \ker L_s - \dim \ker L_s^\dagger \\ &= N(s) - N(-s-2). \end{aligned} \tag{4·4}$$

By the Index Theorem, we may calculate $\operatorname{ind} L_s$ by ignoring the terms in $\dot{\sigma}^0$ so that from (4·4)

$$\operatorname{ind} L_s = \dim \ker (\eth^2: \mathscr{B}(s) \to \mathscr{B}(s+2)) - \dim \ker (\eth^2: \mathscr{B}(-s-2) \to \mathscr{B}(-s)). \tag{4·5}$$

Now from (2)

$$\begin{aligned} \dim \ker (\eth^2: \mathscr{B}(s) \to \mathscr{B}(s+2)) &= 2s+1+2(s+1)+1 \quad (s \geqslant -\tfrac{1}{2}), \\ &= 0 \qquad\qquad\qquad\qquad\quad (s \leqslant -1), \end{aligned}$$

so by (4·5)

$$\begin{aligned} \operatorname{ind} L(s) &= 4(s+1) \qquad (s \geqslant -1), \\ &= -4(s+1) \quad (s \leqslant -1), \end{aligned}$$

and from (4·4) we obtain (4·2). In particular, (4·2) implies that $N(-\frac{1}{2}) \geqslant 2$ and $N(0) \geqslant 4$. ∎

We next observe that, given $U \in \ker L_s$, we can generate solutions $V \in \ker L_{s'}$ where $s' = s+n$ by multiplying U by elements of $\ker \eth_n$, i.e. if A has spin weight n and $\eth_n A = 0$ then $V = AU$ is in $\ker L_{s'}$. The dimension of $\ker \eth_n$ is $2n+1$ so in this way we

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obtain $2n+1$ linearly independent solutions V from one solution U. We shall call this process of generating higher spin weight solutions from lower ones 'trivial generation'.

As a result of this observation, we have the following:

LEMMA 2. *There must be a lowest spin weight $-s_m$, $s_m \geqslant \frac{1}{2}$, below which there are no solutions, i.e. $N(s) = 0$, $s < -s_m$. Further, there can be only one linearly independent solution at spin weight $-s_m$, $N(-s_m) = 1$, unless $s_m = \frac{1}{2}$ when there will be two.*

Proof. If there were solutions at arbitrarily negative spin weight, then trivial generation would give infinitely many spin weight 0 solutions. Thus s_m is well-defined, but $N(-\frac{1}{2}) \geqslant 2$ so that $s_m \geqslant \frac{1}{2}$.

Now consider (4·2) with $s = s_m - 1$:

$$N(s_m - 1) = N(-s_m - 1) + 4s_m = 4s_m. \tag{4·6}$$

If there were two linearly independent solutions U, V with spin weight $-s_m$ then we could trivially generate solutions at spin weight $s_m - 1$ by multiplying by elements of $\ker \eth_n$ with $n = 2s_m - 1$. This kernel has dimension $2n+1 = 4s_m - 1$ so that trivial operation from U and V gives

$$N(s_m - 1) \geqslant 2(4s_m - 1), \tag{4·7}$$

provided no solution is trivially generated by both U and V. This must be the case, for suppose W is trivially generated by both:

$$W = AU = BV, A, B \in \ker \eth_n, \tag{4·8}$$

then A and B are of the form

$$A = a(\bar{\zeta})(1+\zeta\bar{\zeta})^{-n}, \quad B = b(\bar{\zeta})(1+\zeta\bar{\zeta})^{-n},$$

where a and b are polynomials in $\bar{\zeta}$ of degree $2n$. We may cancel any common factors of a and b from A and B in (4·8), removing a factor of $(1+\zeta\bar{\zeta})^{-\frac{1}{2}}$ with each, to leave

$$A'U = B'V, \quad A', \quad B' \in \ker \eth_m,$$

for some $m \leqslant n$. Therefore

$$Y = B'^{-1}U = A'^{-1}V,$$

is a solution with spin weight $-s_m - m$ which contradicts the definition of s_m.

Returning to (4·7) and (4·6), we would therefore have

$$2(4s_m - 1) \leqslant 4s_m,$$

so

$$s_m \leqslant \tfrac{1}{2}.$$

Conversely if $s_m = \frac{1}{2}$ then $N(-\frac{3}{2}) = 0$ and we have at once from (4·2) that $N(-\frac{1}{2}) = 2$. ∎

The Wronskian of two solutions U and V of spin weight m and n is

$$W(U, V) \equiv U\eth V - V\eth U. \tag{4·9}$$

This has spin weight $m+n+1$ and satisfies

$$\eth W = 0. \tag{4.10}$$

Now if $m+n+1 < 0$, the only solution of (4·10) is $W = 0$ and the Wronskian must vanish. If $m+n+1 = 0$ then the Wronskian is constant. Clearly if U and V are both trivially generated by the same solution, then their Wronskian vanishes. The converse is also true.

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LEMMA 3. *If* $U \in \ker L_s$ *and* $V \in \ker L_{s'}$ *and* $W(U, V)$ *is zero then there exists some* $Y \in \ker L_{s''}$ *with*

$$U = AY, \quad V = BY$$

for $A \in \ker \eth_n$, $B \in \ker \eth_m$, $n = s - s''$, $m = s' - s''$.

Proof. We first consider the case when $s = s'$. Then

$$W(U, V) = (1 + \zeta\bar{\zeta})\left(U\frac{\partial V}{\partial \zeta} - V\frac{\partial U}{\partial \zeta}\right).$$

If this is zero, then where neither U nor V is zero,

$$\frac{U}{V} = f(\bar{\zeta}), \tag{4.11}$$

for some function f holomorphic in $\bar{\zeta}$.

We now consider the zeros of U and V. Since U and V are real analytic functions, they can be zero only at isolated points or on closed curves. If U is zero on an arc in the support of V, then by (4·11), $f(\bar{\zeta})$ is zero on that arc and therefore is identically zero in support of V. Then U would be identically zero. Inverting (4·11), the same will be true for V so aside from their common zeroes U and V have only isolated zeroes. Thus $f(\bar{\zeta})$ is meromorphic away from the common zeroes of U and V. If any of the common zeroes of U and V are isolated points, f is meromorphic there too. There remains the possibility that U and V are both zero on a collection of closed curves. If this is the case, then they must vanish to the same order on each arc. For if not, suppose U vanishes to a higher order then f would be zero on an arc and again would vanish identically in an open neighbourhood so that U would vanish identically. Thus if U and V both vanish on an arc, then f can be extended to be a constant on the arc.

We conclude that $f(\bar{\zeta})$ in (4·11) is meromorphic on the $\bar{\zeta}$-sphere and so is a ratio of polynomials in $\bar{\zeta}$ with equal numbers, say n, of zeroes and poles. Thus $f(\bar{\zeta})$ is a ratio of two functions in $\ker \eth_n$, say $f(\bar{\zeta}) = AB^{-1}$, and

$$Y = \frac{U}{A} = \frac{V}{B} \in \ker L_{s'}, s' = s - n.$$

Now if U and V have different spin weights s, s' the condition that the Wronskian vanish is

$$(1 + \zeta\bar{\zeta})\left(U\frac{\partial V}{\partial \zeta} - V\frac{\partial U}{\partial \zeta}\right) - (s - s')\bar{\zeta}UV = 0.$$

Instead of (4·11) we find

$$\frac{U}{V} = f(\bar{\zeta})(1 + \zeta\bar{\zeta})^{-(s'-s)}. \tag{4.12}$$

Arguing as before, we conclude that $f(\bar{\zeta})$ is rational but now, because of the extra factor, the right hand side of (4·12) is a ratio of terms in $\ker \eth_n$ and $\ker \eth_m$ where $s - n = s' - m = s''$ for some s''. ∎

Thus the Wronskian of two solutions vanishes iff the two solutions are trivially generated from a third. If $U \in \ker L_m$ and $V \in \ker L_{-m-1}$ are not trivially generated from

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a third, then their Wronskian is a non-zero constant and we can normalize U and V so that the constant is unity. A pair of solutions normalized in this way will be called a 'fundamental pair'.

We may now proceed to classify the solutions of the generalized Newman equation by the value of s_m:

(i) $s_m = \frac{1}{2}$. This is the generic case. There are two linearly independent solutions at $s = -\frac{1}{2}$, which we can choose to have Wronskian equal to one.

At spin weight s, each of this fundamental pair trivially generates $2(s+\frac{1}{2})+1 = 2(s+1)$ solutions giving $4(s+1)$ together. There can be no duplication by a repetition of the argument of Lemma 2. Now $N(s) = 0$ if $s \leqslant -1$ so by (4·2)

$$N(s) = 4(s+1) \quad (s \geqslant -1).$$

Thus all solutions are trivially generated by the fundamental pair at $s = -\frac{1}{2}$.

(ii) $s_m = 1$. Now

$$N(-1) = 1$$

$$N(s) = 0 \quad (s < -1),$$

so by (4·2)

$$N(s) = 4(s+1) \quad (s > -1). \tag{4·13}$$

The unique solution U at $s = -1$ trivially generates $2(s+1)+1 = 2s+3$ solutions at the general s. At $s = -\frac{1}{2}$ this agrees with (4·13), namely $N(-\frac{1}{2}) = 2$, but at $s = 0$ this gives 3, while (4·13) requires $N(0) = 4$. There must therefore appear an extra solution V at $s = 0$ which is not trivially generated by U. Thereafter V will trivially generate $2s+1$ solutions at $s \geqslant 0$ and jointly U and V trivially generate

$$2s+3+2s+1 = 4(s+1)$$

solutions which by (4·13) is all solutions.

We may normalize U and V to be a fundamental pair.

(iii) $s_m = \frac{3}{2}$. Now

$$N(-\tfrac{3}{2}) = 1$$

$$N(s) = 0 \quad (s < -\tfrac{3}{2}),$$

and by (4·4)

$$N(-\tfrac{1}{2}) = 3$$

$$N(s) = 4(s+1) \quad (s \geqslant 0). \tag{4·14}$$

The solution U at $s = -\frac{3}{2}$ trivially generates $2s+4$ solutions at the general $s \geqslant -\frac{3}{2}$. By (4·14) this accounts for all the solutions at $s = -\frac{1}{2}$ and $s = 0$ but now an extra solution V must appear at $s = \frac{1}{2}$. V trivially generates $2(s-\frac{1}{2})+1 = 2s$ at the general $s \geqslant \frac{1}{2}$ and again U and V jointly generate $4(s+1)$ solutions at $s \geqslant 0$ which by (4·14) is all solutions. Again, U and V give a fundamental pair.

(iv) General s_m. The pattern is now clear. The unique solution U at $s = -s_m$ trivially generates $2(s+s_m)+1$ solutions. For s in the range $-s_m \leqslant s < s_m - 1$ this gives all solutions. An extra solution V appears at $s = s_m - 1$ and trivially generates $2(s-s_m+1)+1$. Thereafter, U and V together generate $2(s+s_m)+1+2(s-s_m+1)+1 = 4(s+1)$, which is all solutions. U and V can be normalized to a fundamental pair.

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These results are illustrated in table 2 which should be compared with table 1.

Table 2. *$N(s)$ for various s_m; compare $N(0)$ with* $\dim H^0$ *and $N(-2)$ with* $\dim H^1$ *in Table* 1

s_m	$N(-2)$	$N(-\frac{3}{2})$	$N(-1)$	$N(-\frac{1}{2})$	$N(0)$	$N(\frac{1}{2})$	$N(1)$
$\frac{1}{2}$	0	0	0	1+1	2+2	3+3	4+4
1	0	0	1	2	3+1	4+2	5+3
$\frac{3}{2}$	0	1	2	3	4	5+1	6+2
2	1	2	3	4	5	6	7+1
$\frac{5}{2}$	2	3	4	5	6	7	8

The relation between these two tables is taken up in Section 6.

5. *The size of the singularity set*

Singularities of the $\mathscr{H}$-space construction arise when the Newman equation has solutions with $s_m > \frac{1}{2}$. In this section we investigate the codimension in the $\mathscr{H}$-space of the various singular sets, justify the identification of $\ker L_0$ with the Zariski tangent space of the $\mathscr{H}$-space and identify the tangent cone. To this end we look at higher order terms in the good cut equation (3·4) expanded about a particular good cut $Z_0(\zeta, \bar{\zeta})$ corresponding to a point x in $\mathscr{M}$.

We suppose

$$Z = Z_0 + \epsilon Z_1 + \tfrac{1}{2}\epsilon^2 Z_2 + \dots \tag{5·1}$$

and substitute into (3·4) to obtain

$$(\eth^2 - \dot{\sigma}^0) Z_1 = 0 \tag{5·2a}$$

$$(\eth^2 - \dot{\sigma}^0) Z_2 = \ddot{\sigma}^0 (Z_1)^2 \tag{5·2b}$$

$$(\eth^2 - \dot{\sigma}^0) Z_3 = 3\ddot{\sigma}^0 Z_1 Z_2 + \dddot{\sigma}^0 (Z_1)^3 \tag{5·2c}$$

$$(\eth^2 - \dot{\sigma}^0) Z_4 = 4\ddot{\sigma}^0 Z_1 Z_3 + 3\ddot{\sigma}^0 (Z_2)^2 + 6\dddot{\sigma}^0 (Z_1)^2 Z_2 + \ddddot{\sigma}^0 (Z_1)^4 \quad \text{etc.} \tag{5·2d}$$

At each order, the equation on Z_n has the form of the Newman equation with a source term on the right-hand side which involves $Z_1, \dots, Z_{n-1}$. The basic theorem which we shall use is that an elliptic equation

$$LU = V,$$

has a solution iff the right-hand side V is orthogonal to the kernel of the adjoint $L^\dagger$(3). For the Newman equation, $\ker L_s^\dagger$ changes with increasing s_m so we again classify according to s_m:

(i) $s_m = \frac{1}{2}$. Here $\ker L_0^\dagger$ is empty, there are no conditions to be satisfied, and (5·2$b, c, d, \dots$) can be solved at each stage for an initial choice of Z_1 in $\ker L_0$. Leaving aside questions of the convergence of (5·1), there are no obstructions to solution and $\ker L_0 \cong T_x(\mathscr{M})$.

(ii) $s_m = 1$. Again $\ker L_0^\dagger$ is empty and $\ker L_0 \cong T_x(\mathscr{M})$. There is a fundamental pair

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of solutions U, V with spin weights -1, 0 respectively and the general spin weight zero solution X is trivially generated by them:

$$X = AU + bV, \quad b \in \mathbb{C}, \quad A \in \ker \eth_1, \tag{5·3}$$

so

$$A = a^m{}_1 Y_{1m} \quad (m = -1, 0, 1).$$

We wish to know in which directions defined by X we can move in $\mathscr{M}$ and retain the existence of a spin weight -1 solution. Suppose $U + \epsilon K$ is a spin weight -1 solution at $Z_0 + \epsilon X$, then we require

$$\eth^2(U + \epsilon K) = \sigma^0(Z_0 + \epsilon X)(U + \epsilon K),$$

so

$$\eth^2 K = \dot{\sigma}^0 K + \ddot{\sigma}^0 UX, \tag{5·4}$$

(As in (5·2), when $\dot{\sigma}^0$, $\ddot{\sigma}^0$ etc. are evaluated at Z_0 we omit the argument.)

Now (5·4) has the form

$$L_{-1} K = \ddot{\sigma}^0 UX,$$

which has a solution K iff the right hand side is orthogonal to the kernel of $L^\dagger_{-1}$, i.e. iff

$$\int \ddot{\sigma}^0 UX\overline{Y} = 0 \quad \text{for all} \quad Y \in \ker L^\dagger_{-1}. \tag{5·5}$$

From the remarks following (4·2), if $Y \in \ker L^\dagger_{-1}$ then $\overline{Y} \in \ker L_{-1}$. But L_{-1} consists only of multiples of U so (5·5) is just

$$\int \ddot{\sigma}^0 U^2 X = 0.$$

Generically this is a linear condition on X, i.e. on the quadruple (a^m, b) in (5·3), so that locally in $\mathscr{M}$, $s_m = 1$ on a hypersurface.

(iii) $s_m = \frac{3}{2}$. Again $\ker L_0 \cong T_x(\mathscr{M})$ but now $\ker L_0$ is trivially generated by the unique solution U with spin weight $-\frac{3}{2}$, i.e. the general spin weight 0 solution X is

$$X = AU$$

$$A = a^m{}_{\frac{3}{2}} Y_{\frac{3}{2}m}, \quad m = -\tfrac{3}{2}, -\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}. \tag{5·6}$$

We wish to know first in which directions X in $\mathscr{M}$ we retain the existence of a spin weight $-\frac{3}{2}$ solution. If $U + \epsilon K$ is such a solution at $Z_0 + \epsilon X$ then following the derivation of (5·4),

$$L_{-\frac{3}{2}} K = \ddot{\sigma}^0 UX. \tag{5·7}$$

To solve (5·7), $\ddot{\sigma}^0 UX$ must be orthogonal to $\ker L^\dagger_{-\frac{3}{2}}$ which is the complex conjugate of $\ker L_{-\frac{1}{2}}$. This in turn is trivially generated by U, i.e. if $Y \in \ker L_{-\frac{1}{2}}$ then

$$Y = BU$$

$$B = b^m{}_1 Y_{1m} \quad (m = -1, 0, 1).$$

Thus (5·7) has a solution iff

$$\int \ddot{\sigma}^0 U^2 BX = 0, \tag{5.8}$$

for all possible B. This is three linear conditions on X, so that locally in $\mathscr{M}$, $s_m = \frac{3}{2}$ on curves.

We may also ask which directions X in $\mathscr{M}$ retain the existence only of a spin weight -1 solution. There are two linearly independent spin weight -1 solutions, say Y_1

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and Y_2, both trivially generated by U. If Y is the general spin weight -1 solution at Z_0, say $Y = C_1 Y_1 + C_2 Y_2$, and if $Y + \epsilon K$ is a spin weight -1 solution at $Z_0 + \epsilon X$ then

$$L_{-1} K = \ddot{o}^0 Y X. \tag{5·9}$$

This has a solution if $\ddot{o}^0 Y X$ is orthogonal to $\ker L_{-1}^\dagger$ which is the complex conjugate of $\ker L_{-1}$ which in turn consists of linear combinations of Y_1 and Y_2. The integrability conditions on (5·9) are therefore

$$\int \ddot{o}^0 Y X Y_i = 0 \quad (i = 1, 2). \tag{5·10}$$

For given X, (5·10) is two conditions on the two coefficients C_1 and C_2 of Y and so is represented by a 2×2 matrix. Solutions exist if the determinant vanishes and this is a quadratic condition on the coefficients a^m in X, (5·6). Comparing (5·10) with (5·8), since X, Y and Y_i are all trivially generated by U, any X satisfying (5·8) automatically satisfies (5·10). Thus locally in $\mathscr{M}$, the curve of points with $s_m = \frac{3}{2}$ is the vertex of a cone with $s_m = 1$.

(iv) $s_m = 2$. Now $N(0) = 5$ and we cannot have the identification $\ker L_0 \cong T_x(\mathscr{M})$. Instead $\ker L_0$ is the Zariski tangent space to $\mathscr{M}$. To find the tangent cone, we consider (5·2b). If U is the spin weight -2 solution, then the general spin weight 0 solution X is

$$X = AU$$

$$A = a^m {}_2Y_{2m} \quad (m = -2, -1, 0, 1, 2). \tag{5·11}$$

We are seeking a power series solution like (5·1):

$$Z = Z_0 + \epsilon X + \tfrac{1}{2}\epsilon^2 Y + \ldots$$

where by (5·2b)

$$L_0 Y = \ddot{o}^0 X^2. \tag{5·12}$$

This has a solution iff $\ddot{o}^0 X^2$ is orthogonal to $\ker L_0^\dagger$ which is the complex conjugate of $\ker L_{-2}$ and this, for the first time, is non-empty. In fact it consists of multiples of U so that the integrability conditions for (5·12) is

$$\int \ddot{o}^0 X^2 U = 0. \tag{5·13}$$

Substituting from (5·11), (5·13) becomes

$$A_{mn}\, a^m a^n = 0, \tag{5·14}$$

where

$$A_{mn} = \int \ddot{o}^0 U^3 \, {}_2Y_{2m} \, {}_2Y_{2n}.$$

Thus in the five dimensional vector space $\ker L_0$ coordinatized by a^n, the tangent cone is given by (5·14).

We may investigate this case further by going to the next order. If X satisfies (5·13), then (5·12) can be solved for $Y = Y_1 + Y_2$ where Y_1 is a particular integral and Y_2 is the complementary function which is trivially generated by U and has five parameters:

$$Y_2 = BU$$

$$B = b^m \, {}_2Y_{2m}. \tag{5·15}$$

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In order to solve (5·2c) for the third order term Z_3 we again have an integrability condition analogous to (5·13), which this time is

$$\int (3\eth^0 X(Y_1+Y_2)+\eth^0 X^3)\, U = 0. \tag{5·16}$$

From (5·14) and (5·15), condition (5·16) takes the form

$$A_{mn}\, a^m b^m + A_{pqr}\, a^p a^q a^r = 0, \tag{5·17}$$

where A_{mn} is as in (5·14).

This is a single condition on the coefficients b^n of Y_2.

Thus the integrability condition for Z_3 reduces the freedom in Z_2 from five parameters to four. This is the pattern at each stage in solving equations (5·2): a five parameter complementary function arises at the n-th order and is reduced to four parameters by the integrability condition at the $(n+1)$-th order. We interpret (5·14), (5·17) and the higher integrability conditions as follows: near a point p with $s_m = 2$, $\mathscr{M}$ is modelled by a singular variety in five dimensions. Choosing coordinates x^m, $m = 1, \ldots, 5$, with $x^m = 0$ at p, the variety is given by an equation

$$F(x^m) \equiv A_{mn}\, x^m x^n + A_{pqr}\, x^p x^q x^r + \ldots = 0. \tag{5·18}$$

Along a curve $x^m(\lambda)$ through p with $\lambda = 0$ at p we find by differentiating (5·18) and setting $\lambda = 0$

$$A_{mn}\, x'^m x'^n = 0, \quad A_{mn}\, x'^m x''^n + A_{pqr}\, x'^p x'^q x'^r = 0 \quad \text{etc.}$$

where $x'^m = \mathrm{d}x^m/d\lambda|_{\lambda=0}$, etc.

Identifying x'^m with a^m and x''^m with b^m, these equations are precisely (5·14) and (5·17).

Leaving aside the question of convergence of the power series (5·1), these arguments serve to prove that the $\mathscr{H}$-space $\mathscr{M}$ is an analytic space and justify the identification of $\ker L_0$ with the Zariski tangent space of $\mathscr{M}$. Thus, for this example, the theorem of Douady (1) and the nature of the singular sets of $\mathscr{M}$ can be understood solely in terms of the existence of solutions to the generalised Newman equation.

To find the codimension of the set of points with $s_m = 2$ we proceed as before. If U is the spin weight -2 solution at Z_0, suppose $U + \epsilon K$ is one at $Z_0 + \epsilon X$. This requires

$$L_{-2} K = \eth^0 X U. \tag{5·19}$$

Since $\ker L^\dagger_{-2}$ is the complex conjugate of $\ker L_0$, the integrability conditions for (5·19) are

$$\int \eth^0 X U Y = 0, \tag{5·20}$$

for all $Y \in \ker L_0$. From (5·11) and (5·14) this means

$$A_{mn}\, a^m = 0.$$

This has no solution for a^m in general and so $s_m = 2$ at isolated points.

(v) $s_m > 2$. In these cases, $\ker L_0$ is trivially generated by the solution U with spin weight $-s_m$ and so has dimension $2s_m + 1$. The number of integrability conditions arising from (5·12) is $\dim \ker L_{-2}$ which is $2(s_m - 2) + 1 = 2s_m - 3$. Thus the Zariski tangent space is $2s_m + 1$ dimensional and the tangent cone is the intersection of $2s_m - 3$ quadratic cones. In particular, the tangent cone is of course always four dimensional.

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6. *Conclusion*

We now pull together what we have said about the jumps in the normal bundle of various sections of deformed twistor space (Section 2) with the existence of solutions to the Newman equation with $s_m > \frac{1}{2}$ (Section 4). The connection is via the explicit construction of a rank two vector bundle of Chern class zero from a pair of fundamental solutions to the Newman equation. This is done in the appendix. If the fundamental solutions have spin weight $-s_m$ and $s_m - 1$ then the vector bundle is $\mathcal{O}(n) \oplus \mathcal{O}(-n)$, where $n = 2s_m - 1$. From the construction, we see that if the Newman equation comes from the good cut equation linearized about a solution Z_0 then the resulting vector bundle is the normal bundle $\mathcal{N}$ of the section $\mathcal{L}$ of deformed twistor space $\mathcal{T}$ corresponding to Z_0 tensored with $\mathcal{O}(-1)$, i.e. $\mathcal{N} \otimes \mathcal{O}(-1)$.

$\mathcal{N}$ itself is therefore $\mathcal{O}(2s_m) \oplus \mathcal{O}(2-2s_m)$ and the number of jumps defined in Section 1 is $n = 2s_m - 1$. This accounts for the relation between Table 1 and 2 and also relates the difference $\dim H^0 - \dim H^1$ to the dimensions of the kernels of the various L_s.

When $s_m = \frac{1}{2}$, the isomorphism (2·7) which defines the conformal structure corresponds to the fact that $\ker L_0$ is trivially generated by $\ker L_{-\frac{1}{2}}$. A null vector tangent to $\mathcal{M}$ is represented in $\ker L_0$ by a simple solution

$$X = AU, \quad U \in \ker L_{-\frac{1}{2}}, \quad A \in \ker \eth_{\frac{1}{2}}.$$

This breaks down when $s_m = 1$ since then $\ker L_0$ is not trivially generated by $\ker L_{-\frac{1}{2}}$.

The unprimed spin bundle $\mathcal{S}$ has fibre isomorphic to $\ker L_{-\frac{1}{2}}$ provided this is two dimensional. From Table 2, this breaks down when $s_m > 1$, so this signals a singularity in $\mathcal{S}$. Finally, as we have said already, the identification $\ker L_0 \cong T_x(\mathcal{M})$ breaks down when $s_m > \frac{3}{2}$.

Explicit examples of the $\mathcal{H}$-space construction are rather rare but in the known class it is possible to find examples of the first type of pathology. In the $\mathcal{H}$-space described in (10) there is a hypersurface with $s_m = 1$, where the normal bundles of the corresponding sections of deformed twistor space have one jump.

To find examples of the other pathologies, one could simply consider the Newman equation

$$\eth^2 V = \dot{\sigma}^0 V \tag{6·1}$$

and try to adjust $\dot{\sigma}^0(\zeta, \bar{\zeta})$ so that solutions existed with $s_m > 1$. That this is a difficult task may be seen by writing (6·1) as

$$\dot{\sigma}^0 = \eth^2 V / V.$$

The problem now is to choose V with spin weight $s = s_0 < -1$ such that the right hand side is a regular spin weight 2 function. Now the numerator has spin weight $s_0 + 2$ and so has at least $2|s_0 + 2|$ zeroes while the denominator has at least $2|s_0|$ zeroes. The problem is to arrange for these to cancel, which is clearly difficult if $s_0 < -1$. Another way to see this is to attempt to factorize (6·1)

$$(\eth^2 - \dot{\sigma}^0) V = (\eth + \psi)(\eth - \psi) V, \tag{6·2}$$

which requires

$$\eth\psi + \psi^2 = \dot{\sigma}^0. \tag{6·3}$$

If we can solve (6·3) for ψ given $\dot{\sigma}^0$ then (6·2) reduces to solving

$$(\eth + \psi)\chi = 0 \quad (\eth + \psi) V = \chi. \tag{6·4}$$

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However, it is a straightforward matter to show that (6·4) can never have a solution χ with spin weight $s < 0$. (This is a consequence of the fact that the complex structure of $\mathbb{P}_1(\mathbb{C})$ is rigid.) Thus if (6·3) can be solved, then (6·1) can only have $s_m \leqslant 1$. In particular, if (6·3) can always be solved then $s_m > 1$ is impossible. Clearly (6·3) can always be solved for $\dot{\sigma}^0$ small enough in an appropriate sense for set $\dot{\sigma}^0 = t\Sigma$, $\psi = t\phi$ then (6·3) becomes

$$\eth\phi + t\phi^2 = \Sigma. \tag{6·5}$$

By standard theory, this can be solved for t in a neighbourhood of the origin in $\mathbb{C}$, so that (6·3) can be solved for small enough $\dot{\sigma}^0$. Thus for small enough $\dot{\sigma}^0$, the pathologies with $s_m > 1$ cannot occur.

7. *Appendix*

We wish to give an explicit construction of a rank 2 vector bundle from a fundamental pair of solutions to the generalized Newman equation. We begin with a discussion of the vector bundles $B(n) = \mathscr{O}(n) \oplus \mathscr{O}(-n)$ on the Riemann sphere of the spinor $\pi_{A'}$. These may be defined by their transition matrices:

$$G \equiv G_B^A = \begin{pmatrix} \bar{\zeta}^n & 0 \\ 0 & \bar{\zeta}^{-n} \end{pmatrix}, \tag{A 1}$$

where $A, B = 0, 1$ and $\bar{\zeta} = \pi_{0'}/\pi_{1'}$.

(G is defined on the overlap $\mathscr{U} \cap \hat{\mathscr{U}}$ where $\mathscr{U}$ and $\hat{\mathscr{U}}$ give the usual two set cover of the Riemann sphere:

$$\mathscr{U} = \{\pi_{A'} | \pi_{1'} \neq 0\}, \quad \hat{\mathscr{U}} = \{\pi_{A'} | \pi_{0'} \neq 0\}).$$

Since these bundles have Chern class zero, they are topologically trivial and it is possible to find matrices g, $\hat{g}$ which are C^∞ functions of ζ and $\bar{\zeta}$, well defined on $\mathscr{U}$ and $\hat{\mathscr{U}}$ respectively such that

$$G = \hat{g}^{-1} g, \tag{A 2}$$

i.e. $\hat{g}_B^A G_C^B = g_C^A$.

Since G is holomorphic in $\pi_{A'}$, we have from (A 2)

$$\bar{\partial}\hat{g}\,\hat{g}^{-1} = \bar{\partial} g\, g^{-1}. \tag{A 3}$$

Thus there is a globally defined matrix-valued one-form Σ with

$$\bar{\partial} g = \Sigma g, \quad \bar{\partial}\hat{g} = \Sigma\hat{g}. \tag{A 4}$$

In terms of the spinor $\bar{\pi}_A$, since g and $\hat{g}$ are homogeneous of degree zero in $\bar{\pi}_A$, we may write (A 4) as

$$\frac{\partial}{\partial \bar{\pi}_A} g_C^B = \bar{\pi}^A \Sigma_D^B g_C^D. \tag{A 5}$$

If $(S^A, \hat{S}^A)$ represents a holomorphic section of $B(n)$, so that

$$\hat{S}^A = G_B^A S^B \quad \text{on} \quad \mathscr{U} \cap \hat{\mathscr{U}},$$

then

$$V^A = g_B^A S^B = \hat{g}_B^A \hat{S}^B,$$

defines a pair of global C^∞ functions, homogeneous of degree zero in $\bar{\pi}_A$. By (A 5) and the holomorphicity of S^A we find

$$\frac{\partial V^B}{\partial \bar{\pi}_A} = \bar{\pi}^A \Sigma_D^B V^D. \tag{A 6}$$

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Conversely, if we are given an equation of the form of (A 6), we may find matrices g and $\hat{g}$ of partial solutions and recover the bundle $B(n)$. The particular value of n is dictated by the nature of solutions to (A 6).

The Newman equation (3·5) was written in the form (A 6) in (2). Specifically (3·5) is

$$\frac{\partial^2 V}{\partial \overline{\pi}_A \partial \overline{\pi}_B} = \dot{\sigma}^0 \overline{\pi}^A \overline{\pi}^B V, \tag{A 7}$$

where $V(\pi_{A'}, \overline{\pi}_A)$ is homogeneous of degree one in the sense that

$$V(\lambda \pi_{A'}, \overline{\lambda} \overline{\pi}_A) = \lambda \overline{\lambda} V(\pi_{A'}, \overline{\pi}_A). \tag{A 8}$$

Now defining

$$V^A = \frac{\partial V}{\partial \overline{\pi}_A},$$

so that by (A 8),

$$V = V^A \overline{\pi}_A,$$

we find for (A 7)

$$\frac{\partial V^B}{\partial \overline{\pi}_A} = \dot{\sigma}^0 \overline{\pi}^A \overline{\pi}^B \overline{\pi}_D V^D. \tag{A 9}$$

This is of the form of (A 6) with

$$\Sigma^B_D = \dot{\sigma}^0 \overline{\pi}^B \overline{\pi}_D. \tag{A 10}$$

To complete the analogy with (A 6) however, we must have V^A in (A 9) homogeneous of degree zero. This means we must consider V in (A 7) to have homogeneity:

$$V(\lambda \pi_{A'}, \overline{\lambda} \overline{\pi}_A) = \overline{\lambda} V(\pi_{A'}, \overline{\pi}_A),$$

so that we are actually considering the generalized Newman equation:

$$L_{-\frac{1}{2}} V = 0.$$

In this case, the holomorphic bundle which results is $\mathcal{N} \otimes \mathcal{O}(-1)$, the normal bundle of a section of deformed twistor space tensored with $\mathcal{O}(-1)$ (since otherwise it would not have Chern class zero, and the splitting (A 2) would not be possible).

Now given a fundamental pair of solutions (U, V) with $s_m = N$ we can construct g and $\hat{g}$ explicitly. In this case, U and V have spin weights $-N$ and $N-1$ respectively. They are represented by pairs of C^∞ functions on the sphere with

$$\hat{U}(\eta, \overline{\eta}) = (\eta \overline{\eta}^{-1})^N U(\zeta, \overline{\zeta})$$

$$\hat{V}(\eta, \overline{\eta}) = (\eta \overline{\eta}^{-1})^{1-N} V(\zeta, \overline{\zeta}), \quad \eta = \zeta^{-1} \quad \text{on} \quad \mathcal{U} \cap \hat{\mathcal{U}}.$$

Recalling the definition of ∂ in terms of ζ and $\overline{\zeta}$ from (2), we define

$$g = \begin{pmatrix} P^{N-1}(\partial V + \overline{\zeta} V) & P^{-N}(\partial U + \overline{\zeta} U) \\ P^{N-1}(V + \overline{\zeta} V) & P^{-N}(U - \zeta \partial U) \end{pmatrix},$$

$$\hat{g} = \begin{pmatrix} \hat{P}^{N-1}(\eta \partial \hat{V} + \hat{V}) & \hat{P}^{-N}(\eta \partial \hat{U} + \hat{U}) \\ \hat{P}^{N-1}(\overline{\eta} V + \partial V) & \hat{P}^{-N}(\overline{\eta} U - \partial U) \end{pmatrix},$$

where $P = 1 + \zeta \overline{\zeta}$, $\hat{P} = 1 + \eta \overline{\eta}$.

Note that g and $\hat{g}$ are well-defined on $\mathcal{U}$ and $\hat{\mathcal{U}}$ respectively and on the overlap

$$g = \hat{g} G,$$

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where

$$G = \begin{pmatrix} \bar{\zeta}^{2N-1} & 0 \\ 0 & \bar{\zeta}^{1-2N} \end{pmatrix}.$$

Thus when $s_m = N$, the bundle $\mathscr{N} \otimes \mathscr{O}(-1)$ is $B(2N-1)$ and so

$$\mathscr{N} = \mathscr{O}(2s_m) \oplus \mathscr{O}(2-2s_m).$$

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