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Hyperbolicity in the volume-preserving scenario

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Abstract. Extending a result of Mañé, Hayashi proved that every diffeomorphism f which has a C^1-neighborhood $\mathcal{U}$ where all periodic points of any $g \in \mathcal{U}$ are hyperbolic is an Axiom A diffeomorphism. Here we prove the analogous result in the volume-preserving scenario.

1. *Introduction and statement of the results*

Let M be a C^∞ d-dimensional Riemannian manifold without boundary, and let $\mathrm{Diff}^1_m(M)$ denote the set of diffeomorphisms which preserve the Lebesgue measure m induced by the Riemannian metric. We endow this space with the C^1 topology.

In the theory of dynamical systems, an important question is whether robust dynamical properties in the phase space lead to differentiable properties of the system. For instance, one of the most important properties that a system can have is stability. This says that any system close enough to the initial one has the same orbit structure as the initial system; in other words, there is a topological conjugacy between this system and the initial one.

In a striking article [**30**], Mañé proved that any C^1 structurally stable diffeomorphism is an Axiom A diffeomorphism. In [**33**] Palis extended this result to Ω-stable diffeomorphisms. Actually, Mañé believed that a weaker property than Ω-stability should be enough to guarantee the Axiom A property. Let us elaborate on this property.

Given a diffeomorphism f over M, a periodic point p of f is *hyperbolic* if $Df^{\tau(p,f)}$ has eigenvalues with absolute value different from 1, where $\tau(p, f)$ is the period of p. In the space of C^1 diffeomorphisms over M, $\mathrm{Diff}^1(M)$, we define the set $\mathcal{F}^1(M)$ to be the set of diffeomorphisms $f \in \mathrm{Diff}^1(M)$ which have a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ such that if $g \in \mathcal{U}$, then any periodic point of g is hyperbolic. In [**23**], Hayashi proved that any diffeomorphism in $\mathcal{F}^1(M)$ is Axiom A, which means that the periodic points are dense in the non-wandering set $\Omega(f)$ and the latter is a hyperbolic set. We recall that in dimension two, this result was proved by Mañé [**29**]. It was also studied for flows without singularities by Gan and Wen in [**22**].

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Observe that in the volume-preserving scenario, the Axiom A condition is equivalent to the diffeomorphism being Anosov, since $\Omega(f) = M$ by Poincaré's recurrence theorem. Hence, it is natural to ask whether Hayashi and Mañé's result still holds in the volume-preserving scenario. Actually, it seems that the arguments of Mañé do hold in this case, especially those relating to perturbations. Moreover, using recent generic results in the volume-preserving context, many of the arguments in the original proof can be avoided. The purpose of the present article is to do this. We want to stress that we follow the same lines as Mañé's original argument, once we have shown that periodic points must have the same index. We observe also that Bessa and Rocha have established analogous results in [**11**] and, together with Ferreira, in [**12**] for the context of incompressible and Hamiltonian flows, but in lower dimensions (three and four, respectively). We shall discuss what kind of results our arguments could prove in the context of incompressible flows in any dimension.

We define the set $\mathcal{F}^1_m(M)$ to be the set of diffeomorphisms $f \in \mathrm{Diff}^1_m(M)$ which have a C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_m(M)$ such that if $g \in \mathcal{U}$, then any periodic point of g is hyperbolic.

If $f \in \mathrm{Diff}^1(M)$, then an f-invariant compact set Λ of M is called a *hyperbolic set* if there is a continuous and Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ such that there are constants $0 < \lambda < 1$ and $C > 0$ satisfying

$$\|Df^k|E^s(x)\| \le C\lambda^k \quad \text{and} \quad \|Df^{-k}|E^u(x)\| \le C\lambda^k$$

for every $x \in \Lambda$ and $k > 0$. We say that f is an *Anosov diffeomorphism* if M is a hyperbolic set for f. The main result of this article is the following theorem.

THEOREM 1.1. *Any diffeomorphism in $\mathcal{F}^1_m(M)$ is Anosov.*

Since the arguments for proving the main theorem involve heterodimensional cycles, it is natural to try to relate this result to the well-known Palis conjecture [**32**] in the volume-preserving scenario. In fact, the arguments needed to do this are due to Crovisier and were outlined in [**17**].

COROLLARY 1.1. *If $f \in \mathrm{Diff}^1_m(M)$ is not an Anosov diffeomorphism, then it can be approximated by a diffeomorphism exhibiting either a heterodimensional cycle if the dimension of M is greater than two or a homoclinic tangency if the dimension of M is two.*

The statement for surfaces was proved by Newhouse in [**31**], since for surfaces any volume-preserving diffeomorphism is symplectic.

This paper is organized as follows. In §2, we recall the Franks lemma in the volume-preserving scenario, which is one of the main tools used in our proof. In §3, we prove that the index for hyperbolic periodic points is constant in a neighborhood of any $f \in \mathcal{F}^1_m(M)$. In §4, we give a proof of our main theorem. In §5, we use Crovisier's arguments to prove the Palis conjecture. In §6, we point out how the arguments in the previous sections could be used to prove some results analogous to those of Gan and Wen [**22**] and Toyoshiba [**37**], and see how this extends a corollary of Bessa and Rocha [**11**] to higher dimensions. Finally,

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in Appendix A, we describe adaptations to the volume-preserving case of an argument due to Mañé. We also make some comments about the symplectic scenario.

2. *A Franks-type lemma*

One of the most useful and basic perturbation results is the Franks lemma [**20**]. This lemma enables us to perform nonlinear perturbations along a finite piece of an orbit by simply using arguments from linear algebra.

We recall a version of this lemma in the volume-preserving scenario, which is contained in the work of Bonatti *et al* [**15**, Proposition 7.4]; see also [**27**].

LEMMA 2.1. *Let $f \in \mathrm{Diff}^1_m(M)$ and let $\mathcal{U}$ be a C^1-neighborhood of f in $\mathrm{Diff}^1_m(M)$. Then there exist a neighborhood $\mathcal{U}_0 \subset \mathcal{U}$ of f and $\delta > 0$ such that if $g \in \mathcal{U}_0(f)$, $S = \{p_1, \ldots, p_m\} \subset M$, and $\{L_i : T_{p_i}M \to T_{p_{i+1}}M\}_{i=1}^{m-1}$ are linear maps belonging to $\mathrm{SL}(d)$ and satisfying $\|L_i - Dg(p_i)\| \le \delta$ for $i = 1, \ldots, m$, then there exists $h \in \mathcal{U}(f)$ such that $h(p_i) = g(p_i)$ and $Dh(p_i) = L_i$.*

3. *Index of periodic orbits*

In this section we study the index of periodic orbits of diffeomorphisms in $\mathcal{F}^1_m(M)$. We recall that the index of a hyperbolic periodic orbit is the dimension of its stable space. The main result of this section is that any diffeomorphism $f \in \mathcal{F}^1_m(M)$ cannot have two periodic hyperbolic saddles with different indices.

PROPOSITION 3.1. *Let $f \in \mathcal{F}^1_m(M)$. Then there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ and a positive integer i such that for every diffeomorphism $g \in \mathcal{U}$ and every hyperbolic periodic orbit p of g, the index of p with respect to g is i.*

Obviously, it is enough to prove that if a volume-preserving diffeomorphism has two hyperbolic periodic points with different indices, then we can perturb this diffeomorphism to get a volume-preserving diffeomorphism with a non-hyperbolic periodic point.

We will prove this by creating heterodimensional cycles; thus the proposition can be seen as the volume-preserving and discrete version of a result of Gan and Wen (see [**22**, Theorem 4.1]). We say that f exhibits a *heterodimensional cycle* if there exist two hyperbolic periodic points p and q with different indices, say $\mathrm{ind}\, p > \mathrm{ind}\, q$, such that there is a non-empty transversal intersection between $W^s(p, f)$ and $W^u(q, f)$, as well as a non-empty quasi-transversal intersection between $W^u(p, f)$ and $W^s(q, f)$, i.e. there is $x \in W^u(p, f) \cap W^s(q, f)$ such that $T_x W^u(p, f) \cap T_x W^s(q, f) = \{0\}$.

The next result is the main technical step in the proof of Proposition 3.1.

PROPOSITION 3.2. *Let $f \in \mathrm{Diff}^1_m(M)$ be a diffeomorphism exhibiting a heterodimensional cycle between two hyperbolic periodic points p and q with indices i and $i + j$, respectively, such that $Df^{\tau(p,f)}(p)$ and $Df^{\tau(q,f)}(q)$ have only real eigenvalues with multiplicity one, where $\tau(r, f)$ denotes the minimum period of the periodic point r. Then, for any neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$, there exists a diffeomorphism $g \in \mathcal{U}$ with a non-hyperbolic periodic point $p \in \mathrm{Per}(g)$.*

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Remark 3.1. The idea underlying this proposition comes from a result of Abdenur *et al* in [**2**]; that is, the idea is to use heterodimensional cycles to build periodic points with intermediate Lyapunov exponents with respect to exponents of the hyperbolic periodic points that form the cycle, like a convex combination of the exponents. To prove this, Abdenur *et al* used some combinatoric data from simple cycles; see [**2**] for precise definitions. However, our proof is based on the creation of good perturbations by using properties of the partial hyperbolic set that arises from the cycle. We think that such techniques can be employed to give a proof of the volume-preserving version of the result of Abdenur *et al*.

Let us prove now Proposition 3.1, assuming Proposition 3.2.

Proof of Proposition 3.1. If Proposition 3.1 were not true, then there would exist two hyperbolic periodic orbits $\tilde{p}$ and $\tilde{q}$ of f with indices i and $i + j$, respectively, for some $j > 0$. We aim to use Proposition 3.2 to reach a contradiction.

First, we will perturb f in order to obtain hyperbolic periodic points with indices i and $i + j$ as in the hypothesis of Proposition 3.2. To do this, we will use the results of Bonatti *et al* in [**15**] on periodic linear systems (cocycles). Let us recall some of these concepts.

A *periodic linear system (cocycle)* is a 4-tuple $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ where f is a diffeomorphism, Σ is an infinite set of periodic points of f, $\mathcal{E}$ is an Euclidian vector bundle defined over Σ, and $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ is such that $A(x) : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a linear isomorphism for each x (here $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x). For the precise definition we refer the reader to the work of Bonatti *et al* [**15**]. See also [**15**] for the definition of *periodic linear systems with transition*.

We say that two hyperbolic periodic points with the same index are *homoclinically related* if the stable manifold of one has a transversal intersection with the unstable manifold of the other, and vice versa. If p is a hyperbolic periodic point of f, we define $H(p, f)$, the *homoclinic class* of p, to be the closure of the set of hyperbolic periodic points homoclinically related to p.

LEMMA 3.1. [**15**, Lemma 1.9] *Let $H(p, f)$ be a non-trivial homoclinic class. Then the derivative Df of f induces a periodic linear system with transitions over* $\mathrm{Per}_h(H(p, f))$, *the set of hyperbolic periodic points homoclinically related to p.*

We say that a periodic linear system with transitions $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ is *diagonalizable* at the point $x \in \Sigma$ if the linear map

$$M_A(x) : \mathcal{E}_x \to \mathcal{E}_x, \quad M_A(x) = A(f^{\tau(x)-1}(x)) \circ \cdots \circ A(f^2(x)) \circ A(x)$$

has only positive real eigenvalues of multiplicity one.

LEMMA 3.2. [**15**, Lemma 4.16] *For every periodic linear system $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ with transitions and every $\varepsilon > 0$, there is a dense subset Σ' of Σ and an ε-perturbation A' of A defined on Σ' which is diagonalizable, that is, $M_{A'}(x)$ has only positive real eigenvalues of multiplicity one for every $x \in \Sigma'$.*

By [**15**, Remark 7.2], if the matrices A have determinant equal to 1 (which is the case for periodic linear systems induced by $f \in \mathrm{Diff}^1_m(M)$), we can take the perturbation A' so that $\det A'(x) = 1$ for every $x \in \Sigma'$.

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Now, by a result of Xia [**39**], we have that, generically in $\mathrm{Diff}^1_m(M)$, all the homoclinic classes are non-trivial. Hence we can assume that $H(\tilde{p}, f)$ and $H(\tilde{q}, f)$ are non-trivial, and then, using Lemmas 3.1 and 3.2, we can find p and q homoclinically related to $\tilde{p}$ and $\tilde{q}$, respectively, and matrices A' with determinant 1 such that $M_{A'}(p)$ and $M_{A'}(q)$ are diagonalizable. We can then use Lemma 2.1 to perturb f to f_1 such that p and q are still hyperbolic periodic points of f_1 and, moreover, $Df_1^{\tau(p,f_1)}(p) = M_{A'}(p)$ and $Df_1^{\tau(q,f_1)}(q) = M_{A'}(q)$. In particular, $Df_1^{\tau(p,f_1)}(p)$ and $Df_1^{\tau(q,f_1)}(q)$ have only real eigenvalues with multiplicity one.

In what follows, we will see how to perturb f_1 to get a heterodimensional cycle between p and q. First, we recall a result of Bonatti and Crovisier [**14**].

PROPOSITION 3.3. (Bonatti and Crovisier [**14**]) *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_m(M)$ such that if $g \in \mathcal{R}$, then $M = H(p, g)$, where $H(p, g)$ is the homoclinic class of a hyperbolic periodic point p of g. In particular, g is transitive.*

Thus, by robustness of p and q and associated properties, we can assume that $f_1 \in \mathcal{R}$; in particular, it is transitive. Therefore, using the connecting lemma of Hayashi for conservative diffeomorphisms (see [**40**]), we can perturb f_1 to f_2 such that $W^u(p, f_2)$ and $W^s(q, f_2)$ have non-empty intersection. Making another perturbation, if necessary, we can assume that this intersection is transversal. Hence, the intersection is robust and, as before, we can also suppose that f_2 belongs to $\mathcal{R}$. Using the connecting lemma once more, we can now perturb f_2 to f_3 so that there is also a non-empty intersection between $W^s(p, f_3)$ and $W^u(q, f_3)$; again, we can assume that this intersection is quasi-transversal, after some perturbation using Lemma 2.1. Thus f_3 exhibits a heterodimensional cycle between p and q. Moreover, since f_3 can be taken as close to f as we want, $Df_3^{\tau(p,f_3)}(p)$ and $Df_3^{\tau(q,f_3)}(q)$ have only real eigenvalues with multiplicity one. Therefore, using Proposition 3.2 for f_3, we can find a diffeomorphism g that is C^1-close to f_3, and thus C^1-close to f, such that g has a non-hyperbolic periodic point, which contradicts the fact that $f \in \mathcal{F}^1_m(M)$. The proof of Proposition 3.1 is now complete. □

As mentioned in Remark 3.1, we will prove Proposition 3.2 by making some 'good' perturbations. To do this, we use the following basic perturbation lemma; see [**39**] for a proof of this fact.

LEMMA 3.3. *Fix $\phi \in \mathrm{Diff}^1_m(M)$. There exist constants $\varepsilon_0 > 0$ and $c > 0$, depending only on ϕ, such that for any $x \in M$, any $\psi \in \mathrm{Diff}^1_m(M)$ which is ε_0-C^1-close to ϕ, and any positive numbers $0 < \delta < \varepsilon_0$ and $0 < \varepsilon < \varepsilon_0$, we have that if $d(y, x) < c\delta\varepsilon$, then there is a $\psi_1 \in \mathrm{Diff}^1_m(M)$ which is ε-C^1-close to ψ such that $\psi_1(\psi^{-1}(x)) = y$ and $\psi_1(z) = \psi(z)$ for all $z \notin \psi^{-1}(B_\delta(x))$.*

Before we prove Proposition 3.2, recall that an f-invariant compact set Λ has a *dominated splitting* if there exist a continuous Df-invariant splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$ and constants $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that for every $x \in \Lambda$ we have

$$\|Df^m(x)|E_i\| \|Df^{-m}(f^m(x))|E_j\| \leq \lambda \quad \text{for every } i < j.$$

Proof of Proposition 3.2. Let $f \in \mathrm{Diff}^1_m(M)$ be as in the hypothesis of the proposition. Hence, let $x \in W^s(p, f) \cap W^u(q, f)$ and $y \in W^u(p, f) \cap W^s(q, f)$ be heteroclinic

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points of the heterodimensional cycle. Since the periodic points are hyperbolic, by the regularization result of Avila [**7**] we can assume that the diffeomorphism f is C^∞. Although we have some change of the stable and unstable manifolds of p and q, recall that compact pieces of these manifolds vary continuously, so by using Lemma 3.3, after some perturbation, we could connect $W^s(p,\ f)$ and $W^u(q,\ f)$ again, recovering the quasi-transversal intersection of the cycle in a neighborhood of x. Now we can use Arbieto and Matheus's pasting lemma [**4**] to linearize the diffeomorphism in neighborhoods U_p and U_q of p and q, respectively. It is important to note that this perturbation keeps unchanged the initial map in the complement of U_p and U_q. Also, as before, we can assume that we still have the quasi-transversal intersection. Hence, we may assume that the initial map f is linear (in local coordinates) in the neighborhoods U_p and U_q.

We assume from now on, just for simplicity, that p and q are fixed points, and we will look at the neighborhoods U_p and U_q in local coordinates. Let $E_1 \oplus \cdots \oplus E_d$ be an orthonormal decomposition of $\mathbb{R}^d$ into one-dimensional subspaces, and recall that $\dim M = d$. Since $Df(p)$ and $Df(q)$ have only real eigenvalues with multiplicity one, by modifying the coordinates in U_p and U_q, if necessary, we can assume that these subspaces are $Df(p)$- and $Df(q)$-invariant. More precisely, if we write a decomposition of $\mathbb{R}^d$ as $E_{1,p} \oplus \cdots \oplus E_{d,p}$ (respectively, $E_{1,q} \oplus \cdots \oplus E_{d,q}$) in U_p (respectively, U_q), and denote by λ_k (respectively, σ_k), $k = 1, \ldots, d$, the ordered eigenvalues of $Df(p)$ (respectively, $Df(q)$), then we can assume that $E_{k,p}$ (respectively, $E_{k,q}$) is the corresponding eigenspace of λ_k (respectively, σ_k). Therefore, these splittings are dominated with respect to $Df(p)$ and $Df(q)$, respectively. To be more precise, we will make the following assumptions: we consider $E_{i,p}(\cdot)$ to be the extension of the direction $E_{i,p}$ in the neighborhood U_p, and similarly for $E_{i,q}(\cdot)$.

Now, since $\operatorname{ind} p = i$ and $\operatorname{ind} q = i + j$, the stable and unstable subspaces at p are $E^s_p = E_{1,p} \oplus \cdots \oplus E_{i,p}$ and $E^u_p = E_{i+1,p} \oplus \cdots \oplus E_{d,p}$, respectively. In the same manner, the stable and unstable subspaces at q are $E^s_q = E_{1,q} \oplus \cdots \oplus E_{i+j,q}$ and $E^u_q = E_{i+j+1,q} \oplus \cdots \oplus E_{d,q}$, respectively. Hence, since f is equal to $Df(p)$ and $Df(q)$ in U_p and U_q, respectively, we have that the local stable and unstable manifolds of p and q are $W^s_{\mathrm{loc}}(p,\ f) = E^s_p \cap U_p$, $W^u_{\mathrm{loc}}(p,\ f) = E^u_p \cap U_p$, $W^s_{\mathrm{loc}}(q,\ f) = E^s_q \cap U_q$ and $W^u_{\mathrm{loc}}(q,\ f) = E^u_q \cap U_q$.

Recalling that x is a point of intersection between $W^s(p,\ f)$ and $W^u(q,\ f)$ while y is a point of intersection between $W^u(p,\ f)$ and $W^s(q,\ f)$, the next step consists of making finitely many perturbations in order to get a dominated splitting for the f-invariant set $O(x) \cup O(y) \cup \{p, q\}$, where $O(x)$ and $O(y)$ are the f-orbits of x and y, respectively. So as not to complicate the notation further, we will continue denoting all of these perturbations by f.

Consider now the subspace $E(y) = T_y(W^u(p,\ f) \cap W^s(q,\ f))$ of dimension j. Since y belongs to the unstable manifold of p and $f = Df(p)$ in U_p, if n is large enough we have that $Df^{-n}(y)(E(y))$ is contained in $E_{i+1,p}(f^{-n}(y)) \oplus \cdots \oplus E_{d,p}(f^{-n}(y))$. Moreover, provided $\dim E(y) = j$, by transversality we can suppose that

$$Df^{-n}(y)(E(y)) \cap E_{i+j+1,p}(f^{-n}(y)) \oplus \cdots \oplus E_{d,p}(f^{-n}(y)) = \{0\}.$$

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Therefore, since this splitting in U_p is dominated, $Df^{-n}(y)(E(y))$ converges to $E_{i+1,p}(p) \oplus \cdots \oplus E_{i+j,p}(p)$ as $n \to \infty$. Hence, we can choose n large enough and use Lemma 2.1 to perturb f so that

$$Df^{-n}(y)(E(y)) = E_{i+1,p}(f^{-n}(y)) \oplus \cdots \oplus E_{i+j,p}(f^{-n}(y)).$$

Observe that this perturbation is local; more precisely, this new map f was modified only in a neighborhood of $f^{-n+1}(y)$ and, moreover, the orbit of y remains intact. This implies that y is still a heteroclinic point. By the same arguments, but now considering the future orbit of y, we can make another perturbation of f so that we also have $Df^n(y)(E(y)) = E_{i+1,q}(f^n(y)) \oplus \cdots \oplus E_{i+j,q}(f^n(y))$. Thus, with these perturbations we have now a Df-invariant subbundle over the set $O(y) \cup \{p, q\}$, which we denote by E.

The λ-lemma says that

$$Df^m(f^{-n}(y))(E_{i+j+1,p}(f^{-n}(y)) \oplus \cdots \oplus E_{d,p}(f^{-n}(y)))$$

converges to $E_{i+j+1,q}(q) \oplus \cdots \oplus E_{d,q}(q)$ as $m \to \infty$; then we can use Lemma 2.1 again to perturb f, keeping the subbundle E still Df-invariant, such that

$$\begin{aligned} &Df^m(f^{-n}(y))(E_{i+j+1,p}(f^{-n}(y)) \oplus \cdots \oplus E_{d,p}(f^{-n}(y))) \\ &\quad = E_{i+j+1,q}(f^{m-n}(y)) \oplus \cdots \oplus E_{d,q}(f^{m-n}(y)). \end{aligned}$$

Similarly, by another perturbation of f, and replacing m and n by large positive integers if necessary, we can also assume that f satisfies

$$\begin{aligned} &Df^{-m}(f^{m-n}(y))(E_{1,q}(f^{m-n}(y)) \oplus \cdots \oplus E_{i,q}(f^{m-n}(y))) \\ &\quad = E_{1,p}(f^{-n}(y)) \oplus \cdots \oplus E_{i,p}(f^{-n}(y)). \end{aligned}$$

Consequently, we have found a Df-invariant splitting over $O(y) \cup \{p, q\}$.

By repeating this process finitely many times inside each invariant subbundle, we can perturb f so that

$$Df^{2n}(f^{-n}(y))(E_{k,p}(f^{-n}(y))) = E_{k,q}(f^n(y)), \quad k = 1, \ldots, d, \quad \text{for } n \text{ large enough.}$$

This process produces a Df-invariant splitting by one-dimensional subbundles over $O(y) \cup \{p, q\}$. Moreover, this is a dominated splitting.

Repeating the same argument using x instead of y, we can perturb f so that we also have

$$Df^{2n}(f^{-n}(x))(E_{k,q}(f^{-n}(x))) = E_{k,p}(f^n(x)), \quad k = 1, \ldots, d \quad \text{for } n \text{ large enough.}$$

Thus, there exists a dominated splitting $T_\Lambda M = E_1 \oplus \cdots \oplus E_d$ for the f-invariant set $\Lambda = O(x) \cup O(y) \cup \{p, q\}$.

Therefore, if U is a small neighborhood of Λ, then the maximal invariant set in U for g close enough to f,

$$\Lambda_U(g) = \bigcap_{k \in \mathbb{Z}} g^k(U),$$

has a dominated splitting by one-dimensional subbundles extending the splitting over Λ for f.

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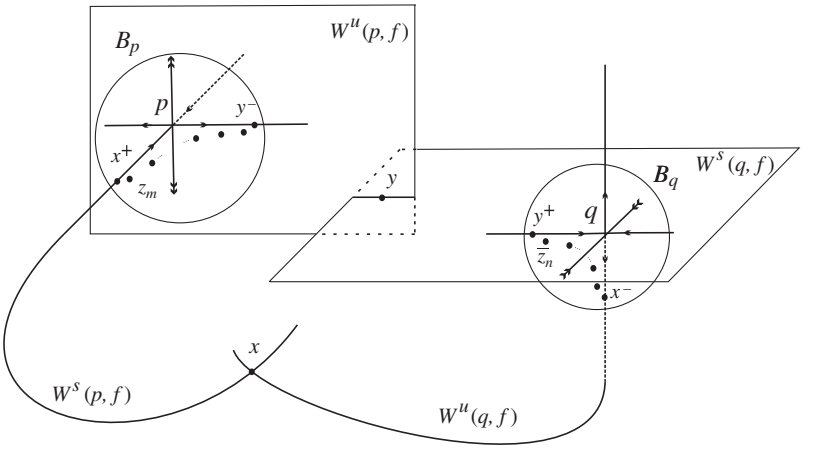


FIGURE 1. Choosing the points z_m in B_p and $\overline{z}_n$ in B_q.

Claim. Given any neighborhood $\mathcal{U}$ of f, there exists a diffeomorphism $g \in \mathcal{U}$ such that g has a periodic point $\tilde{p}$ with at least one real eigenvalue arbitrarily close to 1.

Fix $\varepsilon_0 > 0$ and $c > 0$ for f according to Lemma 3.3. Fix $0 < \varepsilon < \varepsilon_0$ such that if $f_1 \in \mathrm{Diff}^1_m(M)$ is ε-C^1-close to f, then $f_1 \in \mathcal{U}$.

Now we take $B_p \subset U_p \cap U$ and $B_q \subset U_q \cap U$ to be two small balls around p and q, respectively. Here U_p and U_q can be taken to be smaller neighborhoods so that f is equal to $Df(p)$ and $Df(q)$ in these neighborhoods. Since x and y are heteroclinic points, let m_1, m_2, m_3 and m_4 be the smallest positive integers such that $f^{m_1-1}(x), f^{-m_3+1}(y) \in B_p$ and $f^{-m_2+1}(x), f^{m_4-1}(y) \in B_q$. We define $x^+ := f^{m_1}(x)$, $y^- := f^{-m_3}(y)$, $x^- := f^{-m_2}(x)$ and $y^+ := f^{m_4}(y)$. Let $0 < \delta < \varepsilon_0$ be such that

$$f^{-1}(B_\delta(x^+)) \cap B_\delta(x^+) = \varnothing, \quad f(B_\delta(x^-)) \cap B_\delta(x^-) = \varnothing$$

and

$$f^{-1}(B_\delta(y^+)) \cap B_\delta(y^+) = \varnothing, \quad f(B_\delta(y^-)) \cap B_\delta(y^-) = \varnothing,$$

where $B_\delta(z)$ is the ball of radius δ centered at z.

Using the λ-lemma, if m is a sufficiently large positive integer, then we can find z_m which is $c\delta\varepsilon$-near x^+ such that $f^m(z_m)$ is also $c\delta\varepsilon$-near y^- and, moreover, $f^r(z_m) \in B_p$ for $0 \le r \le m$. Analogously, we can find $\overline{z}_n$ satisfying similar conditions upon replacing p by q, x^+ by y^+ and y^- by x^-. See Figure 1.

Hence, the set

$$O_{mn} = \{z_m, \ldots, f^{m-1}(z_m), y^-, \ldots, f^{-1}(y^+), \overline{z}_n, \ldots, f^{n-1}(\overline{z}_n), x^-, \ldots, f^{-1}(x^+)\} \tag{1}$$

is a periodic pseudo-orbit. Using Lemma 3.3, we can perturb f to f_{mn} such that the pseudo-orbit O_{mn} is now a periodic orbit of f_{mn}. Moreover, we remark that this perturbation can be made in such a way that its support is inside $B_p \cup B_q$. Let us denote by p_{mn} a point fixed arbitrarily inside O_{mn}.

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Note that the orbit of p_{mn} contains m elements in B_p and n elements in B_q. By construction, the orbit of p_{mn} is contained in U, and hence $p_{mn} \in \Lambda_U(f_{mn})$. Then, since this set has a dominated splitting by one-dimensional subbundles, we have that $E_k(p_{mn})$, $k = 1, \ldots, d$, are the eigenspaces of $Df_{mn}^{\tau(p_{mn}, f_{mn})}(p_{mn})$. We denote by ρ_k the eigenvalue of $Df_{mn}^{\tau(p_{mn}, f_{mn})}(p_{mn})$ corresponding to each Df-invariant subbundle $E_k(p_{mn})$, $1 \le k \le d$.

From now on, we fix some $k \in (i, i+j)$ and consider the eigenvalues λ_k and σ_k of $Df(p)$ and $Df(q)$, respectively. Note that $|\lambda_k| > 1$ and $|\sigma_k| < 1$. The idea is that, depending on how many elements of the orbit of p_{mn} belong to the neighborhoods B_p and B_q (i.e. m and n, respectively), the absolute value of ρ_k could be greater or smaller than 1.

Let n be a large positive integer. We choose $m = m(n)$ to be the largest possible positive integer such that the periodic points $p_{m-1\,n}$ and p_{mn} of $h = f_{m-1\,n}$ and $g = f_{mn}$, respectively, have eigenvalues corresponding to the invariant direction E_k with modulus smaller than 1 and greater than 1, respectively. More precisely, m is chosen such that

$$\|Dh^{\tau(p_{m-1n},h)}(p_{m-1n})|E_k\| < 1 \quad \text{and} \quad \|Dg^{\tau(p_{mn},g)}(p_{mn})|E_k\| > 1.$$

Observe now that these perturbations h and g of f satisfy $h = g$ in the complement of B_p and $h = g = f$ in the complement of $B_p \cup B_q$.

Using equation (1), we get that the orbits of the periodic points p_{ln}, for $l = m-1, m$, are

$$\{z_l, \ldots, f^{l-1}(z_l), y^-, \ldots, f^{-1}(y^+), \bar{z}_n, \ldots, f^{n-1}(\bar{z}_n), x^-, \ldots, f^{-1}(x^+)\},$$

where z_l and $\bar{z}_n$ were chosen using the λ-lemma, depending on l, as before.

Let $\gamma > 0$ be a small positive number. We can take n large enough so that g and h are $\gamma/2$-C^1-close to f, and the subbundles E_k of g (and h) are $\gamma/2$-close to the subbundles E_k for f in their respective base points. In particular, we will use the same notation E_k for the diffeomorphisms f, g and h, and trust that this will not cause confusion.

If we denote by τ the g-period of $y^- \in O_{mn} = O(p_{mn})$ and define $K = \log \|Df(p)|E_k\|$, then we have

$$\begin{aligned}
0 &< \frac{1}{\tau} \log \|Dg^{\tau}(y^-)|E_k\| \\
&= \frac{1}{\tau} \sum_{t=0}^{\tau-1} \log \|Dg(g^t(y^-))|E_k\| \\
&< \frac{1}{\tau-1} \left(\sum_{t=0}^{\tau-1} (\log \|Df(g^t(y^-))|E_k\| + \gamma) \right) \\
&\le \frac{1}{\tau-1} \left(\sum_{t=0}^{\tau-m-1} \log \|Df(g^t(y^-))|E_k\| + m(\log \|Df(p)|E_k\| + \gamma) + \gamma\tau \right) \\
&< \frac{1}{\tau-1} \left(\sum_{t=0}^{\tau-m-1} \log \|Df(g^t(y^-))|E_k\| + (m-1)(\log \|Df(p)|E_k\|) \right) \\
&\quad + \gamma \left(\frac{m+\tau}{\tau-1} \right) + \frac{K}{\tau-1}. \qquad (2)
\end{aligned}$$

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Here we have used that E_k is a one-dimensional subbundle in the second equality, the continuity of the dominated splitting in the third inequality, and in the fourth inequality the fact that B_p could be taken very small and $g^t(y^-) \in B_p$ for $\tau - m - 1 < t < \tau - 1$.

On the other hand, we also know that y^- is a periodic point of h with period $\tau - 1$. Moreover, the orbit of y^- for h is $O_{m-1\,n} = O(p_{m-1\,n})$. Hence, using analogous arguments, we obtain that

$$\begin{aligned} 0 &> \frac{1}{\tau - 1} \log \|Dh^{\tau-1}(y^-)|E_k\| \\ &= \frac{1}{\tau - 1}\left(\sum_{t=0}^{\tau-2} \log \|Dh(h^t(y^-))|E_k\|\right) \\ &> \frac{1}{\tau - 1}\left(\sum_{t=0}^{\tau-2} (\log \|Df(h^t(y^-))|E_k\| - \gamma)\right) \\ &> \frac{1}{\tau - 1}\left(\sum_{t=0}^{\tau-m-1} \log \|Df(h^t(y^-))|E_k\| + (m-1)(\log \|Df(p)|E_k\|)\right) \\ &\quad - \gamma\left(\frac{m - 1 + \tau}{\tau - 1}\right). \end{aligned} \tag{3}$$

Observe that $g = h$ in the complement of B_p, Moreover, the points of the g-orbit and the h-orbit of y^- also coincide in the complement of B_p, i.e. $g^t(y^-) = h^t(y^-)$ for every $0 \le t \le \tau - m - 1$. So we can use inequality (3) in (2) to obtain the following inequality:

$$0 < \frac{1}{\tau} \log \|Dg^{\tau}|E_k(y^-)\| < \gamma\left(\frac{2m + 2\tau - 1}{\tau - 1}\right) + \frac{K}{\tau - 1}. \tag{4}$$

Therefore, since $m < \tau$ and the period τ goes to infinity as $n \to \infty$, if n is large enough the right-hand side of the inequality will be smaller than 4γ. So, if $\gamma > 0$ is small enough, we can find a periodic point $\tilde{p} = p_{mn}$ for some perturbation g of f with at least one Lyapunov exponent close enough to zero, i.e. $Dg^{\tau(p_{mn},g)}(p_{mn})$ has one eigenvalue close enough to 1.

In the general case, where p and q are hyperbolic periodic points, the proof is completely analogous. The difference is that in this case, U_p and U_q should be neighborhoods of the orbits of p and q, and the numbers m and n are now multiples of the periods of p and q, respectively.

Finally, let us use the claim to prove the proposition.

Let $\mathcal{U}_0 \subset \mathcal{U}$ and $\delta > 0$ be as given by Lemma 2.1 for the diffeomorphism f. Using the claim, we can find a diffeomorphism $\tilde{g} \in \mathcal{U}_0$ such that $\tilde{g}$ has a periodic point $\tilde{p}$ with a large period $\tau = \tau(\tilde{p}, \tilde{g})$ and $D\tilde{g}^{\tau}_{\tilde{p}}$ has one eigenvalue ρ with multiplicity one such that $|\rho| = 1/\alpha$ for some α very close to 1. Now, denoting the eigenspace corresponding to ρ by E, we can choose an orthonormal basis of $T_{\tilde{g}^t(\tilde{p})}M$ as $\{v, v_2, \ldots, v_d\}$, where v is a unit vector in the one-dimensional subspace $E(\tilde{g}^t(\tilde{p}))$. Hence we can build isomorphisms $\tilde{L}_t : T_{\tilde{g}^t(\tilde{p})}M \to T_{\tilde{g}^t(\tilde{p})}M$, $0 \le t \le \tau - 1$, which are close to the identity since α is close to 1, defined by $\tilde{L}_t(v) = \alpha^{1/\tau} v$, $\tilde{L}_t(v_2) = (1/\alpha^{1/\tau})v_2$ and $\tilde{L}_t(v_r) = v_r$ for $3 \le r \le d$. Therefore, defining $L_t : T_{\tilde{g}^t(\tilde{p})}M \to T_{\tilde{g}^{t+1}(\tilde{p})}M$ by $L_t = D\tilde{g}(\tilde{g}^t(\tilde{p}))\tilde{L}_t$, we have a conservative linear isomorphism close to $D\tilde{g}(\tilde{g}^t(\tilde{p}))$ for every t. Thus, we can use Lemma 2.1 for these points

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and these linear isomorphisms, and find a volume-preserving diffeomorphism $g \in \mathcal{U}$ such that $\tilde{p}$ is still a periodic point of g and, moreover, if $v \in E(g^t(\tilde{p}))$, then

$$Dg(g^t(\tilde{p}))(v) = (\alpha)^{1/\tau} D\tilde{g}(\tilde{g}^t(\tilde{p}))(v) \quad \text{for every } 0 \le t < \tau.$$

Then, since the space E is one-dimensional, the eigenvalue of $Dg^\tau(\tilde{p})|E$ has absolute value equal to 1, that is, $\tilde{p}$ is a non-hyperbolic periodic point of g. This completes the proof of Proposition 3.2. □

4. *Proof of Theorem 1.1*

First, we will prove the hyperbolicity of $\overline{\mathrm{Per}(f)}$ for every $f \in \mathcal{F}^1_m(M)$. After that, we will conclude that the diffeomorphism is Anosov.

However, we will need a volume-preserving version of a result of Mañé [**29**, Proposition II.1] (see also the work of Liao [**28**]). This result can be deduced from the Franks lemma, Lemma 2.1, by adapting some arguments of Mañé. We will give the necessary adaptations in Appendix A. Let $\Lambda_i(f)$ be the closure of the set formed by the hyperbolic periodic points of f with index i.

PROPOSITION 4.1. *If $f \in \mathcal{F}^1_m(M)$, then there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_m(M)$ and constants $K > 0$, $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that:*

(a) *for every $g \in \mathcal{U}$ and $p \in \mathrm{Per}(g)$ with minimum period $\tau(p) \ge m$,*

$$\prod_{i=0}^{k-1} \|Dg^m(g^{mi}(p))|E^s_g\| \le K\lambda^k$$

and

$$\prod_{i=0}^{k-1} \|Dg^{-m}(g^{-mi}(p))|E^u_g\| \le K\lambda^k,$$

where $k = [\tau(p)/m]$;

(b) *for all $0 < i < \dim M$ there exists a continuous splitting $T_{\Lambda_i(g)}M = E_i \oplus F_i$ such that*

$$\|Dg^m(x)|E_i\| \|Dg^{-m}(g^m(x))|F_i\| \le \lambda$$

for all $x \in \Lambda_i(g)$. Moreover, $E_i(p) = E^s_g(p)$ and $F_i(p) = E^u_g(p)$ when $p \in \mathrm{Per}(g)$ and $\dim E^s_g(p) = i$;

(c) *for all $p \in \mathrm{Per}(g)$,*

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg^m(g^{mi}(p))|E^s_g\| < 0$$

and

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Dg^{-m}(g^{-mi}(p))|E^u_g\| < 0.$$

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4.1. *Hyperbolicity of* $\overline{\mathrm{Per}(f)}$. We fix $f \in \mathcal{F}^1_m(M)$. By Proposition 3.1, there exist a neighborhood $\mathcal{U}$ of f and an integer s such that for every $g \in \mathcal{U}$, any periodic point g is hyperbolic and has index s. Moreover, by Proposition 4.1, we can suppose that for any $g \in \mathcal{U}$ there exists a dominated splitting $T_{\overline{\mathrm{Per}(g)}}M = E_g \oplus F_g$ over $\overline{\mathrm{Per}(f)}$. We also fix the constants $m \in \mathbb{N}$, $0 < \lambda < 1$ and $K > 0$ given by Proposition 4.1.

We will prove that this splitting is hyperbolic. It is well known that it suffices to prove that

$$\liminf_{n\to+\infty} \|Df^n(x)|E\| = 0 \tag{5}$$

and

$$\liminf_{n\to+\infty} \|Df^{-n}(x)|F\| = 0$$

for all $x \in \overline{\mathrm{Per}(f)}$.

Of course, we only need to prove the first equality, since the second could be obtained in the same way using f^{-1}.

Let us suppose, by way of contradiction, that (5) is not true. Then there exist $c > 0$ and $x \in M$ such that

$$\|Df^{jm}(x)|E\| \geq c \quad \text{for all } j > 0.$$

If δ_a denotes the Dirac measure on the point a, we define the sequence of measures

$$\mu_j = \frac{1}{j}\sum_{i=0}^{j-1} \delta_{f^{mi}(x)}.$$

There exists a sequence $j_n \to +\infty$ such that the measures μ_{j_n} converge to an f^m-invariant probability measure μ in the weak* topology. Moreover,

$$\lim_{n\to+\infty} \frac{1}{j_n} \log \|Df^{mj_n}(x)|E\| \geq 0. \tag{6}$$

In particular, using the continuous function $\phi(x) = \log \|Df^m(x)|E\|$ defined on $\overline{\mathrm{Per}(f)}$, we obtain

$$\begin{aligned}\int_{\overline{\mathrm{Per}(f)}} \phi \, d\mu &= \lim_{n\to+\infty} \frac{1}{j_n} \sum_{i=0}^{j_n-1} \log \|Df^m(f^{mi}(x))|E\| \\ &\geq \lim_{n\to+\infty} \frac{1}{j_n} \log \|Df^{mj_n}(x)|E\| \geq 0.\end{aligned}$$

By Birkhoff's ergodic theorem, we have

$$0 \leq \int_{\overline{\mathrm{Per}(f)}} \phi \, d\mu = \int_{\overline{\mathrm{Per}(f)}} \lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m(f^{mi}(x))|E\| \, d\mu. \tag{7}$$

Let $\Sigma(f) \subset M$ be the total probability set given by the ergodic closing lemma in the volume-preserving scenario; see [**6**].

Now, we define $\nu = (1/m)\sum_{i=0}^{m-1} (f^i)^*\mu$, which is an f-invariant probability measure. Since ν is supported on $\overline{\mathrm{Per}(f)}$, we have that $\nu(\Sigma(f) \cap \overline{\mathrm{Per}(f)}) = 1$. Actually, since $\Sigma(f) \cap \overline{\mathrm{Per}(f)}$ is an f-invariant set, it is not difficult to see that this set is also a set of total probability for μ.

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This, together with (7), implies that there exists a point $y \in \Sigma(f) \cap \overline{\mathrm{Per}(f)}$ such that

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m(f^{mi}(y))|E\| \geq 0. \tag{8}$$

Assertion (c) of Proposition 4.1 implies that y cannot be a periodic point. Thus, $y \notin \mathrm{Per}(f)$.

By (8), we can take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n} \sum_{i=0}^{n-1} \log \|Df^m(f^{mi}(y))|E\| \geq \log \lambda_0 \tag{9}$$

when $n \geq n_0$.

The idea now is to find a hyperbolic periodic point p of a diffeomorphism g close enough that its orbit is 'close' to the orbit of y. With this, we will use Lemma 2.1 to exchange the derivative at the orbit of p, keeping it periodic, such that the inequality (9) will contradict statement (a) of Proposition 4.1.

We begin by making some remarks. First, by domination, there exists a neighborhood U of $\overline{\mathrm{Per}(f)}$ such that the maximal invariant set $\Lambda_U(f)$ of U has a dominated splitting. Moreover, we can suppose that for every $h \in \mathcal{U}$ the maximal invariant set of U with respect to h has a dominated splitting too; we denote this splitting by $T_{\Lambda_U(h)}M = E_h \oplus F_h$. In particular, the subbundles $E_g(x)$ and $F_g(x)$ are continuous in both $g \in \mathcal{U}$ and $x \in \overline{\mathrm{Per}(g)}$. Thus, given $\varepsilon > 0$, upon shrinking $\mathcal{U}$ if necessary, there exist $\eta > 0$ such that if $d(x, z) < \eta$ and $g, h \in \mathcal{U}$, then $E_g(x)$ is ε-close to $E_h(z)$ (we can suppose also that x and z are in the same local chart); and, if $g \in \mathcal{U}$, we have that $E_g(p) = E^s_g(p)$ and $F_g(p) = E^u_g(p)$ for a periodic point p.

Second, if for some $z \in \Sigma(f)$ there exist $g_n \to f$ and periodic points p_n of g_n such that $p_n \to z$ as given by the ergodic closing lemma, with the period of p_n bounded, then $z \in \mathrm{Per}(f)$.

If $\varepsilon > 0$ is fixed, then by the ergodic closing lemma we can approximate f by a diffeomorphism g such that there exists $p \in \mathrm{Per}(g)$ with period n such that

$$d(g^j(p), f^j(y)) < \eta \quad \text{for } j = 0, \ldots, n. \tag{10}$$

As remarked before, if η is small and g is sufficiently close to f, we have that n is large, say $n \geq m(n_0 + 1)$. Moreover, p has index s and $E^s_g(g^i(p))$ (respectively, $E^u_g(g^i(p))$) is ε-close to $E(f^i(y))$ (respectively, $F(f^i(y))$) for any $0 \leq i \leq n$.

We will build isometries $A_i : T_{g^i(p)}M \to T_{f^i(y)}M$, $0 \leq i \leq n$, which are 2ε-close to the identity (in local charts) and such that $A_i(E^s_g(g^i(p))) = E(f^i(y))$ for $0 \leq i \leq n$.

Let us show how to construct A_0; the other cases are similar. Let

$$\{e_1(i), \ldots, e_s(i), e_{s+1}(i), \ldots, e_d(i)\}$$

be an orthonormal basis of T_iM, for $i = y$ and p. Moreover, we require that $\{e_j(i), 1 \leq j \leq s\}$ be an orthonormal basis for $E(y)$ if $i = y$ and an orthonormal basis for $E^s_g(p)$ if $i = p$.

Using inequality (10) and continuity, these bases could be taken such that the linear map $A_0 : T_pM \to T_yM$ satisfying $A_0(e_j(p)) = e_j(y)$, $1 \leq j \leq d$, is 2ε-close to the

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identity map (in local charts). Now, by construction, A_0 is an isometry and, moreover, $A_0(E_g^s(p)) = E(y)$, as desired.

Using these isometries, we can define volume-preserving maps $L_i : T_{g^i(p)}M \to T_{g^{i+1}(p)}M$ as follows:

$$L_i = A_{i+1}^{-1} Df(f^i(y))A_i \quad \text{for } 0 \le i \le n-1.$$

Therefore, L_i is $D\varepsilon$-close to $Df(f^i(p))$ for all $0 \le i \le n-1$, where D depends only on f. So, using Lemma 2.1, if g is sufficiently close to f, we can find $h \in \mathcal{U}$ such that $p \in \mathrm{Per}(h)$ and $Dh(h^i(p)) = L_i$ for $0 \le i \le n-1$. By construction, $E_g^s(p) \subset T_pM$ is invariant for $Dh^n(p)$. Moreover, by domination, we have that $E_g^s(p) \cap F_h(p) = \{0\}$. This implies that $E_g^s(p) = E_h(p)$. We also have that $E_h^s(p) = E_g^s(p)$ by Proposition 4.1.

Finally, since A_i restricted to $E_g^s(g^i(p))$ is an isometry, upon defining $k = [n/m]$ we have that

$$\|Dh^m(h^{jm}(p))|E_h^s\| = \|Df^m(f^{jm}(y))|E\| \quad \text{for all } 0 \le j \le k.$$

Therefore, since n is large, we have that $k \ge n_0$. Thus, using equation (9), we have

$$\prod_{j=0}^{k-1} \|Dh^m(h^{jm}(p))|E_h^s\| = \prod_{j=0}^{k-1} \|Df^m(f^{jm}(y))|E\| \ge \lambda_0^k,$$

which contradicts Proposition 4.1. Thus $\overline{\mathrm{Per}(f)}$ is hyperbolic.

4.2. *f is Anosov.* To prove that f is an Anosov diffeomorphism, we just need to show that $\Omega(f) = \overline{\mathrm{Per}(f)}$, since $\Omega(f) = M$ by Poincaré's recurrence theorem.

First, we remark that since $f \in \mathcal{F}_m^1(M)$, there exists some neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}_m^1(M)$ such that if $H_n(g)$ denotes the set of hyperbolic periodic points of $g \in \mathcal{U}$ with period less than or equal to n, then $\#H_n(g)$ is finite and constant on $\mathcal{U}$. This follows from the fact that every diffeomorphism in $\mathcal{U}$ has only hyperbolic periodic points.

Let us suppose, for contradiction, that $\overline{\mathrm{Per}(f)} \subsetneq \Omega(f)$. Take $x \in \Omega(f) \backslash \overline{\mathrm{Per}(f)}$. By the conservative version of Pugh's closing lemma [**34**], we can fix $k \in \mathbb{N}$ such that all perturbations of f needed to create a periodic point near x are done in a small neighborhood of

$$\bigcup_{-k \le j \le k} f^j(x).$$

Thus, let U be a neighborhood of $\overline{\mathrm{Per}(f)}$ such that $f^j(x) \notin \overline{U}$ for $-k \le j \le k$. Then, we can find $g \in \mathcal{U}$ and $p \in \mathrm{Per}(g)$ such that $p \notin \overline{U}$. Moreover, by the choice of k and U, $f|_U \equiv g$. However, since $p \notin \overline{U}$, we have $H_n(f) \ne H_n(g)$ for some $n \in \mathbb{N}$. This contradicts the previous remark. Therefore, we have $\Omega(f) = \overline{\mathrm{Per}(f)}$ and this completes the proof.

5. *Palis conjecture in the volume-preserving scenario*

In this section, we elaborate on the arguments of Crovisier in [**17**] concerning the Palis conjecture in the volume-preserving scenario, in order to prove Corollary 1.1. We only

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need to prove the corollary for M with dimension greater than two, since the result for surfaces has already been proved by Newhouse in [**31**].

Suppose that $f \in \mathrm{Diff}^1_m(M)$ is not Anosov; then, by Theorem 1.1, we have that f does not belong to $\mathcal{F}^1_m(M)$. Hence, there exists $g \in \mathrm{Diff}^1_m(M)$ close to f with a non-hyperbolic periodic point p.

LEMMA 5.1. *There exists $h \in \mathrm{Diff}^1_m(M)$ that is C^1-close to f and has two periodic points q and r with different indices.*

Proof. First, we remark that, due to Avila's result [**7**], there exists a C^∞ volume-preserving diffeomorphism F which is C^1-close to f.

Let p be a non-hyperbolic periodic point of f; we assume that p is fixed for simplicity. As is well known, we can choose a small neighborhood U of p and bifurcate p, producing a C^1-close diffeomorphism g that has two fixed points q and r with different indices (it is possible that one of these periodic points has period 2, but the proof is the same). Moreover, $g|_{U^c} \equiv f$ and $\det Dg(q)$ is close to 1 (and the same for $\det Dg(r)$).

Now we can select two disjoint neighborhoods $V_1 \subset U$ of q and $V_2 \subset U$ of r, and consider two linear maps A on V_1 and B on V_2 (using local charts) such that $\det A = \det B = 1$, A has the same index as $Dg(q)$ and B has the same index as $Dg(r)$.

Using the same arguments as in [**4**, Theorem 3.6], we can produce a volume-preserving diffeomorphism h which coincides with A on V_1, B on V_2 (in local charts) and F on U^c, upon shrinking the neighborhoods if necessary. Moreover, h is C^1-close to f. This proves the lemma. □

Therefore, as in the proof of Proposition 3.1, we can perturb once more and create a heterodimensional cycle between q and r.

6. *Star flows*

In this section, we discuss how some results from the theory of star flows can be extended to the divergence-free context. Let $\mathfrak{X}^1(M)$ be the set of vector fields on M endowed with the C^1 topology, and let $\mathfrak{X}^1_m(M) \subset \mathfrak{X}^1(M)$ be the subspace of vector fields which are divergence-free. By Liouville's formula, we know that the flow generated by a divergence-free vector field is volume-preserving, so such flows are also called incompressible flows in the literature.

The analogous version of $\mathcal{F}^1_m(M)$ in the flow case comprises the incompressible star flows. We say that X generates an incompressible star flow if there exists a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1_m(M)$ such that if $Y \in \mathcal{U}$, then all of its singularities and periodic orbits are hyperbolic. We denote the set of incompressible star flows by $\mathfrak{X}^*_m(M)$.

The result proved by Gan and Wen in [**22**] is true for incompressible star flows.

THEOREM 6.1. *If $X \in \mathfrak{X}^*_m(M)$, then X is Anosov.*

This can be seen as follows. First of all, Proposition 4.1 can be extended to the context of incompressible flows exactly as Hayashi did in [**24**]. This will imply that there exists a dominated splitting on the set of periodic points for the linear Poincaré flow; see [**24**] for precise definitions. Now, we want to rule out singularities. This will be done using an

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argument similar to that in the paper [**5**] by the first author jointly with Senos and Sodero. For the sake of completeness we will sketch the argument here.

Let $\mathcal{U}$ be the open set in the definition of an incompressible star flow. First, by the general density theorem for incompressible flows [**34**], we have that if Y belongs to a residual subset $\mathcal{R} \subset \mathcal{U}$, the periodic orbits of Y are dense in M. So, we have a dominated splitting on $M - \mathrm{Sing}(Y)$ for the linear Poincaré flow of any vector field Y in this residual set. Now we use the following lemma.

LEMMA 6.1. [**3**, Lemma 2.29] *Given a connected invariant (not necessarily compact) subset Λ with a (C, λ)-dominated splitting for the linear Poincaré flow with respect to a vector field Z, there exists a neighborhood U of Λ and a C^1-neighborhood $\mathcal{V}$ of Z such that the set $\Lambda_W(U)^* := \bigcap_{t\in\mathbb{R}} W_t(U - \mathrm{Sing}(W))$ has a (C', λ')-dominated splitting for the linear Poincaré flow with respect to any vector field $W \in \mathcal{V}$, where C' and λ' are positive constants depending only on $\mathcal{V}$, U and (C, λ).*

Thus, since singularities are hyperbolic, there exists an open and dense subset $\mathcal{V} \subset \mathcal{U}$ such that for every $Y \in \mathcal{V}$ we have a dominated splitting with respect to Y over $M - \mathrm{Sing}(Y)$. We remark that all singularities of X are saddles, since it is a divergence-free vector field.

We say that a singularity is linear if there exist C^1 linearizing coordinates around it. So a result of Vivier [**38**] (see also [**26**]) will imply the following lemma.

LEMMA 6.2. *$X \in \mathfrak{X}^*_m(M)$ has no singularities.*

If X has singularities, then since $\mathcal{V}$ is dense, there exists $Y \in \mathcal{V}$ where the continuation of the singularity exists. Using the pasting lemma and the fact that $\mathcal{V}$ is open, there exists $Z \in \mathcal{V}$ with a linear singularity and a dominated splitting for the linear Poincaré flow of Z on $M - \mathrm{Sing}(Z)$. Now we invoke the following result due to Vivier [**38**].

PROPOSITION 6.1. *If Z is a C^1 vector field with a linear hyperbolic singularity which is a saddle, then there exists no dominated splitting for the linear Poincaré flow on $M - \mathrm{Sing}(Z)$.*

This leads to a contradiction.

Now, the same calculations that we did for heterodimensional cycles (which were inspired by the calculations of Gan and Wen; see [**22**, Theorem 4.1] for more details about the arguments in the flow case) can be done to prove the following proposition.

PROPOSITION 6.2. *If $X \in \mathfrak{X}^*_m(M)$, then X has no heterodimensional cycles.*

Remark 6.2. Indeed, following Pugh and Robinson [**34**], we say that a subset of vector fields S satisfies the *lift axiom* at X on an open subset $U \subset M$ if there exist $\varepsilon > 0$ and a continuous function $\delta : U \to (0, 1)$ such that for any vector $v \in [X(p)]^{\perp}$ with norm $\|v\| \le \delta(p)$, there exists a perturbation $Y \in S$ such that the Poincaré map f_Y of Y between p and $Y_1(p)$ and the Poincaré map f_X between p and $X_1(p)$ satisfy

$$f_X^{-1} \circ f_Y(p) = \exp(\varepsilon v).$$

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Here, we consider the submanifolds transversal to the flow as the images under the exponential map of the subspaces normal to the flow direction. Moreover, the perturbation Y is supported on a flow box of size $\|v\|$ around the piece of orbit $\{X_t(p), 0 \le t \le 1\}$, and several such perturbations are allowed if their supports are disjoint.

The lift axiom plays a key role in concluding Lemma 3.3. In [**34**, §8(c)], it is proved that divergence-free vector fields satisfy the lift axiom.

Moreover, as at the beginning of the proof of the analogous result for diffeomorphisms, we can assume that X is C^∞. Indeed, in the discrete setting we use Avila's result, and in the flow setting we can use [**4**, Theorem 2.2]; see also [**41**]. This allows us to use [**10**, Franks lemma, p. 3133], which requires that the vector field be C^4.

In particular, we can recast the arguments in the discrete scenario so that they hold in the flow setting.

We observe that the following version of Bonatti and Crovisier's result, Proposition 3.3, for incompressible flows can be found in [**9**]. Hence, together with the previous proposition, we obtain the following.

PROPOSITION 6.3. *If $X \in \mathfrak{X}^*_m(M)$, then there exist a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1_m(M)$ and $i \in \mathbb{N}$ such that the index of any periodic orbit of any vector field $Y \in \mathcal{U}$ is i.*

Now we can adapt the arguments of Toyoshiba [**37**] to obtain the following result.

PROPOSITION 6.4. *Suppose that $X \in \mathfrak{X}^*_m(M)$ has no singularities and that there exist a neighborhood $\mathcal{U}$ of X in $\mathfrak{X}^1_m(M)$ and $i \in \mathbb{N}$ such that the index of any periodic orbit of any vector field $Y \in \mathcal{U}$ is i. Then $X|_{\overline{\mathrm{Per}(X)}}$ is hyperbolic and $\overline{\mathrm{Per}(X)} = M$.*

Since we already know that the index of periodic points is constant, and we have a result similar to Proposition 4.1 in the context of incompressible flows, we have that $\overline{\mathrm{Per}(X)}$ has a dominated splitting for the linear Poincaré flow. Now, as we did before, we can prove that in fact $\overline{\mathrm{Per}(X)}$ is a hyperbolic set of X; see Toyoshiba's paper [**37**] for more details about these kinds of arguments in the flow case.

Suppose now that there exists $x \in M - \overline{\mathrm{Per}(X)}$. Without loss of generality, we can suppose $x \in \Sigma(X)$, i.e. that x belongs to the total probability set given by the ergodic closing lemma. Indeed, this follows from the fact that the Lebesgue measure m is invariant, that $M - \overline{\mathrm{Per}(X)}$ is an open set and thus has positive Lebesgue measure and that there are no singularities.

Let U be a neighborhood of x such that $U \cap \overline{\mathrm{Per}(X)} = \emptyset$. By the ergodic closing lemma (see [**6**]), we can choose periodic orbits $\gamma_n \in \mathrm{Per}(Y_n)$ such that γ_n converges to $\overline{\mathrm{orb}(x)}$ in the Hausdorff topology and Y_n converges to X in the C^1 topology as n goes to infinity. Here, $\overline{\mathrm{orb}(x)}$ denotes the closure of the orbit of x. Hence, using Proposition 4.1 (for incompressible flows) we can prove, as before, that $\overline{\mathrm{orb}(x)}$ is a hyperbolic set of X. We remark also that x can be taken to be a recurrent point by Poincaré's recurrence theorem applied to Lebesgue measure. Thus, by the shadowing lemma, there exists a periodic point inside U, which is a contradiction. This proves that $M = \overline{\mathrm{Per}(X)}$. In particular, X is Anosov.

We remark that recently Ferreira announced a different proof of Theorem 6.1 in [**19**].

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If we denote by KS the C^1-interior of Kupka–Smale incompressible vector fields, then $KS \subset \mathfrak{X}^*_m(M)$.

COROLLARY 6.1. *Let $X \in KS$. Then X is Anosov.*

We observe that this corollary was proved in the dissipative case by Gan [**21**]; see also [**37**].

Finally, we observe that if X is a C^1-structurally stable divergence-free vector field, then $X \in \mathfrak{X}^*_m(M)$. Indeed, let us suppose that there exists Y close to X with a non-hyperbolic periodic orbit O. Using the pasting lemma, we can produce a divergence-free vector field Z close to Y such that in a neighborhood of the orbit of O, Z acts like the linearization on O; in particular, it has uncountably many periodic orbits. However, if $\dim(M) > 4$, then by Robinson's result [**36**] there exists a Kupka–Smale vector field close to X. This leads to a contradiction to the stability of X.

COROLLARY 6.2. *Let M be an n-manifold with $n \geq 4$. If $X \in \mathfrak{X}^1_m(M)$ is a C^1-structurally stable divergence-free vector field, then X is Anosov.*

In dimension three, this result can be found in [**11**].

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A. *Appendix. Adaptation of Mañé's argument to the volume-preserving case*

In this appendix, we show the modifications necessary for proving Proposition 4.1. In particular, we will review Mañé's argument from [**29**, pp. 523–540].

First, we recall, in our context, some notions introduced by Mañé. Let $\mathrm{GL}(d)$ be the group of linear isomorphisms and $\mathrm{SL}(d)$ the subgroup of $\mathrm{GL}(d)$ consisting of isomorphisms with determinant equal to 1. By a hyperbolic sequence we mean a sequence of hyperbolic isomorphisms $\xi : \mathbb{Z} \to \mathrm{GL}(d)$. The sequence is periodic if there exists m such that $\xi_{j+m} = \xi_j$ for $j \in \mathbb{Z}$, and the minimal positive m is called the period of the sequence. If the stable spaces of the elements of the sequence are the whole of $\mathbb{R}^d$, then we call it a contracting sequence. If the sequence is formed by volume-preserving isomorphisms, i.e. $\xi : \mathbb{Z} \to \mathrm{SL}(d)$, then we call it a vol-sequence. Also, a family of periodic vol-sequences $\{\xi^\alpha\}$ will be called a vol-family.

A vol-family of periodic sequences $\{\xi^\alpha\}$ is vol-hyperbolic if all of its sequences are hyperbolic and $\sup_{\alpha,j}\{\|\xi^\alpha_j\|\} < \infty$. We define a distance $d(\xi, \eta) = \sup_{\alpha,j}\{\|\xi^\alpha_j - \eta^\alpha_j\|\}$ between two families ξ and η; we also say that they are equivalent if for every α the periods of ξ^α and η^α coincide. Moreover, the family ξ is uniformly vol-hyperbolic if there exists $\varepsilon > 0$ such that every equivalent vol-family η which satisfies $d(\xi, \eta) < \varepsilon$ is vol-hyperbolic.

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The proof of Proposition 4.1 proceeds in the same way as in [**29**], using the Franks lemma in the volume-preserving case (Lemma 2.1) and the following proposition, which is analogous to [**29**, Lemma II.3].

PROPOSITION A.1. *If $\{\xi^\alpha\}$ is a uniformly vol-hyperbolic family, then there exist constants $K > 0$, $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that:*

(1) *if ξ^α has period $n \geq m$ and $k = [n/m]$, then*

$$\prod_{j=0}^{k-1} \left\| \left(\prod_{i=0}^{m-1} \xi^\alpha_{mj+i} \right) \Big|_{E^s_{mj}} \right\| \leq K\lambda^k \quad \text{and} \quad \prod_{j=0}^{k-1} \left\| \left(\prod_{i=0}^{m-1} \xi^\alpha_{mj+i} \right)^{-1} \Big|_{E^u_{m(j+1)}} \right\| \leq K\lambda^k;$$

(2) *for every α and $j \in \mathbb{Z}$,*

$$\left\| \left(\prod_{i=0}^{m-1} \xi^\alpha_{j+i} \right) \Big|_{E^s_j} \right\| \left\| \left(\prod_{i=0}^{m-1} \xi^\alpha_{j+i} \right)^{-1} \Big|_{E^u_{j+m}} \right\| \leq \lambda;$$

(3) *for every α,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| \left(\prod_{i=0}^{m-1} \xi^\alpha_{mj+i} \right) \Big|_{E^s_{mj}} \right\| < 0$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| \left(\prod_{i=0}^{m-1} \xi^\alpha_{mj+i} \right)^{-1} \Big|_{E^u_{m(j+1)}} \right\| < 0.$$

The proof of this proposition is similar to that given in Mañé's paper, so we will only indicate the necessary adaptations.

First, we recall [**29**, Lemma II.7], which will be used to control the stable and unstable parts of the isomorphisms.

LEMMA A.1. *Let $\{\xi^\alpha\}$ be a uniformly contracting family. There exist $K > 0$, $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that:*

(1) *if ξ^α has period $n \geq m$ and $k = [n/m]$, then*

$$\prod_{j=0}^{k-1} \left\| \prod_{i=0}^{m-1} \xi^\alpha_{mj+i} \right\| \leq K\lambda^k;$$

(2) *for every α,*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left\| \prod_{i=0}^{m-1} \xi^\alpha_{mj+i} \right\| < 0.$$

With this we can prove the following lemma.

LEMMA A.2. *If $\{\xi^\alpha\}$ is a uniformly vol-hyperbolic family, then there exist $\varepsilon > 0$, $K > 0$ and $0 < \lambda < 1$ such that if $\{\eta^\alpha\}$ is an equivalent vol-family with $d(\xi, \eta) < \varepsilon$, then η is vol-hyperbolic. Moreover, if n is the period of η^α and $\widetilde{\eta}^\alpha_j : \mathbb{R}^n/E^s_j \to \mathbb{R}^n/E^s_{j+1}$ is the map induced by η^α_j, then*

$$\left\| \prod_{j=0}^{n-1} \eta^\alpha_j / E^s_0 \right\| \leq K\lambda^n \quad \text{and} \quad \left\| \left(\prod_{j=0}^{n-1} \widetilde{\eta}^\alpha_j \right)^{-1} \right\| \leq K\lambda^n.$$

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Proof. If $\varepsilon > 0$ is small and $0 < m < d$, we will define, as Mañé did, a vol-family $\{\phi^\beta\}$ as the family containing every sequence $\phi : \mathbb{Z} \to \mathrm{GL}(d-m)$ that satisfies the following properties: it has the same period as some ξ^α, and $\sup_i \|\phi_i - (\widetilde{\xi}_i^\alpha)^{-1}\| < \varepsilon$. Since $\varepsilon > 0$ is small, this family is uniformly contracting. Indeed, if not, there would exist a sequence $\psi : \mathbb{Z} \to \mathrm{GL}(d-m)$ such that for some β, ψ and ϕ^β have the same period, $\sup_i \|\phi_i^\beta - \psi_i\|$ is small and $\prod_{j=0}^{n-1} \psi_j$ has an eigenvalue with modulus 1 and determinant close to 1 (since it is close to ϕ). Then, we construct a sequence ζ which has the same period as ξ^α and is such that ζ_j is equal to ψ_j^{-1} restricted to $\mathbb{R}^d/E_j^s$ and equal to $\det(\psi_j)(\det \xi^\alpha|_{E_j^s})^{-1}\xi^\alpha$ restricted to E_j^s. Note that ζ is close to ξ. This would contradict the vol-uniform hyperbolicity of $\{\xi^\alpha\}$. Now we can proceed exactly as in the rest of [**29**, proof of Lemma II.8]. □

In particular, as in [**29**], we obtain statements (1) and (3) of Proposition A.1. See also Remark A.1 below.

Next, we discuss the proof of statement (2) of Proposition A.1. First, we observe that the following lemma about angles [**29**, Lemma II.9] holds in the volume-preserving case.

LEMMA A.3. *If $\{\xi^\alpha\}$ is a uniformly vol-hyperbolic family, then there exist $\varepsilon > 0$, $\gamma > 0$ and $m \in \mathbb{N}$ such that if $\{\eta^\alpha\}$ is an equivalent vol-family with $d(\xi, \eta) < \varepsilon$, then for every η^α with period $n \geq m$ the angle between the stable and unstable spaces is bounded away from zero, that is,*

$$\angle(E_0^s(\eta^\alpha), E_0^u(\eta^\alpha)) > \gamma.$$

Proof. Indeed, to obtain a contradiction, we proceed as Mañé did in the non-volume-preserving case. First, we consider a vol-sequence η with large period n and with small angle, such that its period coincides with the period of some ξ^α and $\sup_i \|\eta_i - \xi_i^\alpha\|$ is small.

Then, consider the matrix of $\prod_{j=0}^{n-1} \eta_j$ with respect to the splitting $\mathbb{R}^d = E_0^s(\eta)^\perp \oplus E_0^s(\eta)$. This gives a triangular matrix

$$\begin{pmatrix} A & 0 \\ P & B \end{pmatrix}.$$

Mañé constructed the perturbation $\zeta_j = \eta_j$ for $0 < j \leq n-1$ and

$$\zeta_0 = \eta_0 \begin{pmatrix} I & C \\ 0 & I \end{pmatrix}.$$

For some suitable C, this will guarantee that $\prod_{j=0}^{n-1} \zeta_j$ has an eigenvalue equal to 1 and the sequence is close to η. Note that this is still a vol-sequence, which contradicts the uniform hyperbolicity of ξ^α and finishes the proof of the lemma. □

Finally, if assertion (2) of Proposition A.1 does not hold, then we can select a sequence ξ^α in the family such that there exist unit vectors $v \in E_i^u(\xi^\alpha)$ and $w \in E_i^s(\xi^\alpha)$ satisfying

$$\frac{1}{2}\left\|\prod_{j=0}^{m} \xi_{j+i}^\alpha v\right\| \leq \left\|\prod_{j=0}^{m} \xi_{j+i}^\alpha w\right\|$$

for some i and m that satisfy some suitable inequalities as in [**29**, p. 536].

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Mañé showed that the following sequence has small angles between the stable and unstable spaces. The sequence has the form $\eta^t : \mathbb{Z} \to \mathrm{GL}(d)$ for some t and has period n_0 large, such that for $1 \le j < n_0 - 1$,

$$\eta_i^t = (I + T_i)\xi_i^\alpha (I + P_t), \quad \eta_{i+j}^t = (I + T_{i+j})\xi_{i+j}^\alpha$$

and

$$\eta_{i+n_0-1}^t = (I + S_t)(I + T_{i+n_0-1})\xi_{i+n_0-1}^\alpha.$$

Here, the families of transformations involved, P_t, S_t for $t \in [0, 1]$ and T_k, are the same as in [**29**, p. 537] and satisfy the following estimates for $\varepsilon > 0$ very small:

$$\|P_t\| \le \varepsilon, \quad \|S_t\| \le \varepsilon \quad \text{and} \quad \|T_j\| \le \varepsilon.$$

Hence $\det(I + T_j)$, $\det(I + S_t)$ and $\det(I + P_t)$ are close to 1, as close as we want. Thus, we can divide η^t by these determinants appropriately to obtain a vol-sequence with the same invariant spaces and therefore the same angles. Indeed, as in [**29**, p. 539], η satisfies the following properties:

$$\left\| \prod_{j=0}^{n_0-1} \eta_{i+j}^t \Big|_{E_i^s(\xi^\alpha)} \right\| \le (1 + \varepsilon_1)^{n_0} K \lambda^{n_0} < 1 \quad \text{and} \quad \left(\prod_{j=0}^{n_0-1} \eta_{i+j}^t \right) E_i^u(\xi^\alpha) = E_i^u(\xi^\alpha),$$

where $j \in \mathbb{Z}$ and $(1 + \varepsilon_1)\lambda < 1$, with other inequalities as in [**29**, p. 536]. This implies that the stable and unstable spaces of η^t are the corresponding subspaces for ξ^α at i. Hence, we can divide the stable part of η by the determinant of η^t, while preserving the unstable part of η. This will give the vol-sequence desired.

This vol-sequence gives the contradiction required, since it violates the previous lemma.

Remark A.1. Actually, the same argument could be used to show that if a vol-family is not uniformly hyperbolic, then it is not uniformly vol-hyperbolic. Indeed, it would have a non-hyperbolic sequence sufficiently close; in particular, the determinant would be almost 1. Thus, upon multiplying by the inverse of the determinant in a direction distinct from the non-hyperbolic direction, we obtain a non-hyperbolic vol-sequence close to the original, which is a contradiction.

We finish this appendix by commenting briefly on analogous results in the symplectic scenario. Recall that a manifold is symplectic if there is a non-degenerate closed 2-form ω over M^{2n}. In this case, we can define the set $\mathcal{F}_\omega^1(M)$ using only symplectic diffeomorphisms, just as we did using volume-preserving diffeomorphisms. We denote by $\mathrm{Diff}_\omega^1(M)$ the space of symplectic diffeomorphisms. It was proved by Newhouse in [**31**] that any element of $\mathcal{F}_\omega^1(M)$ is Anosov. In particular, if $m = \omega^n$, then it seems reasonable that a consequence of this result and our main result is that $\mathcal{F}_\omega^1(M) \subset \mathcal{F}_m^1(M) \subset \mathcal{F}^1(M)$.

Moreover, we think that Proposition 4.1 could be obtained directly. Recall that in canonical symplectic coordinates of $\mathbb{R}^{2n}$ we have a canonical map J such that $J^2 = -I$ with the property that a linear map A is symplectic if $A^* J A = J$. One could try to formalize the following argument to prove Lemma A.2. Let us give an example. Let η^α be a family close to ξ^α. Suppose that for some α, some entry of the matrix of $\prod_{j=0}^{m-1} \eta_j^\alpha|_{E^s}$ in an orthonormal basis $\{e_i\}$ of the stable subspace, where m is the period, is very large, say K.

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Observe that the trace of this product is less than n, since we are in the stable subspace. We could then use the basis $\{e_i, Je_i\}$ to make a perturbation, as before, by taking ζ_0 equal to η_0^α multiplied by a matrix like

$$\begin{pmatrix} I + tU & 0 \\ 0 & (I + tU^*)^{-1} \end{pmatrix},$$

where U has all of its entries equal to zero except the entry that is correctly placed, and in this entry we put a small number, say $3n/K$. Notice that, by the remark above, this matrix is symplectic and, moreover, since K is large, ζ is close to η. Set $\zeta_i = \eta_i$ for $1 \leq i < m$. Then if $t = 1$, the trace of $\prod_{j=0}^{m} \zeta_j|_{E^s}$ is bigger than n, and so for some t this perturbation must have an eigenvalue of modulus 1. A similar argument could be used to get the exponential contraction of η in the stable part.

It is worth mentioning that the domination part could be obtained by means of arguments from [**13, 25**]. In fact, Avila *et al* obtained in [**8**, Theorem 3.5] a direct proof that a partially hyperbolic non-Anosov diffeomorphism can be approximated by one with a non-hyperbolic periodic point under the hypothesis of *unbreakability*. Since there are many references and ideas about this subject, we will not elaborate further on it and refer the reader to those references for more details.

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