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The shear-rotation interval of twist maps

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Abstract. Given a twist map on the torus, we prove that the non-existence of a rotational invariant circle implies that the shear-rotation interval is non-trivial, i.e. the existence of orbits with non-zero mean acceleration. Under the same assumption, we prove the existence of infinitely many periodic orbits and positive entropy invariant measures with non-zero shear-rotation number. The method is variational, but relies on topological understanding of the shear-rotation interval of torus homeomorphisms homotopic to a Dehn twist.

1. *Introduction*

Twist maps have been studied as the simplest example of a non-integrable Hamiltonian dynamical system. In the existing literature, most constructions of different types of orbits and all constructions of invariant measures of twist maps (known to the author) deal with orbits and invariant measures with well-defined mean momentum, i.e. rotation number, hence with zero mean acceleration (see [**MF94**] for a general reference on Aubry–Mather theory; [**For96**] for construction of invariant measures). However, estimates by Bellissard and Vaienti [**BV92**] imply that oscillations in momentum are significant when a twist map is far from integrable, hence one may expect that accelerating orbits then fill a significant part of the phase space.

In this paper, we show that non-existence of a barrier to diffusion of a twist map (i.e. a rotational-invariant circle) is a sufficient and necessary condition for the existence of orbits and positive-entropy measures with non-zero mean acceleration (i.e. shear-rotation number). This strengthens the result by Mather [**Mat91**], who, under the same assumption, proved the existence of orbits with unbounded momentum.

1.1. *Twist maps on the torus.*

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus, x, y the canonical coordinates on $\mathbb{R}^2$, $\pi : \mathbb{R}^2 \to \mathbb{T}^2$, the canonical projection. Let $\Omega = dy \wedge dx$ be the canonical 2-form, $\alpha = y\,dx$ be the 1-form, and then $\Omega = d\alpha$. Given a map $f : \mathbb{T}^2 \to \mathbb{T}^2$, its lift is a continuous map $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\pi \circ F = f \circ \pi$.

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Definition 1.1. A C^1-diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a twist diffeomorphism on the torus, if there exists a lift F of f, $(x', y') = F(x, y)$, which is:

(T1) *exact symplectic:* i.e. there exists a C^2 function $\tilde{V} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\alpha - F^*\alpha = d\tilde{V}$, and such that $\tilde{V}(x, y) = \tilde{V}(x + 1, y)$ for all $(x, y) \in \mathbb{R}^2$;

(T2) *orientation-preserving:* i.e. $\det DF > 0$;

(T3) *a twist map:* i.e. $\partial y'/\partial x > 0$.

In the following we fix a lifting F of f satisfying (T1)–(T3).

Remark 1. If f is a twist diffeomorphism on the torus, not all liftings of f satisfy conditions (T1)–(T3); namely we can add $(n, 0)$ to the lifting, but not $(0, n)$. More specifically, condition (T1) allows only those liftings which have 'zero flux' in the y-direction, which is a standard requirement in Aubry–Mather theory (see [**MF94**]).

The standard family $(x, y) \mapsto (x + y + k \sin(2\pi x), y + k \sin(2\pi x))$ (where k is a real parameter) is an example of a twist diffeomorphism on the torus.

Note that (T1) implies that $F^*\Omega = \Omega$ (i.e. F and f are symplectic or area-preserving) and (T2) now implies $\det DF = 1$. Since a homotopy class of the torus homeomorphism is uniquely determined by the induced action on the fundamental group, the proof of the following is straightforward (see, for example, [**Sli99a**, Proposition 2.1]).

PROPOSITION 1.1. *Each twist diffeomorphism on the torus is homotopic to a Dehn twist, i.e. a toral automorphism represented in canonical coordinates by* $L = \left(\begin{smallmatrix} 1 & N \\ 0 & 1 \end{smallmatrix}\right)$, *where* $N \in \mathbb{N}$.

In particular, for all $z \in \mathbb{Z}^2$, $u \in \mathbb{R}^2$,

$$F(u + z) = F(u) + Lz. \tag{1}$$

We call $N = N(f)$ the twist number of f.

1.2. *Shear-rotation numbers and intervals.* We denote the F-orbit of $(x, y) = (x_0, y_0)$ by $(x_n, y_n)_{n\in\mathbb{Z}}$.

Definition 1.2. The rotation number of a point $(x, y) \in \mathbb{R}^2$ is

$$\rho(x, y) = \lim_{n\to\infty} \frac{x_n - x_0}{n}, \tag{2}$$

and the shear-rotation number is $\sigma(x, y) = \lim_{n\to\infty} (y_n - y_0)/n$, if the expressions are convergent. The shear-rotation set $\sigma(f)$ of a map f is the set of all (well-defined) $\sigma(x, y)$, $(x, y) \in \mathbb{R}^2$.

Aubry–Mather theory [**MF94**] showed that every $\rho \in \mathbb{R}$ can be represented as a rotation number of a point of a lift of a given twist diffeomorphism.

Relation (1) implies that $\sigma(x, y) = \sigma(x + a, y + b)$ for all $(a, b) \in \mathbb{Z}^2$, hence we can define the shear-rotation number of a point $(x, y) \in \mathbb{T}^2$. The theory of shear-rotation numbers for homeomorphisms of the torus homotopic to a Dehn twist was developed in [**Doe97, DM97**], the following is [**Doe97**, Theorem 6.7].

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THEOREM 1.2. (Doeff) *Given a homeomorphism f on the torus homotopic to a Dehn twist, its shear-rotation interval $\sigma(f)$ is a non-empty closed interval $\sigma(f) = [a, b]$.*

1.3. *The main result.* We say that a closed, f-invariant curve Γ on $\mathbb{T}^2$ is a rotational invariant circle, if Γ is homotopic to $y = 0$. We say that the rotation interval $\sigma(f)$ of a twist diffeomorphism f is trivial, if $\sigma(f) = \{0\}$.

THEOREM 1.3. *Given a twist diffeomorphism f on the torus, $\sigma(f)$ is not trivial if and only if rotational-invariant circles do not exist, and then $\sigma(f) = [a, b]$, $a < 0 < b$.*

The proof of one implication in Theorem 1.3 is simple and well known.

LEMMA 1.4. *If f is a twist diffeomorphism on the torus such that a rotational-invariant circle Γ exists, then $\sigma(f) = \{0\}$.*

Proof. Let Γ_i, $i \in \mathbb{Z}$, be the lifts of Γ, i.e. the curves such that $\pi\Gamma_i = \Gamma$ and $\Gamma_i = \Gamma_{i-1} + (0, 1)$ for all i. Let $\mathcal{A}_i$ be the closed set 'in between' Γ_i and Γ_{i+1} (note that, since by the Birkhoff theorem the Γ_i are graphs, $\mathcal{A}_i = \{(x, y + \delta) : (x, y) \in \Gamma_i, \delta \in [0, 1]\}$). (T1) implies that Γ_i are F-invariant, hence $\mathcal{A}_i$ are invariant. Since Γ is homotopic to $y = 0$, the set $\{y : (x, y) \in \mathcal{A}_i\}$ is bounded, and for all $(x, y) \in \mathcal{A}_i$, $\sigma(x, y) = 0$. Finally, note that $\cup\mathcal{A}_i = \mathbb{R}^2$. □

Remark 2. Doeff [**Doe97**, Lemma 6.1], proved a similar statement in a more general framework: the existence of a homotopically non-trivial invariant circle of a homeomorphism homotopic to a Dehn twist implies that the shear-rotation interval is a single point.

The converse of Lemma 1.4 and Theorem 1.3 will be proved in the next section.

1.4. *The variational principle.* The main technique in the proof will be variational. The following was defined and used in the well-known Aubry–Mather theory [**MF94**] and is a discrete-time analogue of the variational principle in Hamiltonian dynamics.

Definition 1.3. We say that a C^2 map $V : \mathbb{R}^2 \to \mathbb{R}^2$ is a generating function of a twist diffeomorphism on the torus f, if for all $(x, y) \in \mathbb{R}^2$,

$$y = -V_1(x, x'), \quad y' = V_2(x, x'), \tag{3}$$

where $(x, y) \mapsto (x', y')$ is a lift of f, and indices denote partial derivatives.

PROPOSITION 1.5. *Every twist diffeomorphism on the torus f with twist number N is generated by a generating function $V : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying for all $(x, x') \in \mathbb{R}^2$:*
(G1) $V(x + 1, x' + 1) = V(x, x)$;
(G2) $V_{12}(x, x') < 0$;
(G3) $V_1(x, x' + N) = V_1(x, x') - 1$, $V_2(x, x' + N) = V_2(x, x') + 1$.
Every C^2-map $V : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying (G1)–(G3) for some $N \in \mathbb{N}$ is a generating function of a twist diffeomorphism on the torus f with the twist number N.

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Given a twist diffeomorphism f, we define its generating function with $V(x, x') = \tilde{V}(x, y)$, where $(x', y') = F(x, y)$, and $\tilde{V}$ is the function specified in (T1); for details of the proof of Proposition 1.5 see [**MF94**] or [**KH95**]. We only note that the condition (G3) follows from (1) and (3); (G3) implies the following condition (usually assumed in the Aubry–Mather theory):

(G3*) *Superlinear growth* $\lim_{|\eta|\to\infty} V(x, x+\eta)/|\eta| = \infty$.

We say that a configuration $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$, $\mathbf{x} = (x_i)_{i\in\mathbb{Z}}$ is stationary, if for each $i \in \mathbb{Z}$, $V_2(x_{i-1}, x_i) + V_1(x_i, x_{i+1}) = 0$. We always assume the product topology on $\mathbb{R}^{\mathbb{Z}}$ and the induced topology on the set of stationary configurations $\mathcal{S}$, and the shift $s(\mathbf{x})_i = x_{i-1}$ is then a homeomorphism of $\mathcal{S}$. The map $\iota : \mathbb{R}^2 \to \mathcal{S}$, $(x, y) \mapsto (x_i)_{i\in\mathbb{Z}}$ now conjugates F and s, which justifies our choice to study the dynamical system $(\mathcal{S}, s)$ in the following.

COROLLARY 1.6. *Given a point* $(x, y) \in \mathbb{R}^2$ *and* $\iota(x, y) = \mathbf{x} = (x_i)_{i\in\mathbb{Z}}$, *then its shear-rotation number is*

$$\sigma(x, y) = \lim_{|n|\to\infty} \frac{x_n - x_{n-1}}{N \cdot n}, \tag{4}$$

where N *is the twist-number of the map* f.

Proof. It is easy to deduce from (G1), (G3) the existence of constants C_1, C_2 such that for all $x, x' \in \mathbb{R}^2$,

$$V_2(x, x') \in \left[\frac{x'-x}{N} + C_1, \frac{x'-x}{N} + C_2\right].$$

Now, applying (3) we get

$$\begin{aligned}\sigma(x, y) &= \lim_{|n|\to\infty} \frac{y_n - y_0}{n} = \lim_{|n|\to\infty} \frac{V_2(x_{n-1}, x_n) - V_2(x_{-1}, x_0)}{n} \\ &\in \left[\lim_{|n|\to\infty} \left(\frac{x_n - x_{n-1}}{N \cdot n} + \frac{C_1}{n}\right), \lim_{|n|\to\infty} \left(\frac{x_n - x_{n-1}}{N \cdot n} + \frac{C_2}{n}\right)\right] \\ &= \left\{\lim_{|n|\to\infty} \frac{x_n - x_{n-1}}{N \cdot n}\right\}. \end{aligned}$$

□

Given a configuration $\mathbf{x}$, we can define $\rho(\mathbf{x})$, $\sigma(\mathbf{x})$ using the right-hand sides of (2), (4); now ι preserves rotation and shear-rotation numbers.

1.5. *A priori estimate.* Proposition 1.5, ((G1) and (G3)) implies that the function $(x, y) \mapsto V_1(x, x+y) + V_2(x, x+y)$ is periodic in x, y with periods $1, N$ respectively, hence we can define $s(f) = \max_{(x,y)\in\mathbb{R}^2} |V_1(x, y) + V_2(x, y)|$. If $\mathbf{x} \in \mathcal{S}$,

$$|V_2(x_{i-1}, x_i) - V_2(x_i, x_{i+1})| = |V_2(x_{i-1}, x_i) + V_1(x_{i-1}, x_i)| \le s(f).$$

Now (3) and the definition of a shear-rotation number imply that $\sigma(f) \subset [-s(f), s(f)]$. For a standard map f, which has the generating function $V(x, y) = (x-y)^2/2 + k\sin(2\pi x)/2\pi$ (k is a real parameter), this estimate yields $\sigma(f) \subset [-k, k]$.

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2. *Construction of the accelerating orbit*

2.1. *Angenent sandwich lemma.* We say that a configuration $\mathbf{x} \in \mathbb{R}^{\mathbb{Z}}$ is substationary (respectively superstationary) or that $\mathbf{x} \in \mathcal{S}^-$ (respectively $\mathbf{x} \in \mathcal{S}^+$), if for all $i \in \mathbb{Z}$, $V_2(x_{i-1}, x_i) + V_1(x_i, x_{i+1}) \leq 0$ (respectively ≥ 0). We define the partial ordering on $\mathbb{R}^{\mathbb{Z}}$ with

$$\begin{aligned} \mathbf{x} \geq \mathbf{y} \quad &\Leftrightarrow \quad \forall i,\ x_i \geq y_i, \\ \mathbf{x} \gg \mathbf{y} \quad &\Leftrightarrow \quad \forall i,\ x_i > y_i. \end{aligned}$$

The following lemma was used by Angenent [**Ang90**] in his proof of the positive topological entropy of twist maps; it will be a useful tool here as well.

LEMMA 2.1. (Angenent) *Assume that* $\mathbf{u} \in \mathcal{S}^-$, $\mathbf{v} \in \mathcal{S}^+$, *and* $\mathbf{u} \leq \mathbf{v}$. *Then there exists a stationary configuration* $\mathbf{z} \in \mathcal{S}$, $\mathbf{u} \leq \mathbf{z} \leq \mathbf{v}$.

Proof. The configuration $\mathbf{z}$ is the supremum of the set of all configurations $\mathbf{x} \in \mathcal{S}^-$, such that $\mathbf{x} \leq \mathbf{v}$. Now (G2) easily implies that $\mathbf{z} \in \mathcal{S}$ (for details see [**Ang90**, Theorem 4.2]). □

The usual method of construction of substationary (superstationary) configurations is 'copying' and 'pasting' maxima (respectively minima) of pairs of stationary configurations. Precisely, we have the following.

LEMMA 2.2. *If a configuration* $\mathbf{x}$ *satisfies the following: for each* i, *there exist two stationary configurations* $\mathbf{v}^i$, $\mathbf{w}^i$ *such that* $u_j = \max\{v^i_j, w^i_j\}$ *(respectively* $u_j = \min\{v^i_j, w^i_j\}$*) for* $j = i-1, i, i+1$, *then* $\mathbf{x} \in \mathcal{S}^-$ *(respectively* $\mathbf{x} \in \mathcal{S}^+$*).*

The proof is a simple consequence of (G2) and is left to the reader (or, alternatively, see [**Ang90**] or [**Sli99c**]).

2.2. *Translations.* Given $(p, q) \in \mathbb{Z}^2$, we define a translation $T_{p,q}$ on the space of configurations $\mathbb{R}^{\mathbb{Z}}$ with $T_{p,q}\mathbf{x} = \mathbf{x}^*$,

$$x^*_i = x_{i-q} + p.$$

Given $(a, b) \in \mathbb{Z}^2$, We define a translation $\Theta_{a,b}$ on the space of configurations $\mathbb{R}^{\mathbb{Z}}$ with $\Theta_{a,b}\mathbf{x} = \mathbf{x}^*$,

$$x^*_i = x_i + a + bNi.$$

(G1) and (G3) respectively imply that $T_{p,q}$, $\Theta_{a,b}$ leave the set $\mathcal{S}$ invariant. Rotation number and shear-rotation number are invariants of $T_{p,q}$; $\Theta_{a,b}$ leaves shear-rotation numbers invariant, while $\rho(\Theta_{a,b}(\mathbf{x})) = \rho(\mathbf{x}) + bN$.

If we plot configurations in the Aubry diagram (i.e. we plot x_i as a function of i), $T_{p,q}$ translates a configuration q to the right and p upwards, and the translation $\Theta_{a,b}$ adds a straight line with the slope bN to a configuration $\mathbf{x}$.

Let

$$\Psi^{i,n}_{p,q} = T_{p,q+i}\Theta_{0,n}T_{0,-i} \tag{5}$$

(we add a straight line $x_j \mapsto x_j + n(j-i)N$ to a configuration $\mathbf{x}$, translate it q to the right and p upwards); and then $\mathcal{S}$ is an invariant set of $\Psi^{i,n}_{p,q}$.

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2.3. *Aubry–Mather theory and connecting configurations.* The results listed here are proved assuming (G1), (G2), (G3*), and therefore apply in our situation. For a definition of a minimizing configuration and a proof of Theorem 2.3 below, see, for example, [**MF94**]. We denote the set of minimizing configurations by $\mathcal{M}$; $\mathcal{M} \subset \mathcal{S}$ and $\mathcal{M}$ is $T_{p,q}$-invariant.

THEOREM 2.3. (Aubry–Mather)

(i) *For every $\rho \in \mathbb{R}$, there exists $\mathbf{x} \in \mathcal{M}$, such that $\rho(\mathbf{x}) = \rho$. Every $\mathbf{x} \in \mathcal{M}$ has a well-defined rotation number.*

(ii) *Given $\mathbf{x} \in \mathcal{M}$ with rotation number ρ, $p \in \mathbb{Z}$ and $q \in \mathbb{N}$,*

$$p/q > \rho \quad \Rightarrow \quad T_{p,q}\mathbf{x} \gg \mathbf{x},$$
$$p/q < \rho \quad \Rightarrow \quad T_{p,q}\mathbf{x} \ll \mathbf{x}.$$

(iii) *If $\mathbf{x}, \mathbf{y} \in \mathcal{M}$, then they intersect at most once, i.e. there exists $i \in \mathbb{Z} \cup \{\infty\}$ such that $(x_j \leq y_j \Leftrightarrow j \leq i)$ or $(x_j \geq y_j \Leftrightarrow j \leq i)$.*

Mather [**Mat91**] proved the existence of configurations which connect minimizing configurations in the same Birkhoff region of instability. We will use a slightly more detailed formulation from [**Sli99c**, Proposition 5.4]. We say that $\mathbf{u}, \mathbf{v}$ are α- (or ω-) asymptotic, if $\lim_{i\to-\infty} |u_i - v_i| = 0$ (respectively $\lim_{i\to\infty} |u_i - v_i| = 0$).

THEOREM 2.4. *Assume that rotational-invariant circles do not exist, and choose $\mathbf{x}, \mathbf{y} \in \mathcal{M}$ with irrational rotation numbers, $\rho(\mathbf{x}) < \rho(\mathbf{y})$ (or $\rho(\mathbf{x}) > \rho(\mathbf{y})$). Then there exists a stationary configuration $\mathbf{z}$, α-asymptotic to $\mathbf{x}$, and ω-asymptotic to $\mathbf{y}$, and $\mathbf{z} \gg \mathbf{x}, \mathbf{y}$ (respectively $\mathbf{z} \ll \mathbf{x}, \mathbf{y}$). Furthermore, $\mathbf{z}$ is the smallest configuration greater than $\mathbf{x}, \mathbf{y}$ (respectively $\mathbf{z}$ is the greatest configuration smaller than $\mathbf{x}, \mathbf{y}$).*

2.4. *Construction of accelerating configurations.*

LEMMA 2.5. *Assume that rotational-invariant circles do not exist. Then there exist stationary configurations $\mathbf{x}$, $\mathbf{x}'$ such that $\sigma(\mathbf{x}) > 0$, $\sigma(\mathbf{x}') < 0$.*

Proof. We construct $\mathbf{x}$, $\sigma(\mathbf{x}) > 0$; construction of $\mathbf{x}'$, $\sigma(\mathbf{x}') < 0$ is analogous. Choose a minimizing configuration $\mathbf{u}^0$ with irrational rotation number $\rho > 0$, and denote by $\mathbf{u}^1 = \Theta_{0,2}\mathbf{u}^0$, a minimizing configuration with rotation number $\rho + 2N$. Theorem 2.4 implies the existence of an orbit $\mathbf{y}^0$, α-asymptotic to $\mathbf{u}^0$, ω-asymptotic to $\mathbf{u}^1$, $\mathbf{y}^0 \gg \mathbf{u}^0, \mathbf{u}^1$. Theorem 2.3(iii) implies the existence of $i \in \mathbb{Z}$ such that for each $k \leq i$, $u_k^0 \geq u_k^1$; and for each $k > i$, $u_k^0 < u_k^1$. For given q, $p \in \mathbb{Z}$, we define:

$$\mathbf{u}^{2,q,p} = \Psi_{p,q}^{i,1}\mathbf{u}^0,$$

and then $\mathbf{u}^1$ and $\mathbf{u}^{2,q,p}$ intersect at some $j(q, p)$, i.e. for each $k \leq j(q, p)$, $u_k^1 \leq u_k^{2,q,p}$; and for each $k > j(q, p)$, $u_k^1 > u_k^{2,q,p}$. Denote by $\mathbf{y}^{1,q,p}$ the connecting orbit α-asymptotic to $\mathbf{u}^1$, ω-asymptotic to $\mathbf{u}^{2,q,p}$, and $\mathbf{y}^{1,q,p} \ll \mathbf{u}^1, \mathbf{u}^{2,q,p}$. We define:

$$\mathbf{y}^{2,q,p} = \Psi_{p,q}^{i,1}\mathbf{y}^0,$$
$$\mathbf{y}^{-1,q,p} = \Psi_{-p,-q}^{i,-1}\mathbf{y}^1.$$

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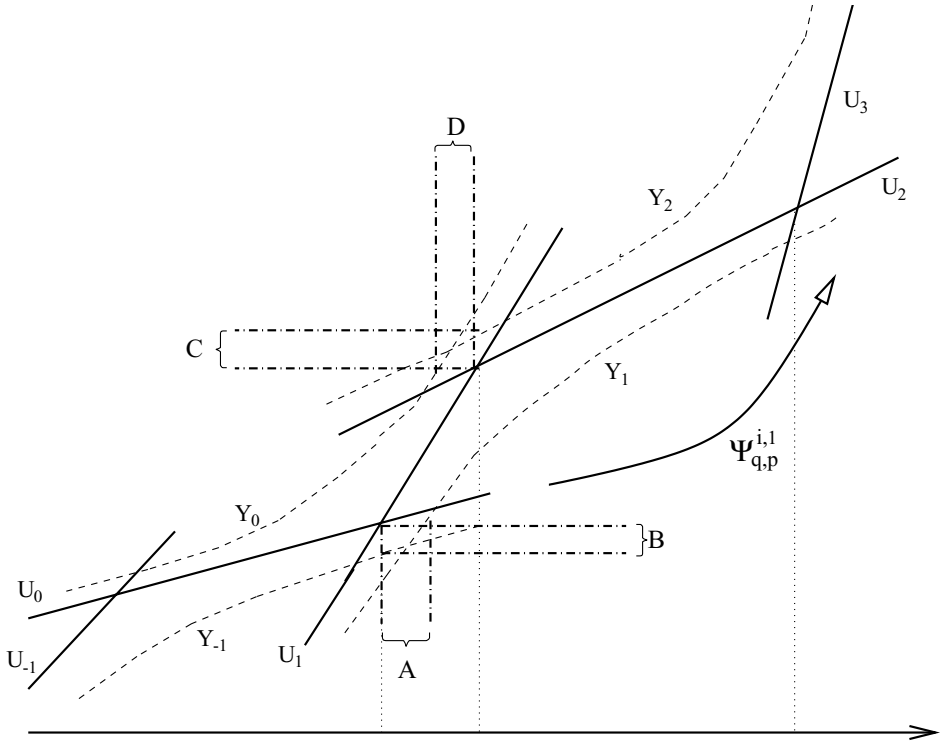


FIGURE 1. Proof of Lemma 2.5.

We want to choose $q,\ p > 0$, large enough, such that we can control where $\mathbf{y}^0$, $\mathbf{y}^{2,q,p}$, and $\mathbf{y}^{-1,q,p}$, $\mathbf{y}^{1,q,p}$ intersect, hence we define the constants (see Figure 1):

$$\begin{aligned}
B^{q,p} &= \sup\{|u_k^0 - y_k^{-1,q,p}|,\ i-2 \le k \le i+3\}\\
A^{q,p} &= \sup\{|u_k^1 - y_k^{1,q,p}|,\ i-2 \le k \le i+3\}\\
C^{q,p} &= \sup\{|u_k^{2,q,p} - y^{2,q,p}|,\ j(q,p)-2 \le k \le j(q,p)+3\}\\
D^{q,p} &= \sup\{|u_k^1 - y_k^0|,\ j(q,p)-2 \le k \le j(q,p)+3\}.
\end{aligned}$$

Step I. Claim: there exists $p, q > 0$ *such that* $A^{q,p}, B^{q,p}, C^{q,p}, D^{q,p} \le 2$. Since $\mathbf{y}^0$ is ω-asymptotic to $\mathbf{v}^1$, there exists j_0 such that for each $p, q > 0$,

$$j(q,p) \ge j_0 \tag{6}$$

implies $D^{q,p} \le 2$. Since $\mathbf{y}^{2,q,p}$ is α-asymptotic $\mathbf{u}^{2,q,p}$, and is a translate of $\mathbf{y}^0$, there exists q_0 such that for each $p, q > 0$,

$$q - j(q,p) \ge q_0 \tag{7}$$

implies $C^{q,p} \le 2$. We find i_1 such that for each $k \le i_1$, $|v_k^1 - y_k^{1,0,0}| < 1$. Since two translates of $\mathbf{u}^1$ do not intersect (Theorem 2.3(ii)), we can find $q_1,\ p_1 > 0, q_1 > i + 3 - i_1$,

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such that $\mathbf{u}^1 \gg T_{p_1,q_1}\mathbf{u}^1 \gg T_{-1,0}\mathbf{u}^1$, and therefore for each $k \le i+3$,

$$|u_k^1 - y_k^{1,q_1,p_1}| = |u_k^1 - (T_{p_1,q_1}\mathbf{y}^{1,0,0})_k| \le |u_k^1 - (T_{p_1,q_1}\mathbf{u}^1)_k| + |u_{k-q_1}^1 - y_{k-q_1}^{1,0,0}| \le 2,$$

and $\mathbf{u}^1 \gg T_{q_1,p_1}\mathbf{u}^1 \gg \mathbf{y}^{1,q_1,p_1}$. Since $\mathbf{y}^{1,q,p}$ is the smallest configuration smaller than $\mathbf{u}^1$ and $\mathbf{u}^{2,q,p}$ (Theorem 2.4) and

$$\mathbf{y}^{1,q,p} \gg \mathbf{y}^{1,q_1,p_1}, \tag{8}$$

we conclude that $\mathbf{u}^1 \gg \mathbf{y}^{1,q,p} \gg \mathbf{y}^{1,q_1,p_1}$, hence $A^{q,p} \le 2$. Denote by $j_1 = j(q_1, p_1)$, and the definition of $j(q, p)$ shows that for each $p, q > 0$,

$$j(q, p) > j_1 \tag{9}$$

implies (8) and $A^{q,p} \le 2$. In an analogous way we deduce the existence of q_1, such that for each $p, q > 0$,

$$q - j(q, p) \ge q_1 \tag{10}$$

implies $B^{q,p} \le 2$. Theorem 2.3(ii) implies that $(q'-q)/(p'-p) > \rho+1$ is equivalent to $\mathbf{u}^{2,q',p'} > \mathbf{u}^{2,q,p}$ and $j(q', p') \ge j(q, p)$. We now choose $p''/q'' \ne p'/q'$ which satisfy (6), (9), i.e. $j(q', p'), j(q'', p'') > j_0, j_1$, and $j(q'', p'') \ge j(q', p')$. Now any p/q such that $(q''-q)/(p''-p) > \rho+1, (q-q')/(p-p') > \rho+1$, with q large enough, satisfies (6), (7), (9), (10) and the claim is true.

We fix p, q, and drop the indices p, q.

Step II. Claim: The configurations $\mathbf{y}^{-1}$, $\mathbf{y}^1$ *intersect between* $i-2$ *and* $i+3$, *and the configurations* $\mathbf{y}^0$, $\mathbf{y}^2$ *intersect between* $j-3$, $j+3$. Since $\mathbf{u^0}$ is minimizing, for each k, $|u_k^0 - u_0^0 - \rho k| \le 1$. Applying relations $\mathbf{y}^1 \ll \mathbf{u}^1$ and $B \le 2$, we get for $k = i-2$

$$y_k^{-1} \ge u_k^0 - 2 \ge u_i^0 - (i-k)\rho - 3 \ge u_i^1 - (i-k)\rho - 3 \ge u_k^1 + (i-k)(\rho+2) - 1 - (i-k)\rho - 3 = u_k^1 > y_k^1.$$

Analogously we conclude that $y_{i+3}^{-1} < y_{i+3}^1$, $y_{j-2}^0 < y_{j-2}^2$, $y_{j+3}^0 > y_{j+3}^2$. Therefore $\mathbf{y}^{-1}$, $\mathbf{y}^1$ intersect between $i-2$, $i+3$, say for the first time at $\tilde{i}$, and $\mathbf{y}^0$, $\mathbf{y}^2$ intersect between $j-2$, $j+3$, say for the first time at $\tilde{j}$.

Step III. Construction of the sandwich (Figure 1). We define

$$u_k^- = y_k^1 \quad \text{for } k = \tilde{i}, \tilde{i}+1, \dots, \tilde{i}+q-1$$
$$u_k^+ = y_k^0 \quad \text{for } k = \tilde{j}-q, \tilde{j}-q+1, \dots, \tilde{j}-1,$$

and extend them to (unique) configurations $\mathbf{u}^-$, $\mathbf{u}^+$ satisfying $\mathbf{u}^- = \Psi_{p,q}^{i,1}\mathbf{u}^-$, $\mathbf{u}^+ = \Psi_{p,q}^{i,1}\mathbf{u}^+$. Now the definition of Ψ implies that $u_{n+q}^- = u_n^- + p + (n-i)N$, and (4) implies that

$$\sigma(\mathbf{u}^-) = \lim_{|n|\to\infty} \frac{1}{nN}\left(\frac{1}{q}(u_{n+q}^- - u_n)\right) = \lim_{|n|\to\infty} \frac{p+nN-iN}{qnN} = \frac{1}{q}.$$

The definitions of $\mathbf{u}^-$, $\mathbf{u}^+$ imply that

$$\sup_{k\in\mathbb{Z}} |u_k^+ - u_k^-| = \sup_{\tilde{i}-q\le k\le\tilde{i}+q} |u_k^+ - u_k^-| < \infty,$$

therefore $\sup_{k\in\mathbb{Z}} |u_k^+ - x_k| < \infty$ and $\sigma(\mathbf{x}) = 1/q > 0$. □

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Theorem 1.3 now follows from Lemmas 1.4 and 2.5.

3. *Further results*

3.1. *Periodic orbits.*

Doeff [**Doe97**] proved the following theorem.

THEOREM 3.1. (Doeff) *Given a homeomorphism f on the torus homotopic to a Dehn twist with shear-rotation interval $[a, b]$, then every rational number $p/q \in (a, b)$ can be represented with a periodic point (x, y), $\sigma(x, y) = p/q$.*

Now Theorem 1.3 implies the following.

COROLLARY 3.2. *Given a twist diffeomorphism f on the torus without a rotational-invariant circle, there exist infinitely many periodic orbits with non-zero shear-rotation number.*

Note that all periodic minimizing configurations and, more generally, all the periodic orbits constructed in the Aubry–Mather theory have shear-rotation number 0.

3.2. *Positive entropy accelerating measures.*

We can naturally define the shear-rotation number of a probability measure on $\mathbb{T}^2$ with $\sigma(\mu) = \int_{\mathbb{T}^2}(y' - y)\, d\mu(x, y)$, where $(x', y') = F(x, y)$. If μ is f-ergodic, μ-a.e. point on $\mathbb{T}^2$ has shear-rotation number $\rho(\mu)$.

The following result is true in a more general setting of homeomorphisms of the torus homotopic to a Dehn twist (see, for example, [**DM97**, proof of Theorem 3.1]).

THEOREM 3.3. *Assume that f is a twist diffeomorphism on the torus. Then for every ρ in the interior of the rotation interval, there exists an f-ergodic measure with the shear-rotation number ρ and positive entropy.*

Remark 3. One can apply a result of J. Kwapisz and strengthen the previous result, namely construct an f-ergodic measure with positive entropy, supported on an invariant compact set, every point of which has shear rotation number ρ (see [**GM99**, Theorem 5.1 and Corollary 5.2]).

3.3. *Connecting orbits.*

For the benefit of the reader, in the following we quote related results on shear-rotation sets and accelerating orbits of a twist diffeomorphism f on the torus. The following is the main result from [**Sli99a**] proving, in particular, existence of an orbit with two arbitrary asymptotic shear-rotation numbers.

THEOREM 3.4. *For any two $\sigma_1, \sigma_2 \in \sigma(f)$, there exists $\mathbf{x} \in \mathcal{S}$ such that*

$$\lim_{n\to-\infty} \frac{x_n - x_{n-1}}{n \cdot N} = \sigma_1, \quad \lim_{n\to\infty} \frac{x_n - x_{n-1}}{n \cdot N} = \sigma_2.$$

For any $\sigma \in \sigma(f)$ and $\rho \in \mathbb{R}$, there exists $\mathbf{x} \in \mathcal{S}$, α-asymptotic to a minimizing configuration with rotation number ρ, and satisfying $\lim_{n\to\infty}(x_n - x_{n-1})/n \cdot N = \sigma$.

This suggests that the complexity of the set of orbits with non-zero shear-rotation number is comparable to the complexity of a Birkhoff region of instability as described in [**Mat91**].

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3.4. *Anti-integrable limit.* Assume that $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies (G1)–(G3), that $P : \mathbb{R} \to \mathbb{R}$ is a 1-periodic C^2-function such that $P'(x) = 0$ implies $P''(x) \neq 0$ and that f_t, $t > 0$, is the twist diffeomorphism on the torus generated by $(x, x') \mapsto tP(x) + V(x, x')$. The following is an application of the idea and techniques of anti-integrable limit introduced in [**Aub92**]; the proof is in [**Sli99b**, Corollary 3.20].

THEOREM 3.5. *Given $C > 0$, there exists t_0 such that for all $t \geq t_0$, $[-C, C] \subset \sigma(f_t)$.*

3.5. *Numerical calculation of the shear-rotation interval.* When pulling (or pushing) a generalized Frenkel–Kontorova chain from one end, starting from a minimizing configuration, the asymptotic configuration will be the one with the largest (respectively the smallest) shear-rotation number [**Sli98**, Theorem 5.6]; this could be used as a numerical procedure for calculation of $\sigma(f)$.

4. *Conclusion*

The break-up of the last rotational-invariant circle in a one-parameter family of twist diffeomorphisms is a rather spectacular event, where dynamically rich invariant sets described earlier are created. We conjecture that the main result, as formulated in the introduction, could be generalized to systems with more degrees of freedom. We also believe that further study of accelerating invariant measures could give an insight into the metric entropy of twist maps which are far from integrable.

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