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The homology of MSpin

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1. *Introduction*

The mod two homology of MSpin, the Spin-cobordism Thom spectrum, has a rich algebraic structure. We will describe it explicitly as a comodule algebra, and give some applications to the ring structure of the Spin-cobordism ring.

As was originally the case for most of the other Thom spectra for the classical groups, the structure of $H_*(\mathrm{MSpin}; Z/2)$ has long been known both as an algebra and as a comodule over the dual Steenrod algebra A, but not as an A-(comodule-)algebra. There are several reasons for wishing to see how the two structures are interwoven. First, the connection turns out to be particularly attractive and easy to describe, and yields further information about the Spin-cobordism ring. Second, the homology is tightly structured as an A-algebra, and contains powerful information about the topology of the ring spectrum. For example, in the case of MSU, computing and utilizing the full A-algebra structure was instrumental in determining the homotopy type of $\mathrm{MSU}_{(2)}$ as a wedge of indecomposables [7]. Finally, the successful display of a relatively simple form for the A-algebra $H_*(\mathrm{MSpin}; Z/2)$ is, we believe, necessary to any similar attempt at the homology A-algebra structure of the next connective Thom spectrum $MO\langle 8\rangle$. Some of what we do here can be carried over to $MO\langle 8\rangle$. There is considerable interest in $MO\langle 8\rangle$ as a tool in obstruction theory [**4**, **5**], but its study presents new difficulties, and understanding all the features of easier cases, such as MSpin, in great detail, seems a prerequisite to success with it.

An outline of the paper is as follows. In Section 2, we first apply a general technique developed in [**8**, **9**], which shows that our homology A-algebra is merely an 'extension' over A of a comodule algebra N over a very small quotient of A. Roughly, this quotient is the dual of the subalgebra generated by the Steenrod squares which annihilate the Thom class. In our case, this is the dual A_1 of the subalgebra $\langle Sq^1, Sq^2\rangle$. Having simplified the algebra to a study over A_1, we then use the A_1^* methods of [**2**], and their determination of the homotopy type of $\mathrm{MSpin}_{(2)}$, to begin closing in on the A_1-algebra structure. In Section 3 we describe the (co-)action on the indecomposable quotient, and in Section 4 we complete the description of the A_1-algebra structure. We do this by producing polynomial generators which build in a simple fashion into split cocyclic A_1-summands in N, each summand corresponding to one of the three types of indecomposable spectra in the homotopy type of $\mathrm{MSpin}_{(2)}$. In Section 5 we present some applications to the ring structure of the Spin-cobordism ring. Our results show that the ring structure is quite complex.

The reader can easily obtain a capsule impression of the entire A_1-algebra structure from the pictures in Theorems 4·4–4·6. They illustrate the situation in blocks surrounding dimensions divisible by 8. There are three cases. First, if $8n$ is a two-power, there

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is an indecomposable split $Z/2$ summand there (indicating a $bo_{(2)}$ wedge summand in $\mathrm{MSpin}_{(2)}$), and a summand of indecomposables just above it dual to $A_1^*/A_1^*Sq^3$ (indicating a $bo_{(2)}\langle 2\rangle$ summand in $\mathrm{MSpin}_{(2)}$). Second, if $8n$ is a sum of two two-powers, the same split summands appear, but the $A_1^*/A_1^*Sq^3$ involves some simple twofold decomposables. The additional indecomposables combine with three explicit decomposables to produce a free A_1 summand (indicating a $KZ/2$ wedge summand in $\mathrm{MSpin}_{(2)}$, the first appearing in dimension 22, with $n = 3$). Finally, if $8n$ is any other 8-multiple, there is still an analogous $A_1^*/A_1^*Sq^3$ summand, and an analogous free A_1 summand (building down this time exactly to the square of a previously chosen indecomposable), but the indecomposable $Z/2$ summand in dimension $8n$ disappears, and instead a second free A_1 summand builds downward from the indecomposable there. Our proof in this last case involves an intricate induction, but the pictures reveal a remarkably simple overall structure.

2. *Reduction to* A_1

We will reduce the problem to analysing a certain comodule algebra over the quotient $A_1 = Z/2[\zeta_1, \zeta_2]/(\zeta_1^4, \zeta_2^2)$ of A (A_1 is dual to the subalgebra generated by Sq^1 and Sq^2, and $\zeta_i = \bar{\xi}_i$).

From now on, all coefficient groups will be $Z/2$ and, as usual, $b_n \in H_nBO$ denotes the standard polynomial generator dual to w_1^n.

Recall [**11**, pp. 230, 292] that the natural maps between classifying spaces induce isomorphisms

$$H^*BSO \cong H^*BO/(w_1)$$

and

$$H^*\mathrm{BSpin} \cong H^*BSO/(w_2, Q_iw_2 : i \geqslant 0) \cong Z/2[w_n : n \neq 1, 2^i + 1],$$

where $Q_i \in A^{2^{i+1}-1}$ is the primitive in dimension $2^{i+1} - 1$ in the Steenrod algebra A^*. The final isomorphism is merely an algebra map, but the first two are Hopf algebra isomorphisms. By Borel's Theorem [6] the homologies and cohomologies of all three spaces are bipolynomial Hopf algebras. We freely identify their homologies with the homologies of their Thom spectra via the Thom isomorphism, and also identify all these with their (monomorphic) images in H_*MO and H_*BO.

In [**9**] it was noted that the map $PH_n\mathrm{BSpin} \to PH_nBO$ of coalgebra primitives is an isomorphism if $n \neq 1, 2^j + 1$. Thus

$$\{p_1^4, p_2^2, p_3, \ldots, p_j, \ldots\} \subset H_*\mathrm{MSpin},$$

where p_j denotes the coalgebra primitive in $H_{2j-1}BO$ (transported to H_*MO). We let M denote $H_*\mathrm{MSpin}$, P denote the ideal in M generated by the polynomial generators $\{p_1^4, p_2^2, p_3, \ldots, p_j, \ldots\}$, and N be the quotient M/P. In [**9**] it was also shown that

LEMMA 2·1. *N is a quotient A_1-comodule of M, and the composite $M \xrightarrow{\psi} A \otimes M \xrightarrow{1\otimes\pi} A \otimes N$ is an A-algebra isomorphism onto $A \square_{A_1} N$.*

Thus the analysis of M reduces to understanding the A_1-algebra structure of N.

We begin our analysis of N by noting that

LEMMA 2·2. *$A \square_{A_1} N$ is isomorphic to the 'twisted' A-algebra $(A \square_{A_1} Z/2) \otimes^{\Delta} N$.*

Here $\otimes^{\Delta}$ denotes the multiplicative coaction on the tensor product, dual to the diagonal action of A^*.

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Proof. This follows from [**10**, lemma 3·1] provided the A_1-algebra structure of N is induced from an A-algebra structure. The argument used in [**3**] shows that $Z/2[p_1^4, p_2^2, p_3, \ldots, p_j, \ldots] \subset H_*\,\mathrm{BSpin}$ is the image of $H_*\,\Omega^2\Sigma^2\,\mathrm{BSpin}^7$, and thus P is a sub A-comodule of $H_*\,\mathrm{BSpin}$. The Thom isomorphism is an A_1-comodule map on $H_*\,\mathrm{BSpin}$, so it induces an A-algebra structure on N extending the A_1-coaction. ∎

In [**2**] Anderson, Brown, and Peterson prove a splitting theorem for A_1^* modules. We will use this to show that N is a direct sum of cocyclic A_1 comodules, by applying their theorem to the dual N^*. Recall that a graded A_1^* module L has *isomorphic homologies* if the natural map

$$H(L) = (\mathrm{Ker}\,Q_0 \cap \mathrm{Ker}\,Q_1/\mathrm{Im}\,Q_0 \cap \mathrm{Im}\,Q_1) \xrightarrow{\lambda_i} H(L, Q_i)$$

is an isomorphism for $i = 0, 1$.

Lemma 2·3. *N^* has isomorphic homologies.*

Proof. First we note that the dual isomorphism $(A^*/A^*\bar{A}_1^*) \otimes^{\Delta} N^* \cong M^*$ carries $Z/2 \otimes N^*$ via the identity into M^*. Since the generators of the ideal P are coalgebra primitives, any square in M^*, in particular $w_{2i}^2, i \geqslant 2$, lies in N^*. Moreover w_{2i}^2 is in the kernel of Q_0 and Q_1.

Using the notation and results of [**2**, Theorems 6·1 and 6·3], we have

$$H_*(M^*, Q_0) \cong Z/2[w_{2i}^2 : i \neq 2^j] \otimes Z/2[x_{2^i} : i \geqslant 2]$$

and

$$H_*(M^*, Q_1) \cong Z/2[w_{2i}^2 : i \neq 2^j - 1] \otimes Z/2[y_{2^i-2} : i \geqslant 3].$$

(We have reindexed their x's and y's to reflect their degrees.) Since x_{2^i} and y_{2^i-2} are indecomposable in M^*, this shows that $\{w_{2i}^2 : i \geqslant 2\}$ represent algebraically independent elements in $H_*(M^*, Q_k)$ for $k = 0$ and 1, and hence also in $H_*(N^*, Q_k)$. Together, the Künneth theorem and the known ranks of $H_*(A^*/A^*\bar{A}_1^*, Q_k)$ [**2**, proposition 6·6] then show that

$$H_*(N^*, Q_k) \cong Z/2[w_{2i}^2 : i \geqslant 2] \tag{1}$$

for $k = 0$ and 1.

We now adapt the argument in [**11**, p. 308f] to show that $N^* \subset M^*$ has isomorphic homologies.

Since $Z/2[w_i^2 : i \geqslant 2] \subset \mathrm{Ker}\,Q_0 \cap \mathrm{Ker}\,Q_1 \cap N^*$, each λ_i is epic. Suppose $x \in (\mathrm{Ker}\,Q_0 \cap \mathrm{Ker}\,Q_1 \cap N^*)$ and $\lambda_0([x]) = 0$. From (1) we have $x = Q_1 y + f(\{w_{2i}^2 : i \geqslant 2\})$ where $y \in N^*$ and f is a polynomial. Since $x = Q_0 z$ with $z \in N^*$, this gives $f(\{w_{2i}^2 : i \geqslant 2\}) \in Q_0 N^* + Q_1 N^*$. But the Wu formulae show that, in $M^* = Z/2[w_n : n \neq 1, 2^j + 1]$, $Q_0 M^* + Q_1 M^*$ lies in the ideal generated by the odd Stiefel–Whitney classes, and thus (although w_{2^j+1} is not necessarily zero in M^*), it lies in the ideal generated by $\{w_{2i+1} : i \neq 0, 2^j\}$. Thus f is zero, so $x \in Q_0 N^* \cap Q_1 N^*$, and therefore λ_0 is an isomorphism. Now since λ_1 is epic, with both groups having the same graded rank, it too is an isomorphism. ∎

If we follow [**1**] by letting J (for 'joker') denote the dual of $\Sigma^{-2}A_1^*/A_1^*Sq^3$, we can now prove

Theorem 2·4. *As an A_1-comodule, N is a sum of suspensions of the comodules $Z/2$, A_1, and J. The $Z/2$ summands all lie in dimensions $\equiv 0$ (mod 8), and the J summands all begin in dimensions $\equiv 2$ (mod 8).*

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Proof. Lemma 2·3 shows that N^* satisfies the hypothesis of Theorem 4·2 of [2], producing the splitting. The sparseness of $H_*(N^*, Q_0)$ ensures, as in [2, Lemma 5·5], that the fourth type of summand in their theorem does not appear. The splitting of $\mathrm{MSpin}_{(2)}$ given in their Theorem 2·2 shows that the $Z/2$ and J summands occur only in the dimensions claimed. ∎

3. *The indecomposable quotient*

In this section the A_1-coaction on the indecomposable quotient QN is computed. We should note that QN is a quotient A_1-comodule since the augmentation ideal is an A_1-subcomodule of N. Henceforth the word 'primitive', without further qualification, will refer to this A_1-coaction.

First we will describe the behaviour on indecomposables in homology of the induced maps

$$\mathrm{MSpin} \xrightarrow{i'} MSO \xrightarrow{i''} MO$$

and their composite i. The map Qi''_* of the indecomposable quotients is an isomorphism precisely in dimensions which are not a power of two, since the coalgebra primitives in the ideal (w_1) are precisely the two-powers of w_1. Likewise, since the generators of the ideal $(w_2, Q_i w_2 : i \geqslant 0)$ are coalgebra primitives, the coalgebra primitives the ideal contains are precisely their two-powers, and so Qi'_* is an isomorphism precisely in dimensions n for which $\alpha(n)$, the number of ones in the dyadic expansion of n, is greater than two. The same holds for the composite Qi_*. We will also frequently use the fact that squares in H_*BO lie in the image of H_*BSO, and squares in H_*BSO lie in the image of $H_*\mathrm{BSpin}$. This is because the ideals (w_1) and $(w_2, Q_i w_2 : i \geqslant 0)$ are generated by coalgebra primitives, and the coproducts are cocommutative.

Now we are ready to begin defining a set of indecomposables in $H_*\mathrm{MSpin}$ which have a simple A_1-coaction in the quotient QN. We choose indecomposables y_n in dimensions n with $\alpha(n) \leqslant 2$ as follows. Let $y_{2j} = b^4_{2j-2}$, and, if $i > j > 0$, let $y_{2^i+2^j} = (x_{2^{i-1}+2^{j-1}})^2$, where the x_n are the polynomial generators for H_*MSO described in [8]. If, on the other hand, n is a dimension with $\alpha(n) > 2$, and y_n is any indecomposable in $H_n\mathrm{MSpin}$, we have $i_*(y_n) = b_n +$ other terms. We let Sq^1 and Sq^2 also denote the operations in homology dual to Sq^1 and Sq^2 in cohomology. First we note that since the Thom isomorphism is an A_1-map on $H_*\mathrm{MSpin}$, we may compute Sq^1 and Sq^2 using the action in $H_*\mathrm{BSpin}$. In $H_*\mathrm{BSpin}$ it is enough to know the leading term b_n, since Sq^1 kills b_1, and Sq^2 kills b_2 and b_1^2. We find that

LEMMA 3·1. *Let y_n be indecomposable in $H_n\mathrm{MSpin}$, $\alpha(n) > 2$. Then*

(1) *$i_*(Sq^1 y_n)$ is indecomposable in H_*MO iff n is even.*

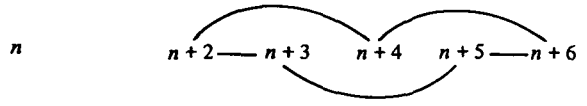
(2) *$i_*(Sq^2 y_n)$ is indecomposable in H_*MO iff $n \equiv 0, 1 \bmod 4$.*

We can now combine this lemma with the y_n we have explicitly defined for $\alpha(n) \leqslant 2$ to completely describe the coaction on the indecomposable quotient. There are three cases, as follows.

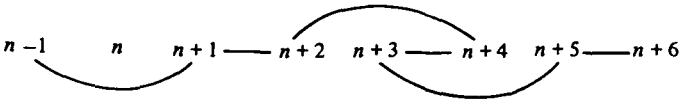
THEOREM 3·2. *Let $n \equiv 0 \bmod 8$. The Sq^1 and Sq^2 actions on N modulo decomposables are described by the following pictures, where $n+k$ denotes the class of the indecomposable in dimension $n+k$.*

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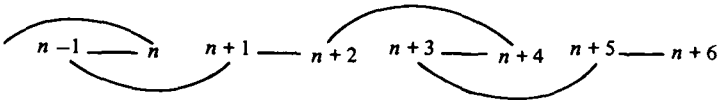
(*a*) $n = 2^j$



(*b*) $\alpha(n) = 2$



(*c*) $\alpha(n) > 2$



where the lines denote a Sq^1 coaction and the curves a Sq^2 coaction. All the nonzero coactions are shown in the pictures.

Proof. Part (c) and all of part (*b*) except the coaction in dimension n follow from Lemma 3·1. For this case note that $n = 2^i + 2^j$, $i > j > 2$ and we chose $(x_{2^{i-1}+2^{j-1}})^2$ as the indecomposable in dimension n. Then Sq^1 on this is zero, and Sq^2 is $(x_{2^{i-1}+2^{j-1}-1})^2$ since (from [8]) $Sq^1 x_{2k} = x_{2k-1}$ for $k \neq 2^j$. Thus $Sq^2 y_n$ is decomposable in $H_*\text{MSpin}$ since $\alpha(n-2) > 2$. This leaves (*a*). Now $b_{2^{j-1}}^4$ is primitive, and Sq^1 on the indecomposable $(x_{2^{j-1}+1})^2$ in dimension $n+2$ is exactly zero. Since $H_{2^{j+2}}(H_*(\text{MSpin}), Sq^1) = 0$, and Sq^1 takes decomposables to decomposables, Sq^1 of the indecomposable in dimension $n+3$ must be indecomposable. Now since $\alpha(n+3)$, $\alpha(n+5)$, and $\alpha(n+6)$ are all greater than 2, Lemma 3·1 says that the indecomposable in dimension $n+6$ coacts via Sq^1 to an indecomposable and then via Sq^2 to the indecomposable in dimension $n+3$. Since $Sq^2Sq^2 = Sq^1Sq^2Sq^1$ the rest follows. ∎

4. *The polynomial generators*

We will now construct polynomial generators for N with specific A_1 coaction. The coaction formulae for the indecomposable in dimension n depend only on $\gamma(n) = n - 2^k$, where k is the largest integer for which $n - 2^k \geqslant 0$. We will use the information on indecomposables from the previous section, the Sq^1 homology of N, and its splitting into a direct sum of comodules of type either $Z/2$, J, or A_1. (We will construct specific summands of these types. The construction gives only submodules; that these are indeed summands follows from Theorems 2·4 and 3·2, since each has an indecomposable at the top.) The construction will proceed in three steps, corresponding to the three cases in Theorem 3·2.

The information that we will need from the splitting is contained in the following three lemmas. The proof of each of these is by examining the structure of each summand, noting that $Z/2$ summands begin only in dimensions $\equiv 0 \bmod 8$ and the J summands only in dimensions $\equiv 2 \bmod 8$.

LEMMA 4·1. *Let $a \in N_n$, $n \not\equiv 0 \bmod 8$, with $Sq^1 a = 0 = Sq^2 a$. Then there is a $b \in N_{n+4}$ with $Sq^2Sq^2 b = a$. Furthermore, if also $n \not\equiv 2 \bmod 8$, then there is a $c \in N_{n+6}$ with $Sq^2Sq^2Sq^2 c = a$.*

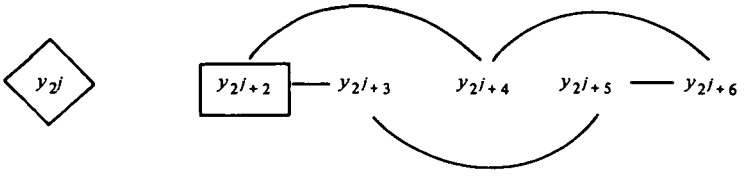
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LEMMA 4·2. *Let $a \in N_n$, $n \not\equiv 0, 2 \bmod 8$, with $Sq^2a = 0$. Then there is a $b \in N_{n+3}$ with $Sq^2Sq^1b = a$.*

LEMMA 4·3. *Let $a \in N_n$, $n \not\equiv 0, 2, 3 \bmod 8$, with $Sq^2a = 0 = Sq^2Sq^1a$. Then there is a $b \in N_{n+5}$ with $Sq^2Sq^1Sq^2b = a$.*

We are now ready for the construction up from a power of two. The coaction formulae will be described in pictures similar to the last section, with the full names of the generators written. In addition, the primitives at the bottom of $Z/2$ and J summands will be enclosed in diamond shaped and rectangular boxes, respectively.

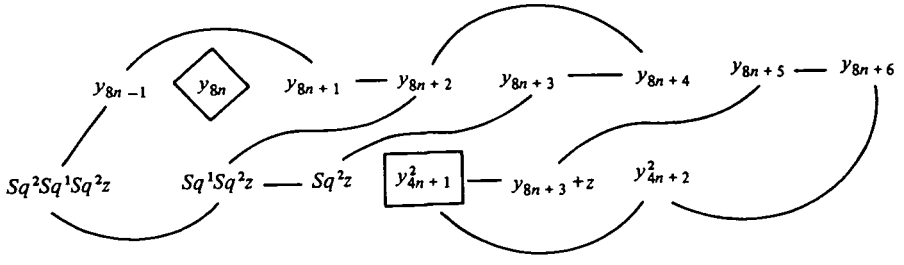
THEOREM 4·4. *There are polynomial generators $y_n \in N_n$, $n = 2^j$, $2^j + k$, $2 \leqslant k \leqslant 6$, $j \geqslant 3$, such that y_{2^j} gives a $Z/2$ summand, and $\{y_{2^j+k}: 2 \leqslant k \leqslant 6\}$ gives one J summand. In particular:*



Proof. Choose $y_{2^j} = b_{2^{j-3}}^4$, $y_{2^j+2} = (x_{2^{j-1}+1})^2$, and $y_{2^j+4} = (x_{2^{j-1}+2})^2$. Then y_{2^j} and y_{2^j+2} are primitive. Hence by Lemma 4·1 there is a $b \in N_{2^j+6}$ with $Sq^2Sq^2b = y_{2^j+2}$. Both b and Sq^2b must be indecomposable. Let $d = Sq^2b + y_{2^j+4}$. Then $Sq^2d = 0$. By Lemma 4·2 there is a c with $Sq^2Sq^1c = d$. Let $y_{2^j+6} = b + Sq^1c$, and define $y_{2^j+5} = Sq^1y_{2^j+6}$ and $y_{2^j+3} = Sq^2y_{2^j+5}$. Then $Sq^2y_{2^j+6} = y_{2^j+4}$ and the rest follows. |

We now proceed to construct the remaining generators in blocks of eight. There are two cases, depending on whether or not the block contains a sum of two powers of two.

THEOREM 4·5. *There are polynomial generators $\{y_{8n+k}, 1 \leqslant k \leqslant 6, n \geqslant 3, \alpha(8n) = 2\}$ which build into a $Z/2$ summand in dimension $8n$, a J summand on y_{4n+1}^2, and an A_1 summand starting in dimension $8n-2$. Let $z = y_{4n+2}y_{4n+1}$. Then we have the following picture:*



Proof. We have defined y_k for $\gamma(k) \leqslant 6$. To construct the block around dimension $2^j + 2^i$ we proceed inductively. Fix i and j, with $j > i \geqslant 3$, and assume inductively that the previous blocks have been constructed. Let $8n = 2^j + 2^i$. Then $4n = 2^{j-1} + 2^{i-1}$, so the blocks around $4n$ have been constructed, either inductively (if $i > 3$) or in the previous theorem. Let $z = y_{4n+2}y_{4n+1}$. If n is odd, then $i = 3$, and y_{4n+2} is at the top of a J summand starting in dimension $2^{j-1} + 2$. If n is even, $y_{4n+2} = Sq^2y_{4n+4}$, where y_{4n+4} is at the top of an A_1 summand, and in both cases $Sq^1y_{4n+2} = y_{4n+1}$, and $Sq^2y_{4n+1} = y_{4n-1}$. Hence $Sq^1z = y_{4n+1}^2$ and $Sq^2Sq^1Sq^2z = y_{4n-1}^2 + Sq^2y_{4n+2}Sq^1y_{4n-1}$ are

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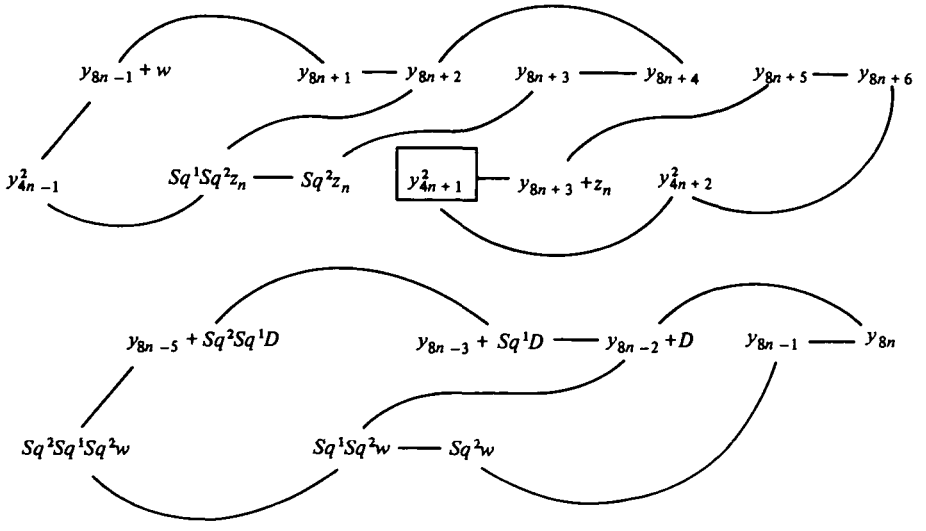
both primitive. These will be at the bottom of a J and an A_1 summand respectively. By Lemma 4·1 there is an $a \in N_{8n+6}$ with $Sq^2Sq^2a = y_{4n+1}^2$. Since $Sq^2y_{4n+2}^2 = y_{4n+1}^2$, by Lemma 4·2 we can adjust a to obtain a y_{8n+6} with $Sq^2y_{8n+6} = y_{4n+2}^2$. Define $y_{8n+5} = Sq^1y_{8n+6}$, and $y_{8n+3} = Sq^2y_{8n+5} + z$. Then $Sq^1y_{8n+3} = Sq^2Sq^2y_{8n+6} + Sq^1z = 0$. Also $Sq^2y_{8n+3} = Sq^2z$. Since $Sq^1y_{8n+3} = 0$ and $H_{8n+3}(N, Sq^1) = 0$ there is a y_{8n+4} with $Sq^1y_{8n+4} = y_{8n+3}$. Define $y_{8n+2} = Sq^2y_{8n+4}$, and $y_{8n+1} = Sq^1y_{8n+2}$. Define $y_{8n-1} = Sq^2y_{8n+1}$. Then $Sq^2y_{8n-1} = 0$ and $Sq^1y_{8n-1} = Sq^1Sq^2Sq^1Sq^2y_{8n+4} = Sq^2Sq^1Sq^2z = y_{4n-1}^2 + Sq^1y_{4n-1}Sq^2y_{4n+2}$ from above. Since $\alpha(4n) = 2$, $Sq^1y_{4n} = 0$, and there is no decomposable hitting y_{4n-1}^2 via Sq^1. Hence y_{8n-1} and (using Theorem 3·2) all the other y_i's defined above must be indecomposable. We have left $y_{2^j+2^i}$ for last, since this gives a $Z/2$ summand. Note that $a = (x_{2^{j-1}+2^{i-1}})^2$ is indecomposable in N, $Sq^1a = 0$ and $Sq^2a = (x_{2^{j-1}+2^{i-1}-1})^2 = b$. Now $Sq^1b = 0 = Sq^2b$, so Lemma 4·1 gives a $c \in N$ with $Sq^2Sq^2Sq^2c = b$. Let $y_{2^j+2^i} = a + Sq^2Sq^2c$, which is indecomposable since Sq^2Sq^2c is decomposable by Theorem 3·2. |

We now proceed to the general case. The first step is to construct a useful subset of the positive integers. Let $W_2 = \{n > 0 : \alpha(n) > 2\}$, and let $W \subset W_2$ be defined inductively by $k \in W$ if either k is even or $k = 2m+1$, where $m \in W$. An alternate description of W is

$$\{k > 0 : k \equiv 2^j - 1 \bmod 2^{j+1}, \alpha(k-(2^j-1)) > 2 \quad \text{for some} \quad j \geqslant 0\}.$$

The j in this description is unique, so we define $\phi(k) = (k-(2^j-1))/2^j$, and $\psi(k) = 2^j$, for $k \in W$. Note that if $k \in W$ then $(2k+1) \in W$, $\phi(2k+1) = \phi(k)$, and $\psi(2k+1) = 2\psi(k)$. Then we have the following theorem.

THEOREM 4·6. *There are polynomial generators* $\{y_{8n+k}, -1 \leqslant k \leqslant 6, \alpha(n) > 2\}$ *which build into a* J *summand on* y_{4n+1}^2, *an* A_1 *summand on* y_{4n-1}^2, *and an* A_1 *summand building down from* y_{8n}. *Let* $z_n = y_{4n+2}y_{4n+1}$ *if* $n \notin W$, *and* $y_{4n+2}y_{4n+1} + y_{8n+3-4\phi(n)\psi(n)}(y_{2\phi(n)}^2 + Sq^2y_{4\phi(n)+2})^{\psi(n)}$ *if* $n \in W$. *Let* $w = y_{4n}y_{4n-1}$, *and let* D *be any decomposable solution to* $Sq^2D = Sq^1Sq^2w$. *Then* $Sq^2Sq^1Sq^2z_n = y_{4n-1}^2$ *and we have the following picture*:



Proof. We use induction on n. The first block to be constructed corresponds to $n = 7$. Note that $Sq^1y_{4n+2} = y_{4n+1}$, so y_{4n+1}^2 is primitive. Just as in the proof of the previous theorem, we can find $y_{8n+6} \in N_{8n+6}$ with $Sq^2y_{8n+6} = y_{4n+2}^2$. A straightforward

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computation with Theorem 3·2 shows that there is no decomposable hitting y_{4n+2}^2 via Sq^2, and therefore y_{8n+6} must be indecomposable. Define $y_{8n+5} = Sq^1 y_{8n+6}$, and $y_{8n+3} = Sq^2 y_{8n+5} + z_n$. This fills in the J summand.

Now $Sq^1 z_n = y_{4n+1}^2$, so $Sq^1 y_{8n+3} = 0$. Since $H_{8n+3}(N, Sq^1) = 0$, there is an indecomposable y_{8n+4} with $Sq^1 y_{8n+4} = y_{8n+3}$. Since $Sq^2 Sq^2 y_{8n+5} = 0$, we have $Sq^2 y_{8n+3} = Sq^2 z_n$. Thus we fill in the A_1 summand down from y_{8n+4}. The bottom of this summand is clearly $Sq^2 Sq^1 Sq^2 z_n$, and we need this to be y_{4n-1}^2.

There are three cases to consider: $n \notin W$; $n \in W$, n even; and $n \in W$, n odd. We compute the last, the others being similar but easier. Since n is odd and in W, $n = 2m+1$, with $m \in W$. Then

$$Sq^2 Sq^1 Sq^2 (y_{4n+2} y_{4n+1}) = y_{4m+1}^2 y_{4m+2}^2 + (Sq^2 y_{8m+5})^2$$
$$= y_{8m+3}^2 + y_{8m+3-4\phi(m)\,\psi(m)}^2 (y_{2\phi(m)}^2 + Sq^2 y_{4\phi(m)+2})^{2\psi(m)}.$$

Also $$Sq^2 Sq^1 Sq^2 y_{8n+3-4\phi(n)\,\psi(n)} = Sq^2 Sq^1 Sq^2 y_{16m+11-8\phi(m)\,\psi(m)} = y_{8m+3-4\phi(m)\,\psi(m)}^2.$$

Since $$y_{2\phi(m)}^2 + Sq^2 y_{4\phi(m)+2}$$

is primitive, we have the result.

Finally, we turn to defining y_{8n}. Since $Sq^1 y_{4n} y_{4n-1} = y_{4n-1}^2$, then $Sq^1 y_{8n-1} = 0$. Hence there is an indecomposable c in N_{8n} with $Sq^1 c = y_{8n-1}$. By Theorem 3·1, $Sq^2 c$ is indecomposable, and so $Sq^2 c = y_{8n-2} + d$, for some decomposable d in N_{8n-2}. Now $Sq^2 Sq^2 c = y_{4n-2}^2 + Sq^2 d = Sq^1 Sq^2 Sq^1 c = Sq^1 Sq^2 (y_{4n} y_{4n-1})$. Thus $Sq^2 d = y_{4n-2}^2 + Sq^1 Sq^2 (y_{4n} y_{4n-1})$. If D is any decomposable with $Sq^2 D = Sq^2 d$ then by Lemma 4·3 we can find an element a in N_{8n+1} with $Sq^2 Sq^1 a = D + d$, and then $Sq^2(c + Sq^1 a) = y_{8n-2} + D$. ∎

5. *Primitives in N and the ring structure of* Ω^{Spin}

From the results of Section 4 one can easily identify the following primitive elements of N.

(1) $$p(n) = \begin{cases} y_{4n} & \text{for } n \text{ even}, \ \alpha(n) \leqslant 2 \\ y_{2n}^2 + Sq^2 y_{4n+2} & \text{for } n \text{ even}, \ \alpha(n) > 2. \end{cases}$$

(2) $$p(n, n) = \begin{cases} y_{4n}^2 & \text{for } n = 2^j + 1, \ j \geqslant 1 \\ y_{2n}^4 & \text{for } n \text{ odd}, \ n \neq 2^j + 1. \end{cases}$$

(3) $$p(m, n) = \sum_{i=-2}^{2} a_{4m+i} a_{4n-i} \quad \text{for } m, n \text{ odd},$$

where, for any odd l, $\{a_{4l+i} : -2 \leqslant i \leqslant 2\}$ are the classes we have defined in Theorems 4·4, 4·5, and 4·6 which generate the split J summand building down from y_{4l+2}.

(4) $$q(2m+1) = \begin{cases} y_{2^{j+3}+2} & \text{for } m = 2^j \\ y_{4m+1}^2 & \text{for } m \neq 2^j. \end{cases}$$

Furthermore the p's are all $Z/2$ summands, and the q's are at the bottom of J summands.

Let $h: \Omega_*^{\mathrm{Spin}} \to H_* \mathrm{MSpin}$ be the Hurewicz map. So the image of h lies in PN. Let $I \subset \Omega^{\mathrm{Spin}}$ be the ideal upon which all KO-Pontryagin numbers vanish [2]. Then h restricted to I is an isomorphism onto the set of all primitives in N which are at the bottom of an A_1 summand, or equivalently, the image of $Sq^2 Sq^2 Sq^2$.

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In [2] the ring Ω^{Spin}/I was computed as follows (with subscripts reindexed to reflect degree): Let R be the integral polynomial ring

$$Z[b_4, \ldots, b_{4i}, \ldots, Y_{10}, \ldots, Y_{8i+2}, \ldots, \alpha]$$

modulo the relations

(1) $2\alpha = 0$
(2) $2Y_{8i+2} = 0$
(3) $\alpha Y_{8i+2} = 0$
(4) $\alpha^3 = 0$
(5) $Y_{8i+2}b_4 = b_{8i+4}\alpha^2$
(6) $Y_{8i+2}b_{8j+4} = b_{8i+4}Y_{8j+2}$
(7) $Y_{8i+2}Y_{8j+2} = 0.$

Then Ω^{Spin}/I is isomorphic to the subring of R generated by $\{Y_{8i+2}\}$, α, $\{b_{8i}\}$, $\{2b_{8i+4}\}$, $\{(b_{8i+4}b_{8j+4})\}$.

From the proof in [2] it is clear that all the relations except the last hold in Ω^{Spin}. The generators in Ω^{Spin} can be chosen so that $h(Y_{8i+2}) = q(2i+1)$, $h(2b_{8i+4}) = 0$, $h(b_{8i}) = p(2i)$ and $h(b_{8i+4}b_{8j+4}) = p(2i+1, 2j+1)$. The partitions used to index the primitive p's and q's indicate exactly how they correspond to the $\text{MSpin}_{(2)}$ wedge summands described in [2].

From the form of $q(2i+1)$ we see:

THEOREM 5·1. *The Y_{8i+2} generate a $Z/2$ polynomial subring of Ω_*^{Spin}.*

Moreover we can discern an interesting feature of the elements $\{(b_{8i+4}b_{8j+4})\}$ in Ω_*^{Spin}. In general the two elements $(b_{8i+4}b_{8j+4})(b_{8k+4}b_{8l+4})$ and $(b_{8i+4}b_{8l+4})(b_{8j+4}b_{8k+4})$ are different in Ω^{Spin}, since the formulae for $p(2i+1, 2j+1)p(2k+1, 2l+1)$ and $p(2i+1, 2l+1)p(2j+1, 2k+1)$ are quite different. However, in Ω_*^{Spin}/I the two elements clearly agree, so they differ by an element of order two. Thus we have

THEOREM 5·2. *The subalgebra of Ω_*^{Spin} generated by $\{(b_{8i+4}b_{8j+4})\}$ is not an integral polynomial ring, nor is it the obvious polynomial ring with deleted generators (i.e. it is not the subalgebra of $Z[b_{8i+4}: i \geqslant 1]$ generated by $\{b_{8i+4}b_{8j+4}: i, j \geqslant 1\}$).*

Theorems 5·1 and 5·2 show which relations in Ω^{Spin}/I exist merely due to factoring by I.

Finally, we see that if $B \subset \Omega^{\text{Spin}}$ is the subring generated by the Y_{8i+2}'s, b_{8i}'s, and $(b_{8i+4}b_{8j+4})$'s, then there are no more relations among them in $Z/2$ homology than in $\Omega^{\text{Spin}} \otimes Z/2$. In other words:

THEOREM 5·3. *$h: B \otimes Z/2 \to PN$ is a monomorphism whose image is the subring of the polynomial ring $Z/2[y_i]$ generated by the $p(n)$, $p(m, n)$, and $q(2i+1)$.*

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