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POTENTIALISM DEMODALIZED

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Abstract. Potentialism holds that certain objects are successively generated in an incompletable process. While it is natural to analyze this view modally, there are theorems that connect the resulting modal analysis of potentialism with the non-modal languages of ordinary mathematics. By extending this approach to plural languages, this article proves a far stronger result about definitional equivalence. This opens the door to a new and entirely non-modal explication of potentialism, using a restricted plural logic. Some advantages of this "demodalized" explication are discussed. It is certainly simpler and more user-friendly than the extant modal analysis, as illustrated by an application to potentialist set theory. More ambitiously, I suggest that the explication might also enable potentialists to sidestep the tricky question of which mathematical objects are actual.

§1. Introduction. Aristotle famously held that only the notion of potential infinity is legitimate, not that of actual infinity. Consider the natural numbers. According to Aristotle, a natural number exists provided it is instantiated by some collection of concrete objects. He takes the natural numbers to be potentially infinite. No matter how many such numbers have been instantiated, it is possible to instantiate an even larger natural number, say, by cutting one of the instantiating objects in two. But Aristotle denies that it is possible to complete this process of instantiating ever larger natural numbers. Thus, the natural numbers are *merely* potentially infinite or, as I will also put it, *incompletable*.

A broadly Aristotelian view of infinity was dominant for more than two millenia. The decisive turning point came only with Cantor's pioneering work, starting in the 1870s, which convinced much of the mathematical community that it is coherent and mathematically fruitful to accept a host of different actual infinities, the natural numbers being just one example.

When confronted with the set-theoretic paradoxes in the 1890s, though, even Cantor suggests that certain collections must be regarded as merely potential, not actual. I have in mind his famous distinction between "consistent" and "inconsistent" multiplicities. In a well-known 1899 letter to Dedekind, he writes that

> a multiplicity can be such that the assumption that *all* of its elements 'are together' leads to a contradiction, so that it is impossible to

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> conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities*. [...] If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into '*one* thing', I call it a *consistent multiplicity* or a 'set'. [9, pp. 931–932]

The collection of all sets provides a paradigm example of an inconsistent multiplicity, which cannot be regarded as "one finished thing." For no matter how many sets have been defined, these sets provide a basis for defining yet another set.

Thus, despite their profound differences, Aristotle's view of the natural numbers and Cantor's conception of the totality of all sets have one thing in common. Both are examples of *potentialism*, loosely defined as the view that certain mathematical objects are successively generated (or defined) and that this generation is incompletable.[1] Let *actualism* be the opposing view, which finds no place for potentiality or incompletability in mathematics.

These loose characterizations prompt some hard questions. Here are three particularly important ones:

(Q1) How should potentialism be analyzed?
(Q2) What is the correct logic, according to potentialism, for ordinary non-modal reasoning about an incompletable domain?
(Q3) What are the broader mathematical consequences of adopting a potentialist view?

A natural approach to (Q1) is to use the resources of modal logic to formalize the informal glosses provided above. Such modal analyses of potentialism have been developed in [17] (see also [14, 15, 25]) and will be reviewed in the next section.

(Q2) and (Q3) require some clarification. Mathematical reasoning is ordinarily formalized in non-modal languages. But we can translate from a non-modal language to a corresponding modal one by rendering '$\forall$' and '$\exists$' as '$\Box\forall$' and '$\Diamond\exists$', respectively. The two questions ask what logical and mathematical principle are validated in the non-modal language, relative to this translation. Let us start with the case of first-order logic. A "mirroring theorem" shows that potentialists who accept a certain classical modal logic in their modal analysis of potentialism are entitled to use classical first-order logic in their non-modal language.[2] Thus, as far as first-order logic is concerned, it makes no difference whether the domain is actual or merely potential.

One might suspect, therefore, that the difference between actualism and potentialism is "merely philosophical," with no effect on logic or mathematics.[3] That would be incorrect. The two views have a sharp disagreement expressible in the non-modal language of ordinary mathematics. The disagreement emerges as soon as we go beyond

[1] The form of potentialism discussed in this article is *liberal* potentialism in the sense of [17, sec. 3]. Liberal potentialism contrasts with *strict* potentialism, which adds to the claim about successive generation of *objects* an analogous claim about *truths*, namely, that truths too are successively grounded or "made true" as the generative process unfolds.

[2] By contrast, strict potentialists (in the sense of the previous note) typically only accept an intuitionistic modal logic. If so, it turns out, they are only entitled to intuitionistic first-order logic in the non-modal language.

[3] See, e.g., [6, 13, 24].

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first-order logic and consider the logic of plurals, which encapsulates reasoning about *many objects considered simultaneously*. Plural logic thus allows us to talk about what Russell [22, sec. 74] called "classes as many"—that is, the many objects that might constitute a class, without yet reifying that class.[4] When a domain is actual—as actualists insist any domain is—there is a plurality of all of its objects. It has been known for some time, however, that plural logic needs to be restricted when reasoning non-modally about an incompletable or merely potential domain.[5] For there is no plurality of all objects in such a domain, essentially, because there is no single possible world at which all possible objects are available.

The question of the correct plural logic for non-modal reasoning about an incompletable domain still lacks a complete answer. This article provides the desired answer. A plural mirroring theorem determines the restricted plural logic that is appropriate for non-modal reasoning about a merely potential domain. Better yet, this restricted plural logic, slightly tweaked, is shown to be *definitionally equivalent* to the modal logic of plurals typically used to analyze potentialism. From a purely formal point of view, there is thus no difference between the restricted plural logic and the usual modal logic of plurals.

This finding opens up some new possibilities concerning (Q1). Alongside the familiar modal analysis of potentialism, we can now develop an entirely non-modal explication based on a restricted (or "critical") logic of plurals.[6] This non-modal explication of potentialism has indisputable practical advantages. It is simpler and more user-friendly than the extant modal analyses. Most mathematicians and many philosophers find formalized modal reasoning unfamiliar and hard. As an illustration of the new, non-modal alternative, I present a pleasingly simple "demodalized" version of potentialist set theory.

A harder question is whether the non-modal explication of potentialism has philosophical advantages as well. The answer will depend on how the non-modal language is interpreted. One option is to begin with the modal analysis of potentialism and then invoke the definitional equivalence to provide a handy non-modal shorthand for statements that remain modal in character. Although this maneuver offers the mentioned practical benefits, it would be philosophically sterile. For better or worse, all the commitments made in the modal language would remain unchanged. A more ambitious option is to develop a *self-standing* non-modal view that still retains the core potentialist commitment to successive generation of objects in an incompletable process. The definitional equivalence ensures that there is no formal obstacle to this project: formally speaking, we already know how to "demodalize" potentialism. Yet the modal and non-modal versions of potentialism might differ in philosophically significant ways. For example, the non-modal version might enable potentialists to sidestep the tricky question of which mathematical objects are actual.

[4] Thus, this "logic" has considerable mathematical content. Indeed, from a mathematical point of view, plural logic over set theory can be regarded as class theory, where the classes in question are understood combinatorially, without any definability requirement.

[5] See [14] and [17, sec. 7]. Concerning plural logic, see [3] for a seminal contribution, as well as [11] for a recent survey and critical discussion.

[6] The definitional equivalence provides a bridge between my use of modal analyses in some works (e.g., [15, 16]) and my use of an analysis based on Critical Plural Logic in other works (e.g., [11]): there is no logical conflict between these analyses. I discuss their philosophical relation in §7.

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I proceed as follows. First, I review the mentioned strategy for determining the correct first-order logic for non-modal reasoning about merely potential domains (§2). Next, I extend this strategy to plurals. I define the systems to be compared (§3) and explain the plural mirroring theorem and the stronger result about definitional equivalence (§4). Then, building on this analysis, I identify my preferred plural logic for non-modal reasoning about a merely potential domain (§5). Finally, I explain how this restricted plural logic enables a non-modal explication of potentialism (§6), whose potential advantages are discussed (§7). All proofs that are primarily of technical interest are relegated to appendices.

§2. The modal analysis of potentialism. The modal analysis of potentialism is natural and fairly straightforward. Consider the Aristotelian claim that the natural numbers are merely potentially infinite. This claim has a positive half, which can be formalized as follows:

$$\Box\forall m\Diamond\exists n\textsc{Succ}(m,n), \tag{1}$$

where $\textsc{Succ}(m,n)$ states that n is the immediate successor of m. Next, the negative half of the claim states that it is impossible for all of the natural numbers to exist simultaneously:

$$\neg\Diamond\forall m\exists n\textsc{Succ}(m,n). \tag{2}$$

Clearly, then, the modal language provides a nice way to distinguish the merely potentially infinite from the actually infinite.

Another interesting case is the Cantorian view of sets canvassed above. Any multiplicity of objects that "exist together" can, according to Cantor "be gathered together into '*one* thing"', namely, their set. Let us use plural variables to represent consistent multiplicities, that is, multiplicities of objects that "exist together." Then it is natural to take Cantor to be committed to the following view:

$$\Box\forall xx\Diamond\exists y\textsc{Set}(xx,y), \tag{3}$$

where '$\textsc{Set}(xx,y)$' states that y is the set whose elements are xx. As Cantor is well aware, though, it is impossible to conceive of the multiplicity of all sets as "one finished thing." This naturally corresponds to the claim that it is impossible for the domain to be closed under the transition from a consistent multiplicity to the corresponding set:

$$\neg\Diamond\forall xx\exists y\textsc{Set}(xx,y). \tag{4}$$

I claimed above that the Aristotelian and the Cantorian views are both examples of potentialism, despite their important differences. I can now be more specific. There is a striking similarity between the pair of Aristotelian theses, (1) and (2), and the pair of Cantorian ones, (3) and (4). The first member of each pair states that necessarily it is possible to continue some generative process, while the second member of each pair states that it is impossible to complete this process. The first members of each pair have the logical form $\Box\forall\Diamond\exists$, whereas the second members of each pair have the logical form $\neg\Diamond\forall\exists$.

Although the modal analysis of potentialism is immensely natural, there is no getting around the fact that mathematicians tend to prefer non-modal languages. Despite his invocation of modal notions, even Cantor did not attempt a rigorous development of

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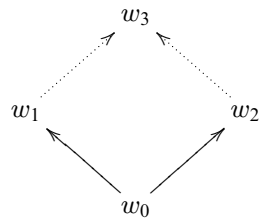
his ideas using modality. So potentialists need a way to translate from the non-modal languages ordinarily used in mathematics to the modal language in which our modal analysis of potentialism is provided.

Fortunately, an attractive translation is available. At the heart of potentialism lies the idea that the existential quantifier of ordinary non-modal mathematics has an implicit modal character. When Aristotelians say that a number has a successor, they mean that it *potentially* has a successor, that is, that it is *possible* to generate a successor. This suggests that the appropriate translation of $\exists$ is $\Diamond\exists$. Likewise, $\forall$ should be translated as $\Box\forall$. To be clear: I am not proposing these translations as a contribution to empirical semantics. Rather, they are proposed as a useful rational reconstruction, which explicates an implicit potentialist conception of the language of mathematics.

Thus understood, the quantifiers of ordinary non-modal mathematics function as devices for generalizing over absolutely all objects, not only the ones available at some stage, but also any that we may go on to generate. In our modal language, these strong generalizations are effected by the strings $\Box\forall$ and $\Diamond\exists$. Although strictly speaking a composite of a modal operator and a quantifier proper, these strings can be shown to behave logically just like quantifiers ranging over all entities at all possible worlds. I therefore refer to them as *modalized quantifiers*.

The proposal is thus that each quantifier of the non-modal language be translated as the corresponding modalized quantifier. Each connective is translated as itself. Let us call this the *potentialist translation*, and let $\varphi^{\Diamond}$ represent the translation of φ. We say that a formula is *fully modalized* just in case all of its quantifiers are modalized. Clearly, the potentialist translation of any non-modal formula is fully modalized.

It is generally agreed that the modal logic used to analyze any form of potentialism should be at least S4. There is a natural accessibility relation $\leq$ among the stages of the generation: one stage accesses another just in case the latter can be obtained from the former by generating more objects. This accessibility relation is obviously reflexive and transitive. Additionally, it is reasonable to assume that the accessibility relation has the *convergence property* that if $w_0 \leq W_1, w_2$, then there is w_3 such that $w_1, w_2 \leq w_3$, or diagrammatically:



That is, if at some stage one has a choice of what objects to generate, it does not matter which option one chooses, since the other option can always be realized later. The license to generate certain mathematical objects never goes away and can always be realized later. When this convergence property is assumed, it will license the adoption of one more axiom in addition to the usual ones of S4, namely:

$$\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi. \tag{G}$$

The modal logic that is obtained by adding axiom G to S4 is known as S4.2.

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Say that a formula φ is *stable* if the necessitations of the universal closures of the following two conditionals hold:

$$\varphi \rightarrow \Box\varphi \qquad\qquad \neg\varphi \rightarrow \Box\neg\varphi.$$

Intuitively, a formula is stable just in case it never "changes its mind," in the sense that, if the formula is true (or false) of certain objects at some world, it remains true (or false) of these objects at all "later" worlds as well.

We can now state the key result—sometimes known as "the mirroring theorem"—which answers the question about the correct logic for non-modal reasoning about merely potential domains. Let $\vdash$ be the relation of classical deducibility in a non-modal first-order language $\mathcal{L}$. Let $\mathcal{L}^\Diamond$ be the corresponding modal language, and let $\vdash^\Diamond$ be deducibility in this language corresponding by $\vdash$, S4.2, and axioms asserting the stability of all atomic predicates of $\mathcal{L}$.

Theorem 1 (Classical first-order mirroring). *We have*

$$\varphi_1, \ldots, \varphi_n \vdash \psi \quad \text{iff} \quad \varphi_1^\Diamond, \ldots, \varphi_n^\Diamond \vdash^\Diamond \psi^\Diamond.$$

In technical parlance, the potentialist translation provides a faithful interpretation *of classical first-order logic in the system represented by the relation* $\vdash^\Diamond$.[7]

The mirroring theorem has a simple moral. Consider a first-order language $\mathcal{L}$. Suppose we are interested in logical relations between formulas in the range of the potentialist translation, in a classical modal theory that includes S4.2 and the stability axioms. Then we may delete all the modal operators and proceed by ordinary classical first-order logic in $\mathcal{L}$. For under the mentioned assumptions, the modalized quantifiers $\Box\forall$ and $\Diamond\exists$ behave logically just like ordinary first-order quantifiers, except that they generalize across all possible worlds rather than a single world. In short, relative to the potentialist translation and under the mentioned assumptions, classical first-order logic is valid for reasoning in $\mathcal{L}$ about merely potential domains.

It does not follow, however, that the modal first-order language $\mathcal{L}^\Diamond$ offers no advantages. There are formulas that lie outside of the range of the potentialist translation; (2) and (4) are examples. Such formulas can exploit the extra expressive resources afforded by the modal language. By using such formulas, we can develop theories and explanations that have no analogues in the non-modal language $\mathcal{L}$. The modal language $\mathcal{L}^\Diamond$ thus allows us to look at the subject matter under a finer resolution, which can be very useful. For other purposes, including much of ordinary, non-foundational mathematics, the finer resolution isn't needed and can simply be "turned off" via the mirroring theorem.

There remains work to be done, however. Theorem 1 answers the question of the correct *first-order* logic in $\mathcal{L}$ for non-modal reasoning about an incompletable domain. But the theorem says nothing about *extensions* of first-order logic. A central aim of this article is to state and prove a plural mirroring theorem that determines the correct plural logic for non-modal reasoning about such a domain. I prove a stronger result concerning plural logic as well. Theories that formalize the appropriate modal and non-modal reasoning about a merely potential domain are *definitionally equivalent*.

[7] See [15, Theorem 5.4]. There is also an intuitionistic version of the theorem, whose difference is that the logic on both sides is intuitionistic, not classical. See [17, Theorem 2].

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This means that plural reasoning on either side can be translated to reasoning on the other side.

I just noted that for *first-order* logic there are formulas outside of the range of the potentialist translation—thus giving the modal language a potentially useful expressive advantage vis-à-vis the corresponding non-modal language. For *plural logic*, by contrast, the definitional equivalence shows that the modal language enjoys no such expressive advantage.

§3. Logics of modals and plurals. Before I can state the desired theorems, I need to define the two formal systems to be compared. In this section, I describe the modal logic of plurals appropriate for modal reasoning about an incompletable domain, as well as a restricted plural logic appropriate for non-modal reasoning about the same. In the next section, these two systems are connected by means of our plural mirroring theorem and the definitional equivalence.

3.1. Minimal plural logic. We begin with a minimal non-modal plural logic.[8]

Laws of identity. Identity is reflexive and subject to Leibniz's Law, as usual.

Tautologies. Every substitution instance of a tautology is an axiom.

Modus ponens. If φ and $\varphi \to \psi$ are theorems, so is ψ.

Universal instantiation (UI). Every instance of

$$\forall x\, \varphi(x) \to \varphi(t)$$
$$\forall xx\, \varphi(xx) \to \varphi(tt),$$

where t and tt are any singular and plural terms, respectively, is an axiom.

Universal generalization (UG). Let v be either a singular or a plural variable. Suppose v does not occur free in φ. Then, if $\varphi \to \psi(v)$ is a theorem, so is $\varphi \to \forall v\, \psi(v)$.

We now turn to axioms specific to plurals.

Indiscernibility. Every instance of the plural indiscernibility scheme is an axiom, provided that 'xx' and 'yy' are not captured by any quantifiers in φ:

$$\forall z(z \prec xx \leftrightarrow z \prec yy) \to (\varphi(xx) \leftrightarrow \varphi(yy)).$$

Deviating from natural language, we *omit* what is often included as an axiom, namely, that every plurality is non-empty: $\forall xx \exists y\ y \prec xx$.[9]

The principles adopted so far describe how pluralities behave. We now turn to principles concerning what pluralities there are. Since pluralities are arbitrary (or combinatorial) collections, it makes sense to adopt the following.

Plural choice. Assume some objects xx parametrize a family of non-overlapping non-empty collections, that is, $(\forall x \prec xx)(\exists y \prec yy)\psi(x, y)$ and $\psi(x, y) \wedge \psi(x', y) \to x = x'$. Then there are some objects zz containing precisely one member of each of these collections, that is, $(\forall x \prec xx)(\exists!\, y \prec zz)\psi(x, y)$.

Let *Minimal Plural Logic* be the logic based on these axioms and rules.

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[8] Every logical axiom in what follows can (by the rules of Universal Generalization and Necessitation, stated below) be freely universally generalized and necessitated. For readability, I omit initial universal quantifiers and boxes. Finally, all axiom *schemes* are understood to include instances from any language in which they figure, including the modal language to be adopted shortly.

[9] See [11, p. 63] for a discussion and defense of this decision.

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3.2. Plural comprehension principles. Next, it is customary to adopt the following unrestricted comprehension principle.

Plural comprehension. Every instance of the following plural comprehension scheme is an axiom, provided that 'xx' does not occur free in φ:

$$\exists xx \forall y (y \prec xx \leftrightarrow \varphi(y)). \tag{P-Comp}$$

Let *Traditional Plural Logic* be the result of adding this axiom scheme to Minimal Plural Logic.

We need a restricted plural logic as well, to be used in non-modal reasoning about an incompletable domain. The guiding idea is that, in a setting where objects are successively generated, each plurality is—in Cantor's words—"finished." That is, each plurality consists of some objects that are all available at some particular stage of the generation. To be a plurality is therefore not a trivial matter. For example, the condition of being self-identical or of being a set fails to define a plurality; for it is impossible to finish the generation of objects that satisfy these conditions.

Let *Basic Plural Logic* (BPL) be the result of supplementing Minimal Plural Logic with Plural choice and the following three axioms or axioms schemes. First, every single object defines a "singleton plurality":

$$\forall x \exists xx \forall y (y \prec xx \leftrightarrow y = x). \tag{P-Sing}$$

Next, any two pluralities xx and yy have a pairwise union:

$$\exists zz \forall x (x \prec zz \leftrightarrow x \prec xx \vee x \prec yy). \tag{P-Union}$$

Finally, we add an axiom scheme of plural separation:

$$\exists yy \forall x (x \prec yy \leftrightarrow x \prec xx \wedge \varphi(x)). \tag{P-Sep}$$

I will eventually recommend a somewhat stronger, but still restricted, plural logic, which can also be used for non-modal reasoning about an incompletable domain. This system, called *Critical Plural Logic*, will be described in §5.

3.3. The modal logic S4.2. Next, we adopt some well-known axioms and inference rules governing the modal operators. This choice is motivated by the desired applications to potentialism, as discussed in §2.

$$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \tag{K}$$

$$\Box\varphi \to \varphi \tag{T}$$

$$\Box\varphi \to \Box\Box\varphi \tag{4}$$

$$\Diamond\Box\varphi \to \Box\Diamond\varphi. \tag{G}$$

We also add the necessity of distinctness, $t_1 \neq t_2 \to \Box t_1 \neq t_2$, which in the absence of the 5 or B axioms cannot be derived. Finally, we add the rule:

Necessitation. If φ is a theorem, so is $\Box\varphi$.

Recall that this rule cannot be applied to derivations from non-logical hypotheses; that is, it does not follow from $\Sigma \vdash \varphi$ that $\Sigma \vdash \Box\varphi$. This restriction will be important below.

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As is well known, the axioms and rules adopted allow us to derive the Converse Barcan Formula and its plural analogue:

$$\Box\forall x\,\varphi(x) \to \forall x\Box\varphi(x).$$

For our purposes, this is unproblematic, since we are anyway only interested in models where the domains are non-decreasing along the accessibility relation.

3.4. The interaction of modals and plurals. The most exciting part of the modal logic of plurals concerns the interaction of these two expressive devices.[10] Our guiding idea here is that a plurality is nothing over and above its members. To talk about a "plurality" is just shorthand for plural talk about "its" many members. So when a plurality is tracked across possible worlds, this must be done entirely in terms of its members. There is no other material on which to base the tracking, unlike teams, committees, or other groups, which have some nature over and above their members that permits a less straightforward form of tracking. The upshot is that a plurality—unlike, say, a committee—has its members by necessity. Wherever a plurality exists at all, it has the very same members.

We thus lay down that being one of some objects is a matter of necessity, and likewise for not being one of some objects. That is, we adopt axioms stating that the one-of relation $\prec$ is stable:

$$x \prec yy \to \Box x \prec yy \qquad (\textsc{Stb}{\prec})$$

$$x \not\prec yy \to \Box x \not\prec yy. \qquad (\textsc{Stb}{\not\prec})$$

If there are non-logical predicates in our language, we adopt analogous stability axioms for them too. Additionally, we adopt a version of the Barcan Formula for singular quantifiers restricted to a plurality:

$$\forall x(x \prec yy \to \Box\theta) \to \Box\forall x(x \prec yy \to \theta). \qquad (\textsc{Inext}{\prec})$$

This can be seen as stating that a plurality yy is inextensible, in the sense that it cannot gain new members at more populous worlds—something that is not ensured by ($\textsc{Stb}{\not\prec}$), which only says of *currently available* non-members of yy that they cannot become members at later worlds.

We can now derive that the among-relation $\preccurlyeq$ too is stable:

$$xx \preccurlyeq yy \to \Box xx \preccurlyeq yy$$

$$xx \not\preccurlyeq yy \to \Box xx \not\preccurlyeq yy.$$

Finally, we would like a version of the Barcan Formula for plural quantifiers restricted to a plurality:

$$\forall xx(xx \preccurlyeq yy \to \Box\theta) \to \Box\forall xx(xx \preccurlyeq yy \to \theta). \qquad (\textsc{Inext}{\preccurlyeq})$$

This can be seen as stating that a plurality does not gain new subpluralities at more populous worlds. Since ($\textsc{Inext}{\preccurlyeq}$) does not follow from the other axioms and rules, we add it as our final axiom scheme.[11]

[10] A consensus is emerging concerning the correct modal logic of plurals (see [21, chap. VIII], [27], [28, pp. 246–248], [11, chap. 10], and [19]).

[11] The observation that this axiom scheme needs to be added is due to [19].

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Let *Modal Traditional Plural Logic (M-TPL)* be the result of adding to Traditional Plural Logic the modal logic S4.2 and the interaction principles just described.

§4. Plural mirroring and definitional equivalence. As observed in the introduction, it is well known that Traditional Plural Logic needs to be restricted when reasoning non-modally about a merely potential domain. We are now in a better position to understand why. The problem is particularly clear in connection with TPL's commitment to a universal plurality: $\exists xx \forall y\ y \prec xx$. The potentialist translation of this theorem is

$$\Diamond \exists xx \Box \forall y (y \prec xx). \tag{5}$$

That is, there potentially are some objects that are necessarily all-inclusive. But this claim is unacceptable, given the potentialist assumption that it is always possible for there to be more objects than there in fact are

$$\Box \forall xx\, \Diamond \exists y (y \not\prec xx). \tag{6}$$

For it is easy to see that (5) is inconsistent with (6) in the modal plural logic M-TPL.

It is a separate—and harder—question *how* Traditional Plural Logic should be restricted when reasoning non-modally about a merely potential domain. The first of my two main theorems provides the answer.[12]

Theorem 2 (Plural mirroring). *The potentialist translation $\varphi \mapsto \varphi^{\Diamond}$ provides a faithful interpretation of BPL in M-TPL. That is, we have*

$$\varphi_1, \ldots, \varphi_n \vdash^{BPL} \psi \quad \textit{iff} \quad \varphi_1^{\Diamond}, \ldots, \varphi_n^{\Diamond} \vdash^{M\text{-}TPL} \psi^{\Diamond}.$$

The upshot is that potentialists who employ the usual modal logic of plurals M-TPL in their modal language are thereby entitled to the BPL, defined in §3.2, in their non-modal language.

What if we add principles stronger than M-TPL in the modal language? Then we might justify stronger principles on the non-modal side as well. In §5, I provide some examples, which are plausible on a potentialist view and therefore justify a stronger—yet still restricted—plural logic for use in the non-modal language.[13]

The proof of Plural Mirroring is given in Appendix A. The left-to-right direction is proved by induction on the length of proofs, as in the case of first-order logic. The right-to-left direction is trickier. The strategy we used in the first-order case does not work in the present setting.[14] So we need a new strategy.

Let me canvass the solution, while leaving details for Appendix A. A key element is to define a translation in the reverse direction. It is far from obvious that such a translation is possible; after all, $\mathcal{L}^{\Diamond}$ adds modal resources, which are not available in $\mathcal{L}$.

[12] This theorem was first presented in a talk I gave to the Oslo Logic Seminar on 19 November 2015.

[13] There are other examples that would *not* appeal to potentialists. Suppose we added the Barcan Formula, BF, on the modal side. Then we would obtain a faithful interpretation of Traditional Plural Logic. To see this, we first use P-Comp to find xx such that $\forall y(y \prec xx \leftrightarrow \varphi^{\Diamond}(y))$. Since both formulas flanking the biconditional are non-contingent, this can be strengthened to $\forall y \Box(y \prec xx \leftrightarrow \varphi^{\Diamond}(y))$, whence BF gives us the desired $\Box \forall y(y \prec xx \leftrightarrow \varphi^{\Diamond}(y))$.

[14] That proof was based on the observation that, for every proof in the modal system, the corresponding argument with all modal operators deleted, is a valid proof in the non-modal system. This strategy fails in our present setting because we have unrestricted plural comprehension in M-TPL but not in BPL.

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It turns out, however, that $\mathcal{L}$, despite being non-modal, allows us to simulate talk about possible worlds with respect to the modality present in $\mathcal{L}^{\Diamond}$. The key is to observe that a possible world can be represented by the plurality of objects in its domain.[15] This works because, first, all atomic predicates are stable, and second, the accessibility $\leq$ between possible worlds is convergent. Together, these assumptions ensure that the truth-value of an atomic predication cannot vary between any two worlds accessible from our reference world.

Our next aim is to show that this reverse translation provides a *faithful* interpretation of M-TPL in BLP. To obtain such a result, we need to tweak each theory to allow it to say of some objects that they are all the objects that actually exist. We call the tweaked systems *BPL** and *M-TPL**.

Having translations from $\mathcal{L}$ to $\mathcal{L}^{\Diamond}$ and back again, we can compose them to define two "round-trip" translations, from one language via the other and back. Our last task is to show that each "round-trip" translation of a formula is equivalent, in the relevant setting, to the original formula.

All the results just described are combined into our overarching theorem.[16]

THEOREM 3 (Definitional equivalence). *BPL** *and M-TPL** *are definitionally equivalent. That is, there are translations in both directions such that*:

(i) *each translation interprets one theory in the other*;
(ii) *their two "round-trip" translations are trivial modulo deductive consequence in the relevant theory.*

(See Appendix A for a proof.) This is a very strong form of equivalence.

Let me conclude with an observation that will be important in §7. The non-modal language offers a choice that is not available in the corresponding modal language. Do we recognize a distinction between actual and potential existence of mathematical objects? If we do, we can adopt the tweaked system BPL*, which draws this distinction. If we do not, we can stick with the original system BPL, which affords no such distinction. In the modal language, by contrast, there is no such choice: here we have both $\exists$ and $\Diamond\exists$, which express actual and potential existence. In sum, whereas the modal language builds in a distinction between actual and potential existence, the non-modal language allows us to choose whether or not to draw that distinction.

§5. Critical plural logic. We have observed that potentialists are entitled to BPL in the non-modal language in virtue of accepting the traditional modal logic of plurals and the potentialist translation. Let us now investigate whether they might also be entitled to a stronger, yet still restricted plural logic. This question is not answered by

[15] See Appendix A.2. Versions of this idea are used to define reverse translations in [15, sec. 8.3], [25, pp. 719f], [18, p. 851], and [6, sec. 6].

[16] See Theorem B.3 of Button [6] for a related result, which establishes the (near-)definitional equivalence of a modal potentialist set theory with a non-modal second-order set theory whose second-order entities are bounded (cf. also the results of Studd [26, appendix C.3]). (Button would be entitled drop the qualification 'near' by making tweaks analogous to those indicated by '$*$'.) My theorem is more general in two respects. First, whereas these other results are concerned with a potentialist set theory, mine is not specific to sets but connects a plural logic with a modal logic of plurals—and thus answers the question of the correct plural logic for non-modal reasoning about a merely potential domain. Second, these other results rest on assumptions that mine avoids, e.g., that the generation is linear and maximalist (in the sense that we always generate all the objects we are capable of generating).

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the plural mirroring theorem. After all, potentialists often add stronger principles on the modal side in order to justify stronger non-modal principles.[17]

We note that the operation of adding matching axioms—that is, adding φ on the non-modal and $\varphi^{\Diamond}$ on the modal side —preserves the results about mirroring and definitional equivalence. In this way, it is routine to extend the definitional equivalence of Theorem 3 to connect the set theory of §6.2 below with that of Linnebo [15].

Our search for a stronger non-modal plural logic acceptable to potentialists will focus on the *Critical Plural Logic* identified and defended by Florio and Linnebo [10, 11]. As before, we begin with Minimal Plural Logic, which has no plural comprehension. Critical Plural Logic is obtained by adding four principles about what pluralities there are. First, we add the plural singleton principle (P-Sing), which is also part of BPL. Next, we add an adjunction principle for pluralities, to the effect that adjoining an object to a plurality yields another plurality:

$$\forall xx \forall y \exists zz \forall u (u \prec zz \leftrightarrow u \prec xx \vee u = y). \tag{P-Adj}$$

Then, we add a union principle that generalizes the pairwise union (P-Union) of BPL. The idea is that the union of "plurality-many pluralities" is in turn a plurality. To make this idea precise, consider a plurality xx each of whose members serves as a "tag" on some plurality of objects. Then, our generalized union principle says that there exists a plurality whose members include all and only the objects that occur as a member of one of the "tagged" pluralities. We can formulate this as the following schematic principle. Suppose there are xx that

$$\forall x (x \prec xx \rightarrow \exists yy \, \forall z (z \prec yy \leftrightarrow \psi(x, z))).$$

That is, for each $x \prec xx$ there is a plurality of all those objects z that are "tagged" by x, in the sense that $\psi(x, z)$. Then, there are zz that comprise all and only the objects that are "tagged" by some member of xx:

$$\forall y (y \prec zz \leftrightarrow \exists x (x \prec xx \wedge \psi(x, y))).$$

Finally, we adopt a schematic principle to the effect that every plurality can be closed under function application:

$$\forall x \exists! \, y \, \psi(x, y) \rightarrow \forall xx \exists yy (xx \preccurlyeq yy \wedge \forall x \forall y (x \prec yy \wedge \psi(x, y) \rightarrow y \prec yy)). \tag{P-Inf}$$

This can be seen as a plural principle of infinity. Suppose ψ describes the function that generates some potential infinity. Consider the potential infinity of objects obtainable from some complete "base" xx by iterated application of that function. (P-Inf) states that this potential infinity is completed by a single plurality yy that extends xx and is closed under the function. Clearly, this axiom scheme will not appeal to austere Aristotelian potentialists, who reject all forms of actual infinity.

What is the relation between Basic and Critical Plural Logic? The former has two axioms that are not present in the latter, namely, Pairwise Union and Plural Separation. But Critical Plural Logic proves these two axioms as theorems.[18] As for the reverse

[17] Some potentialists add a plural replacement principle on the modal side to justify adding a plural replacement on the non-modal side. Others add modal reflection principles, which justify a non-modal principle of infinity or more (see, e.g., [15, 23, 25]).

[18] These results—and the first one from the next note—are proved in [11, sec. 12.5].

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inclusion, it is easy to verify that the two extra axioms of Critical Plural Logic, namely, Generalized Union and Plural Infinity, are not theorems of BPL. Thus, Critical Plural Logic strictly extends its basic cousin.[19]

Should potentialists go beyond BPL and accept the two additional axioms of Critical Plural Logic when reasoning non-modally? Let us begin with Generalized Union. This should be accepted, for two reasons. First, the axiom has considerable intrinsic plausibility. Suppose we can complete the generation of some collection of "tags." Suppose further that for each of the "tags" we can complete the generation of all the objects that receive that "tag." Then, plausibly, we can also complete the generation of all of the "tagged" objects.

Second, potentialists can support Generalized Union by deriving its translation from a plausible generalization of the modal axiom G.[20] We begin by observing that G is equivalent, modulo S4, to the principle that any two possible necessities are possibly both necessary[21]:

$$\Diamond\Box\varphi \wedge \Diamond\Box\psi \rightarrow \Diamond(\Box\varphi \wedge \Box\psi). \qquad (\mathrm{G}')$$

Thus, G can be seen as a principle about the compossibility of possible necessities. By repeated application of G', it follows that any finite number of possible necessities are compossible. This prompts the question of how far we can go: how large a family of possible necessities are compossible? Suppose we allowed absolutely all possible necessities to be co-realized. Our potentialism would then collapse into actualism (albeit prefixed by an unimportant '$\Diamond$'). A more plausible idea is that any plurality-indexed family of possible necessities are compossible:

$$(\forall x \prec xx)\Diamond\Box\varphi(x) \rightarrow \Diamond(\forall x \prec xx)\Box\varphi(x). \qquad \text{(Super-G)}$$

It can be shown that (Super-G) entails the potentialist translation of Generalized Union.[22] Since (Super-G) has considerable plausibility from a poten-

[19] Critical Plural Logic has some other useful theorems as well. It proves Plural Replacement. Moreover, assuming a pairing operation, it proves that any pluralities xx and yy have a Cartesian product $xx \times yy$.

[20] Tim Williamson asked in a discussion in 2011 whether such a generalization might be possible. My answer is properly worked out in this paragraph.

[21] To show that G' entails G, suppose we have a failure of G. That would mean $\Diamond\Box\varphi$ and $\Diamond\Box\neg\varphi$, which by G' would yield (in S4) a contradiction. For the reverse direction, observe first that G' is valid in every Kripke model whose accessibility relation is reflexive, transitive, and convergent. Since S4.2 is complete with respect to such frames, S4.2 proves G', whence S4 proves $\mathrm{G} \rightarrow \mathrm{G}'$, as desired.

[22] Consider the potentialist translation of the antecedent of Generalized Union. Apply (Super-G) to obtain $\Diamond(\forall x \prec xx)\exists yy\, \Box\forall z(z \prec yy \leftrightarrow \psi^{\Diamond}(x, z))$, i.e., that it is possible to simultaneously have all of the "tagged" objects. By an application of plural comprehension within the scope of the outermost '$\Diamond$', we obtain the possible existence of zz defined as the union across $x \prec xx$ of all the yy such that $\Box\forall z(z \prec yy \leftrightarrow \psi^{\Diamond}(x, z))$. I claim that these zz satisfy the condition from the potentialist translation of the consequent of Generalized Union, i.e., $\Box\forall y(y \prec zz \leftrightarrow \Diamond\exists x(x \prec xx \wedge \psi^{\Diamond}(x, y))$. Clearly, any $y \prec zz$ is such that $\exists x(x \prec xx \wedge \psi^{\Diamond}(x, y))$ and thus also possibly such. By (Inext$\prec$), this generalization over y can be necessitated. Next, given (Stb$\prec$), the right-to-left of the embedded biconditional is equivalent to $\Box\forall x(x \prec xx \wedge \psi^{\Diamond}(x, y) \rightarrow y \prec zz)$, which is immediate from the definition of zz.

Separately, I note that the 'amalgamation principle' of Berry [1, sec. 8.12] is closely related to the potentialist translation of Generalized Union.

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tialist point of view, so too does (the potentialist translation of) Generalized Union.[23]

It remains to consider the axiom scheme of Plural Infinity. As explained, this axiom can be seen as a completability principle that enables us to pass from any potential infinity of the described kind to a corresponding actual infinity. Although rejected by austere Aristotelian potentialists, completability principles are part and parcel of the more relaxed Cantorian, or set-theoretic, form of potentialism. Thus, Plural Infinity will be attractive on at least some important forms of potentialism.[24]

In sum, potentialists have good reason to adopt, on the non-modal side, not only Basic but also Critical Plural Logic—or, equivalently, by the definitional equivalence, to adopt the corresponding modal principles.

§6. Potentialist set theory "demodalized". In light of the definitional equivalence of BPL and Modal Plural Logic, a choice opens up concerning how potentialist ideas should be developed. In the past 15 years, there has been a good deal of work developing such ideas in modal frameworks, often using plurals. The result about definitional equivalence makes available an alternative approach that eschews modality in favor of BPL or extensions thereof. As an illustration of this alternative, I will now outline a purely plural explication of set-theoretic potentialism—or at least of its mathematical content.

6.1. Critical plural set theory. The central idea of set-theoretic potentialism is that, whenever there are some objects xx, these can be used to define a set $\{xx\}$. In the non-modal setting—where the modality is replaced by a critical view of what pluralities there are—this idea admits of a very simple and natural expression, namely, as a plural version of Frege's Basic Law V:

$$\{xx\} = \{yy\} \leftrightarrow \forall z(z \prec xx \leftrightarrow z \prec yy). \tag{SA}$$

The label 'SA' abbreviates *set abstraction*.

We wish to add that all sets are generated in this way in a well-founded manner. We do so by adopting the following set induction scheme:

(SI) Suppose that every urelement is φ and that, for any xx each of which is φ, $\{xx\}$ too is φ. Then everything is φ.

After all, if there were other ways to generate sets than by the transition from some antecedently available objects xx to their $\{xx\}$, then the antecedent of (SI) would not suffice for the consequent.

The induction scheme (SI) has some very useful consequences. It entails the usual formulation of the axiom of Foundation. Also, it entails that every set is obtained from some plurality: $\forall x \exists uu(x = \{uu\})$.

Let us now combine the principles (SA) and (SI), which describe how pluralities define sets, with Critical Plural Logic, which provides an account of what pluralities

[23] To gauge the strength of (Super-G), it is not hard to prove that its non-modal correlate is a strong plural collection principle: $(\forall x \prec xx)\exists yy\, \psi(x, yy) \rightarrow \exists zz(\forall x \prec xx)(\exists yy \preccurlyeq zz)\psi(x, yy)$. Since this collection principle is strictly stronger than Generalized Union, the same relation obtains between their modal correlates.

[24] Since Plural Infinity belongs to a larger family of completability principles, anyone attracted to it has a reason to explore and harness this larger family of plural principles as well.

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there are. This yields a simple and natural theory, which we call *Critical Plural Set Theory (CPS)*. Remarkably, this theory proves nearly all of standard ZFC set theory.

THEOREM 4. *Critical Plural Set Theory proves Extensionality, Foundation, Empty Set, Pairing, Union, Separation, Replacement, Infinity, and Choice.*[25]

This leaves only a single axiom of ZFC, namely, Powerset.

6.2. Two options concerning Powerset. We have two options. One is to justify Powerset as well by adding to Critical Plural Set Theory. Suppose we require that it be possible "simultaneously" to exercise all the possibilities for set generation that were available at some stage. In particular, for any set x, it is possible to generate more objects so as to make available all of x's subsets as well[26]:

$$\forall x \exists yy \forall z (z \prec yy \leftrightarrow z \subseteq x). \qquad \text{(POWER)}$$

In effect, every set has a "power plurality" that includes all of its subsets. By applying set formation to this power plurality, we obtain the desired powerset.

The alternative option is to renounce Powerset. If we do so, it is natural to take the further step of adding to Critical Plural Set Theory an axiom to the effect that every set can be made countable, in the sense that there is a surjection from the natural numbers onto the set:

$$\forall x \exists f : \mathbb{N} \twoheadrightarrow x. \qquad \text{(COUNT)}$$

This yields an attractive countabilist set theory.[27] Via the definitional equivalence, the theory comes both in the present non-modal version and as its modal analogue.[28]

We obtain a model of the theory by letting the pluralities be all and only the countable subpluralities of the set $H(\omega_1)$ of hereditarily countable sets. In fact, this model nicely illustrates how Generalized Union works. The axiom allows us to merge any countable family of possible worlds (or pluralities) into a single world (or plurality). But on the countabilist view, there are uncountably many worlds (or pluralities) that extend the plurality of the natural numbers. Intuitively, while any countable collection of sets can co-exist, the entire uncountable collection of subsets of $\mathbb{N}$ cannot. This collection is a proper class. Thus, we have a form of *width potentialism*: however many subsets of $\mathbb{N}$ have been generated, it is always possible to generate more.

To summarize, in its non-modal, purely plural garb, potentialist set theory has a core, CPS, based on the following two components:

[25] A slight variant of this result is proved in [11, sec. 12.7]. See Appendix B for an analogous result concerning Basic rather than Critical Plural Logic.

[26] This is a non-modal counterpart of the principle (ED-$\subseteq$) adopted by [15, pp. 217–218] to obtain Powerset.

[27] Countabilist set theory often adds an axiom scheme of Collection: $(\forall x \in a)\exists y\, \psi(x, y) \rightarrow \exists b(\forall x \in a)(\exists y \in b)\psi(x, y)$. This too follows if we assume (SUPER-G) or its non-modal correlate (see fn. 23).

[28] There has recently been considerable interest in potentialist versions of countabilism (see, e.g., [4, 23]). It is noteworthy that (the modal version of) the present theory, unlike the mentioned alternatives, is based on an absolutely orthodox modal plural logic. See [20] for discussion.

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(i) Critical Plural Logic, which provides an account of what objects can co-exist and thus make up a single plurality;
(ii) the two principles (SA) and (SI), to the effect that all and only pluralities form sets.

To this core we can add either (Power) or (Count), resulting in either ZFC or countabilist set theory.

Note also that the modular structure of our theory allows component (ii), concerning set abstraction, to be replaced or supplemented with other forms of abstraction.[29]

§7. Which is the better explication?. The result about definitional equivalence yields two formally equivalent ways to explicate potentialist ideas. Are there reasons to favor one explication over the other?

In the introduction, I advertised some substantial *practical* benefits of the non-modal explication. This approach is simpler and more user-friendly. I hope to have illustrated these benefits in the preceding section, where I developed, in a single section, what is ordinarily the material of an entire article.

What about *philosophical* benefits? As noted, the answer will depend on how the non-modal language is interpreted. One option is that the non-modal formulas are used merely as a convenient shorthand for more long-winded modal statements. Then there can be no philosophical benefits. For clearly, the non-modal formulation would then inherit all the philosophical pros and cons of the official modal statements. Alternatively, suppose the non-modal language is given an autonomous interpretation, which makes no recourse to the modal language or its interpretation.[30] On this approach, the definitional equivalence would have a very different status. It would provide a useful *formal alignment* of two theories—which might nonetheless differ as to their *content*. When both languages have been independently interpreted, the translations that subserve the equivalence need not preserve meaning or content. After all, definitional equivalence is a proof-theoretic notion, which need not preserve semantic content—or, for that matter, philosophically important properties.[31]

We need to ask, therefore, whether there is an autonomous interpretation of the non-modal language that renders the demodalization version of potentialism true. We also need to know whether the resulting view can still be regarded as a form of potentialism. These questions are too large to be settled here. Instead, I will examine how the debate plays out when applied to an important challenge to potentialism.

I have in mind *the problem of actuality*.[32] In its usual modal explication, potentialism has at its disposal two notions of existence: plain and potential, expressed by '$\exists$' and '$\Diamond\exists$', respectively. These expressive resources give rise to some tricky questions. Potentialists tell us a lot about what mathematical objects *potentially* exist. They also face the question, however, of what mathematical objects *actually* exist.[33] Furthermore, given any proposed dividing line between actual and merely potential mathematical

[29] See [8, 12].
[30] Such an interpretation is attempted in [11].
[31] See [11, pp. 58–59] for discussion and a number of examples.
[32] See, e.g., [24, sec. 4.2], [5], [2, sec. 7.1], and [7].
[33] Note that this question does not need an actuality operator. It suffices to ask questions using '$\exists$', not '$\Diamond\exists$'.

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objects, we would like to know why the line is drawn precisely there. What would it take for some merely potential mathematical object to become actual?

There are two main lines of response to the problem of actuality. Many potentialists happily accept a distinction between actual and merely possible mathematical objects. They believe that there might have been more mathematical objects than there in fact are. Aristotle provides a clear example. He takes the existence of a mathematical object to be contingent on the existence of a concrete instantiation. A number, for example, requires instantiation by a suitably numerous plurality of concrete objects. Since there might have been more concrete objects than there actually are, it follows that there might have been more numbers than there actually are. Structurally analogous views have been espoused by Putnam, Hellman, Scambler, Studd, and Linnebo, albeit with different modalities in mind.[34]

Other potentialists are uncomfortable with any dividing line between actual and merely possible mathematical objects.[35] These philosophers are attracted to the core potentialist idea of successive generation of objects in an incompletable process. They also agree that this idea naturally lends itself to explication in a modal language. As observed, however, this modal language draws a distinction between actual and possible existence. The philosophers in question regard this distinction as "surplus structure," that is, as an unintended byproduct of the decision to develop their view in a modal language.

The result about definitional equivalence can be useful for both sides of this intramural debate. Consider first potentialists who accept the distinction between actual and merely possible mathematical objects. Although they take the distinction to be philosophically important, they admit that it does no serious mathematical work. From a purely mathematical point of view, the distinction is only a distraction. It can therefore be useful to adopt the non-modal explication, interpreted via the translation that figures in the definitional equivalence. When one refrains from singling out some objects *dd* as the actual ones—by using BPL, not BLP*—then this explication expresses all and only what is mathematically significant about the potentialist view.

Next, consider potentialists who reject the distinction between actual and merely possible mathematical objects. How should they develop their view? A relaxed and straightforward approach is to retain the modal explication—despite what these potentialists regard as its surplus structure. For surplus structure can be unproblematic, provided that we are mindful of its status as such. Indeed, this approach is akin to the use of coordinate systems in mathematical physics. Coordinate systems too have surplus structure, by allowing us to talk about being located at the origin, although this notion has no physical content. So long as this surplus structure is not taken seriously, however, the use of coordinate systems is benign and hugely useful.

A more austere approach is to eliminate the surplus structure by adopting a non-modal explication of potentialism (without singling out some objects *dd* as the

[34] Putnam and Hellman are concerned with non-mathematical realizations of set-theoretic structures. The actual world realizes some such structures, although there might have been larger ones. Finally, Linnebo [16] and Studd [26] are concerned with a linguistic interpretation of the modality, where to be actual is a matter of being in the range of the quantifiers on the actual interpretation of the language.

[35] For example, Linnebo [14] suggests that we take an "external view on the [...] process" of defining sets and that it "would be wrong to assign to ourselves any particular stage of the process."

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actual ones).[36] As observed, this response requires that the non-modal language be given an autonomous interpretation. This approach has been attempted—as applied to the Critical Plural Set Theory of the preceding section.[37]

Some will no doubt question whether the resulting view is still a form of potentialism. Isn't potentialism inherently a modal view? A quarrel about labels is rarely rewarding. For what it is worth, potentialism is here—and in many earlier writings—defined as the view that mathematical objects are successively generated and that this generation is incompletable. Both of these core aspects of potentialism, which are obviously present on the modal explication, are retained on the non-modal explication as well. We saw in the previous section how the definition of sets can be nested, such that "new" sets are defined in terms of (or "generated" from) "old" objects already defined. Furthermore, the non-modal explication retains the crucial potentialist idea of incompletability. However many objects have been defined, these objects can be used to define (or "generate") yet more sets. For on the view in question, every plurality, however large, defines a set.[38]

As observed, potentialists disagree about the problem of actuality. Reflecting this disagreement, some potentialists adopt the modal explication as their official one, whereas others adopt the non-modal one. This disagreement should not distract from a central and robust lesson of this article. *Either type of potentialist may find it convenient to use either type of explication and can justifiably do so.* Thanks to the definitional equivalence, we can go back and forth between the two explications. Potentialists who embrace a notion of mathematical actuality may still use the non-modal explication for convenience. This explication is simpler, more workable, and in many ways more elegant. Conversely, potentialists who reject the notion of mathematical actuality may still find it helpful to invoke modality as a heuristic. As we have seen, doing so provides a very intuitive way to explain and study the central potentialist notion of incompletability.

§A. Proofs of plural mirroring and definitional equivalence. We break the proofs of Plural Mirroring and Definitional Equivalence into smaller steps.

A.1. The left-to-right of plural mirroring.

LEMMA A.1. *The potentialist translation* $\varphi \mapsto \varphi^{\diamond}$ *provides an interpretation of BPL in M-TPL.*

Proof. Since the analogous result for first-order logic has already been proved, we need only consider the principles that BPL adds. We begin with the plural comprehension principles. First, the plural singleton principle translates to

$$\diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow y = x). \tag{A.1}$$

[36] Interestingly, while the problem of actuality arises already in first-order languages, the attractive solution described here relies essentially on our results about plural languages.

[37] This is the view of [11, chap. 12]. My own journey has thus been one from what I just called a "relaxed" rejection of actuality, in [14] (cf. fn. 35), through an interpretational understanding of actuality, in [16] (cf. fn. 34), to the present "austere" rejection of actuality.

[38] This non-modal "collapsing" claim is defended in [11, chap. 12], based, in large part, on a liberal theory of definitions. They argue, further, that this theory does not sanction an analogous "collapse" of an intensional collection, such as the entire universe of sets, into a set (pp. 274–275).

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M-TPL proves that there exist xx such that:

$$\forall y(y \prec xx \leftrightarrow y = x). \tag{A.2}$$

Since we have $y = x \rightarrow \Box y = x$, we obtain $\Box\forall y(y \prec xx \rightarrow y = x)$ by invoking (INEXT$\prec$). For the reverse direction, we observe that $\forall y(y = x \rightarrow \varphi(y))$ is provably equivalent to $\varphi(x)$, whence $\Box(\forall y(y = x \rightarrow \varphi(y)) \leftrightarrow \varphi(x))$. Let $\varphi(x)$ be $x \prec xx$, which by (A.2) is true and thus by (STB$\prec$) is necessary. Via the necessary material equivalence, this entails $\Box\forall y(y = x \rightarrow y \prec xx)$, as desired. Hence, M-TPL proves (A.1).

Next, Pairwise Union translates as

$$\Diamond\exists zz\Box\forall x(x \prec zz \leftrightarrow x \prec xx \vee x \prec yy). \tag{A.3}$$

M-TPL proves that there exist zz such that

$$\forall x(x \prec zz \leftrightarrow x \prec xx \vee x \prec yy). \tag{A.4}$$

Using (STB$\prec$), this entails

$$\forall x(x \prec zz \rightarrow \Box(x \prec xx \vee x \prec yy)),$$

which, in turn, by (INEXT$\prec$), entails

$$\Box\forall x(x \prec zz \rightarrow (x \prec xx \vee x \prec yy)).$$

For the reverse direction, we use the equivalence of $p \vee q \rightarrow r$ with $(p \rightarrow r) \wedge (q \rightarrow r)$. This enables us to prove $\Box\forall x(x \prec xx \rightarrow x \prec zz) \wedge \Box\forall x(x \prec yy \rightarrow x \prec zz))$, which in turn implies

$$\Box\forall x((x \prec xx \vee x \prec yy) \rightarrow x \prec zz)$$

as desired. Thus, M-TPL proves (A.3).

Finally, Plural Separation translates as

$$\Diamond\exists yy\Box\forall x(x \prec yy \leftrightarrow x \prec xx \wedge \varphi^{\Diamond}(x)). \tag{A.5}$$

Using (P-COMP), we prove that there are yy such that:

$$\forall x(x \prec yy \leftrightarrow x \prec xx \wedge \varphi^{\Diamond}(x)). \tag{A.6}$$

Using (STB$\prec$) and the stability of $\varphi^{\Diamond}$, as well as (INEXT$\prec$), we prove

$$\Box\forall x(x \prec yy \rightarrow x \prec xx \wedge \varphi^{\Diamond}(x)). \tag{A.7}$$

We next wish to prove the necessitation of the converse conditional. This conditional is equivalent to $x \prec xx \rightarrow (\varphi^{\Diamond}(x) \rightarrow x \prec yy)$. We observe that the embedded conditional is stable, because both its antecedent and its consequent are stable (and any truth-functional combination of stable formulas is stable). Applying (INEXT$\prec$) yet again, we prove the desired formula. Thus, M-TPL proves (A.5).

It remains only to show that Indiscernibility and Plural Choice translate as theorems of M-TPL. This is straightforward, using the techniques illustrated above. □

A.2. Two reverse translations. As explained in the main text, §4, the proof of the right-to-left of Plural Mirroring relies on a translation in the reverse direction. Let us now be precise.

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We define a translation from $\mathcal{L}^{\Diamond}$ to $\mathcal{L}$ as follows. Let vv be a plural variable that doesn't occur in φ. Intuitively, this variable represents the domain of the possible world under consideration. First, we define $[\varphi]_{vv}$ by the following recursion clauses:

- $[\varphi]_{vv} \quad \mapsto \quad \varphi$ for φ atomic;
- the translation $\varphi \mapsto [\varphi]_{vv}$ commutes with the connectives;
- $[\forall y\, \psi]_{vv} \quad \mapsto \quad (\forall y \prec vv)[\psi]_{vv}$;
- $[\forall yy\, \psi]_{vv} \quad \mapsto \quad (\forall yy \preccurlyeq vv)[\psi]_{vv}$;
- $[\Box\varphi]_{vv} \quad \mapsto \quad (\forall uu \succcurlyeq vv)[\varphi]_{uu}$.

Next, we pick a designated plural variable, '*dd*', whose values we think of as making up the domain of the actual world. We call the translation $\varphi \mapsto [\varphi]_{dd}$ *the relativization translation*.

For some purposes a variant of the translation may be preferred. While every world corresponds to a unique plurality, namely, its domain, not every plurality corresponds to a world. For the domain of each world can be "generated from below," which is not the case for arbitrary pluralities. The plurality consisting of just {Socrates}, for example, does not correspond to a world. For every world that contains this singleton must also contain Socrates, who is a necessary stepping stone for the generation of the singleton. Let us therefore add to $\mathcal{L}$ a plural predicate 'W', for *world plurality*, intended to be true of xx just in case xx make up the domain of a possible world.[39] Accordingly, we modify the translation of $\Box\varphi$ as follows:

- $[\Box\varphi]_{xx} \quad \mapsto \quad \forall yy(yy \succcurlyeq xx \wedge Wyy \rightarrow [\varphi]_{yy})$.

We call this *the world relativization translation*.

When we use this alternative translation, some other adjustments are needed as well. First, we extend the potentialist translation such that it translates 'Wxx' as '$\forall y(y \prec xx)$'. Then, we add to BPL an axiom stating that every plurality is contained in a world plurality:

$$\forall xx \exists yy(Wyy \wedge yy \succcurlyeq xx). \tag{P-in-W}$$

We also observe that the potentialist translation of (P-in-W), namely:

$$\Box\forall xx \Diamond \exists yy(\forall y(y \prec yy) \wedge yy \succcurlyeq xx) \tag{P-in-W$^{\Diamond}$}$$

is a theorem of M-TPL.[40]

A.3. The round-trip translation from $\mathcal{L}$. Having defined translations from $\mathcal{L}$ to $\mathcal{L}^{\Diamond}$ and back again, it is natural to inquire about the corresponding "round-trip" translation from $\mathcal{L}$ to itself. Very pleasingly, we can prove that this "round-trip" translation is equivalent to the identity translation.

Lemma A.2 (Round-trip translation from $\mathcal{L}$). *For the plain* (*alternatively*: *the world*) *relativization translation, we have that BPL* (*alternatively*: *BPL* + (*P-in-W*)) *proves* $\varphi \leftrightarrow [\varphi^{\Diamond}]_{dd}$ *for every formula* φ *of* $\mathcal{L}$ *with no occurrences of 'dd'*.

[39] In the special case of sets, the world pluralities are all and only the transitive pluralities (see Appendix B).

[40] Using the necessitation of (P-Comp), M-TPL proves $\Box\forall xx \exists yy(\forall y(y \prec yy) \wedge yy \succcurlyeq xx)$ and thus also the logically weaker (P-in-W$^{\Diamond}$).

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Henceforth we always implicitly assume that our formulas contain no occurrences of '*dd*' whenever a relativization translation is involved.

Proof. We proceed by induction on syntactic complexity. We observe that every atomic formula translates as itself, which also translates back again as itself. The cases of negation and disjunction are also straightforward, since both translations commute with these connectives. More interestingly, consider the quantifiers $\forall x$ and $\forall xx$. First, $\forall x\,\varphi$ translates as $\Box\forall x\varphi^{\Diamond}$, which translates back again as $(\forall yy \sqsupseteq dd)(\forall x \prec yy)[\varphi^{\Diamond}]_{yy}$. By the induction hypothesis (*IH*), the latter formula is equivalent to $(\forall yy \sqsupseteq dd)(\forall x \prec yy)\varphi$. (IH applies because, since nothing is assumed about dd, what goes for dd also goes for yy.) To see that this formula, in turn, is equivalent to the formula with which we started, we use the ability to adjoin any object x to any given plurality dd to obtain a new plurality (alternatively: some world plurality extending that new plurality). The case of the plural quantifier is analogous. Here the reasoning relies on the ability to form the pairwise union of any given objects xx and dd. ⊣ □

A.4. Reverse plural mirroring formulated. Our next question is whether the two relativization translations from $\mathcal{L}^{\Diamond}$ to $\mathcal{L}$ provide an interpretation of M-TPL in BPL, perhaps even a faithful interpretation. As things stand, the translations fall short in both respects. To see that the translations fail to yield an interpretation, observe that the translation of UI is $[\forall x\varphi(x) \rightarrow \varphi(t)]_{dd}$, which in turn is $(\forall x \prec dd)[\varphi(x)]_{dd} \rightarrow [\varphi(t)]_{dd}$. But this is valid only if we make the non-logical assumption that $t \prec dd$. Thus, a theorem is mapped to a non-theorem. The case of plural UI is analogous: the validity of its translation would require the non-logical assumption $tt \preccurlyeq dd$.

Next, there is the question of faithfulness, that is, whether every non-theorem is mapped to a non-theorem. Consider the $\mathcal{L}^{\Diamond}$-formula stating that a plurality dd is universal:

$$\forall x(x \prec dd). \qquad \text{(Dom)}$$

This is not a theorem of M-TPL. But its relativization translation, $(\forall x \prec dd)(x \prec dd)$, is a trivial theorem of BPL. Thus, a non-theorem is mapped to a theorem.

But there is no cause for alarm. The problems just noted arise because the two theories in question fail to express the intuitive significance of dd as making up the domain of the actual world. The solution is to extend the two theories so as to express this significance. On the non-modal side, we wish to add all statements of the form $t \prec dd$ and $tt \preccurlyeq dd$, for any terms t and tt, to express that dd comprise all the available objects. The idea is that, whenever we introduce some objects into an argument, we thereby regard them as actual. Call the mentioned statements *the auxiliary axioms*, and let Aux be their set. Let BPL* be the result of adding Aux, as non-logical axioms or hypotheses, to BPL. On the modal side, we would like to add (Dom) to express that every (actual) object is among dd. But (Dom) cannot be added as a *logical* axiom; for its necessitation, $\Box\forall x(x \prec dd)$, would be unacceptable to a potentialist. The solution is, once again, to add (Dom) as a non-logical axiom or hypothesis. Let M-TPL* be the resulting theory.

We can now formulate the result to be proved at the end of this appendix.

THEOREM 5 (Reverse Plural Mirroring). *The plain relativization translation provides a faithful interpretation of M-TPL* is BPL*, that is*:

$$Aux, [\varphi_1]_{dd}, \ldots, [\varphi_n]_{dd} \vdash^{BPL} [\psi]_{dd} \quad \Leftrightarrow \quad Dom, \varphi_1, \ldots, \varphi_n \vdash^{M\text{-}TPL} \psi.$$

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The same goes for the world relativization translations, provided we add (P-in-W) on the non-modal side.

A.5. Miscellaneous proofs. In Appendix A.1 we proved the left-to-right direction of Plural Mirroring. We will now prove the reverse direction, as well as the two directions of Reverse Plural Mirroring. We begin with the following lemma.

LEMMA A.3 (Right-to-left of Reverse Plural Mirroring). *For the plain translation we have*

$$Aux, [\varphi_1]_{dd}, \dots, [\varphi_n]_{dd} \vdash^{BPL} [\psi]_{dd} \quad \Leftarrow \quad Dom, \varphi_1, \dots, \varphi_n \vdash^{M\text{-}TPL} \psi.$$

The same goes for the world relativization translations, provided we add (P-in-W) on the non-modal side.

Proof. Consider first the plain translation. We already observed that $\forall x(x \prec dd)$ translates as $(\forall x \prec dd)(x \prec dd)$, which is a logical truth of $\mathcal{L}$. We proceed to prove in BPL the translation of each axiom of M-TPL and that the translation of each rule preserves provability in BPL*.

The cases of the laws of identity, tautologies, and modus ponens are straightforward.

The cases of singular and plural UI were discussed above. Here we make essential use of the auxiliary axioms. Consider next UG, which allows us to infer from $\varphi \to \psi(x)$ that $\varphi \to \forall x \psi(x)$. This translates as the rule stating that from $[\varphi]_{dd} \to [\psi(x)]_{dd}$ we may infer $[\varphi]_{dd} \to (\forall x \prec dd)[\psi(x)]_{dd}$. This is a valid derived rule.

Next are the axioms of non-modal plural logic. The axiom schemes of plural indiscernibility and plural choice are straightforward. Instances of the axiom scheme of plural comprehension translate as instances of plural separation.

We now turn to the modal logic. Axiom T translates as $[\Box\varphi \to \varphi]_{dd}$. Unpacking, this is $(\forall xx \succcurlyeq dd)[\varphi]_{xx} \to [\varphi]_{dd}$, which is trivial. The same goes for the translations of K and 4. More interesting is the G axiom, which translates as

$$(\exists xx \succcurlyeq dd)(\forall yy \succcurlyeq xx)[\varphi]_{yy} \to (\forall xx \succcurlyeq dd)(\exists yy \succcurlyeq xx)[\varphi]_{yy}. \tag{A.8}$$

Assume the antecedent of (A.8). Then there are $aa \succcurlyeq dd$ such that, for any $yy \succcurlyeq aa$, we have $[\varphi]_{yy}$. To prove the consequent, consider any $bb \succcurlyeq dd$. We need to find $cc \succcurlyeq bb$ such that $[\varphi]_{cc}$. Let cc be the pairwise union of aa and bb. This union has the desired property.

The rule of Necessitation states that, if φ is proved with no non-logical premises, then $\Box\varphi$ too is a theorem. This translates as the rule that, if $[\varphi]_{dd}$ is proved with no non-logical premises, then BPL* proves $[\Box\varphi]_{dd}$, i.e., $(\forall yy \succeq dd)[\varphi]_{yy}$, as well. This follows because 'dd' can be universally generalized so as to obtain $\forall dd[\varphi]_{dd}$, which is equivalent to our target.

It remains only to consider the axioms describing the interaction of plurals and modals. Consider first the stability axiom $y \prec xx \to \Box y \prec xx$. Since $[\varphi]_{vv}$ is equivalent to φ when this formula is quantifier-free, the mentioned axiom translates as something trivial. The same goes for negative stability. Consider next (INEXT$\prec$). It is routine to verify that its translation

$$[\forall x(x \prec xx \to \Box\theta) \to \Box\forall x(x \prec xx \to \theta)]_{dd} \tag{A.9}$$

is provable. The same goes for (INEXT$\preccurlyeq$).

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Suppose we used the world relativization translation instead of the plain relativization translation. Then, all of the above claims would still go through, using (P-in-W). This concludes the proof. □

We are now ready to prove the right-to-left direction of Plural Mirroring (Theorem 2).

LEMMA A.4 (Right-to-left of Plural Mirroring). *The potentialist translation is faithful, in the sense that we have*

$$\varphi_1, \dots, \varphi_n \vdash^{BPL} \psi \quad \Leftarrow \quad \varphi_1^\diamond, \dots, \varphi_n^\diamond \vdash^{M\text{-}TPL} \psi^\diamond.$$

Proof. Assume the right-hand side. Since none of the formal systems involves any special assumption about dd, we may assume, without loss of generality, that none of the φ_i or ψ involve dd. (Otherwise, just reletter so as to avoid the use of dd.) By Lemma A.3, our assumption entails

$$\mathrm{Aux}, [\varphi_1^\diamond]_{dd}, \dots, [\varphi_n^\diamond]_{dd} \vdash^{\mathrm{BPL}} [\psi^\diamond]_{dd}.$$

By Lemma A.2, we have

$$\mathrm{Aux}, \varphi_1, \dots, \varphi_n \vdash^{\mathrm{BPL}} \psi.$$

Since the φ_i's and ψ do not contain 'dd', any appeal to an auxiliary axiom can be eliminated, which establishes the left-hand side of the desired implication. □

Of course, Lemmas A.1 and A.4 jointly complete the proof of Theorem 2 (Plural Mirroring).

We showed above that the two "round-trip" translations from $\mathcal{L}$ to $\mathcal{L}^\diamond$ and back again are equivalent, modulo BPL, to the identity translation. We now observe that an analogous result holds for the two "round-trip" translations that begin and end at $\mathcal{L}^\diamond$, with an intermediate step at $\mathcal{L}$.

LEMMA A.5 (Round-trip translation from $\mathcal{L}^\diamond$). *We have* $Dom \vdash^{M\text{-}TPL} \varphi \leftrightarrow ([\varphi]_{dd})^\diamond$.

Proof. We proceed by induction on syntactic complexity. Both translations are trivial on atomic formulas. And both translations commute with the logical connectives.

Let us now verify that the above induction holds for universally quantified formulas provided that it holds for the formulas being quantified. First, we observe that $[\forall x \varphi(x)]_{dd}$ is $(\forall x \prec dd)[\varphi]_{dd}$. Next, we apply the potentialist translation to obtain $\Box\forall x(x \prec dd \rightarrow ([\varphi]_{dd})^\diamond)$. I claim that this is equivalent to $(\forall x \prec dd)\Box([\varphi]_{dd})^\diamond$. For the left-to-right, use CBF to push the outer box through the quantifier, use axiom K to distribute it over the conditional, and invoke (STB$\prec$). For the reverse direction, use (INEXT$\prec$). Since every formula in the range of the potentialist translation is stable, this is in turn equivalent to $(\forall x \prec dd)([\varphi]_{dd})^\diamond$. By IH and (Dom), this is equivalent to the formula with which we started.

The case of the univeral plural quantifier is analogous. (This argument uses (INEXT$\preccurlyeq$).) This complete our proof. □

LEMMA A.6 (Left-to-right of Reverse Plural Mirroring). *We have*

$$Aux, [\varphi_1]_{dd}, \dots, [\varphi_n]_{dd} \vdash^{BPL} [\psi]_{dd} \quad \Rightarrow \quad Dom, \varphi_1, \dots, \varphi_n \vdash^{M\text{-}TPL} \psi.$$

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Proof. Assume the left-hand side. By Lemma A.1, we obtain

$$\text{Aux}^{\diamond}, ([\varphi_1]_{dd})^{\diamond}, \ldots, ([\varphi_n]_{dd})^{\diamond} \vdash^{\text{M-TPL}} ([\psi]_{dd})^{\diamond}. \tag{A.10}$$

Since the potentialist translation of each auxiliary axiom is a deductive consequence of (Dom), the claim follows by invoking Lemma A.5. □

Of course, Lemmas A.3 and A.6 jointly complete the proof of Theorem 5 (Reverse Plural Mirroring).

A.6. Definitional equivalence. Summing up, we have established two results about faithful interpretability and two about "round-trip" translations. All these results are combined into the overarching theorem of definitional equivalence, Theorem 3, stated in §4.

§B. Potentialist set theory based on Basic Plural Logic. In the main text we developed a demodalized form of potentialist set theory based on Critical Plural Logic. Let us now examine the analogous theory based on Basic Plural Logic.

THEOREM B.1. *BPL and the principles* (*SA*) *and* (*SI*) *prove the following axioms of set theory*: *Extensionality*, *Foundation*, *Empty Set*, *Pair*, *Separation*, *and Choice*.

Union too is easily obtained. Recall from §A.2 the notion of a world plurality, defined as a plurality that comprises the domain of some possible world. Since the possible worlds represent stages of a generative process, a world plurality thus consists of all and only the objects available at some stage of the generation. Which are the world pluralities in our current example of set-theoretic potentialism? First, we know that every set x is the set obtained from some plurality uu. Thus, for a world plurality to contain a set x, it must also contain each of uu from which x was generated. It follows that every world plurality is transitive, where yy are said to be transitive just in case $x \in y \prec yy \rightarrow x \prec yy$. Conversely, for any transitive plurality, we can produce a chronicle of how exactly these objects were generated. Thus, in the special case of set-theoretic potentialism, the world pluralities are precisely the transitive pluralities. Since every plurality is contained in a world plurality, we therefore adopt

$$\forall xx \exists yy (xx \preccurlyeq yy \wedge \text{transitive}(yy)). \tag{P-in-Tr}$$

When this axiom is added, we can prove the axiom of Union.[41]

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[41] *Proof.* Given x, find xx such that $x = \{xx\}$. Now find transitive yy containing xx. Use (P-SEP) to cut yy down to all and only the elements of elements of x. The set of all these is the union of x.

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